Acta Appl Math (2013) 123:73–97
DOI 10.1007/s10440-012-9715-1

Strong Law of Large Numbers for a Class of Superdiffusions

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Received: 9 September 2011 / Accepted: 2 April 2012 / Published online: 24 April 2012

Abstract In this paper we prove that, under certain conditions, a strong law of large numbers holds for a class of superdiffusions X corresponding to the evolution equation $\partial_t u_t = L u_t + \beta u_t - \psi(u_t)$ on a domain of finite Lebesgue measure in $\mathbb{R}^d$, where L is the generator of the underlying diffusion and the branching mechanism $\psi(x,\lambda) = \frac{1}{2}\alpha(x)\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) n(x, \mathrm{d}r)$ satisfies $\sup_{x\in D}\int_0^\infty (r\wedge r^2) n(x, \mathrm{d}r) < \infty$.

Keywords Superdiffusion · Martingale · Point process · Principal eigenvalue · Strong law of large numbers

Mathematics Subject Classification 60J68 · 60G55 · 60G57

The research of Y.-X. Ren is supported by NSFC (Grant Nos. 10971003 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education.
The research of R. Song is supported in part by a grant from the Simons Foundation (208236).

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1 Introduction

1.1 Motivation

Recently many people (see [3, 4, 9–12, 26] and the references therein) have studied limit theorems for branching Markov processes or super-processes using the principal eigenvalue and ground state of the linear part of the characteristic equations. All the papers above, except [11], assumed that the branching mechanisms satisfy a second moment condition. In [11], a $(1+\theta)$-moment condition, $\theta > 0$, on the branching mechanism is assumed instead.

In [1], Asmussen and Hering established a Kesten-Stigum $L\log L$ type theorem for a class of branching diffusion processes under a condition which is later called a positive regular property in [2]. In [19, 20] we established Kesten-Stigum $L\log L$ type theorems for superdiffusions and branching Hunt processes respectively.

This paper is a natural continuation of [19, 20]. Our main purpose of this paper is to establish a strong law of large numbers for a class of superdiffusions and our main tool is the stochastic integral representation of superdiffusions.

Throughout this paper, we will use the following notations. For any positive integer k, $C_b^k(\mathbb{R}^d)$ denotes the family of bounded functions on $\mathbb{R}^d$ whose partial derivatives of order up to k are bounded and continuous, $C_0^k(\mathbb{R}^d)$ denotes the family of functions of compact support on $\mathbb{R}^d$ whose partial derivatives of order up to k are continuous. For any open set $D \subset \mathbb{R}^d$, the meanings of $C_b^k(D)$ and $C_0^k(D)$ are similar. $\mathcal{B}(D)$ stands for the family of Borel functions on D, $\mathcal{B}^+(D)$ stands for the family of non-negative Borel functions on D, and $\mathcal{B}_b^+(D)$ stands for the family of non-negative bounded Borel functions on D. We denote by $\mathcal{M}_F(D)$ the space of finite measures on D equipped with the topology of weak convergence. We will use $\mathcal{M}_F(D)^0$ to denote the subspace of nontrivial measures (i.e., nonzero measures) in $\mathcal{M}_F(D)$. The integral of a function φ with respect to a measure μ will often be denoted as $\langle\varphi, \mu\rangle$.

For convenience we use the following convention throughout this paper: For any probability measure P, we also use P to denote the expectation with respect to P.

1.2 Model

In this paper, we will always assume that D is a domain of finite Lebesgue measure in $\mathbb{R}^d$. Suppose that $a_{ij} \in C_b^1(\mathbb{R}^d)$, $i, j = 1, \ldots, d$, and that the matrix (a_{ij}) is symmetric and satisfies

$$\kappa|\upsilon|^2 \le \sum_{i,j} a_{ij}\upsilon_i\upsilon_j, \qquad \text{for all } x \in \mathbb{R}^d \text{ and } \upsilon \in \mathbb{R}^d$$

for some positive constant κ. We assume that b_i, $i = 1, \ldots, d$, are bounded Borel functions on $\mathbb{R}^d$. Under these assumptions, there is a diffusion process $(\xi, \Pi_x, x \in \mathbb{R}^d)$ corresponding to the operator

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla.$$

We will use $(\xi^D,\ \Pi_x,\ x \in D)$ to denote the process obtained by killing ξ upon exiting from D, that is,

$$\xi_t^D = \begin{cases} \xi_t, & \text{if } t < \tau, \\ \partial, & \text{if } t \ge \tau, \end{cases}$$

where $\tau = \inf\{t > 0; \xi_t \notin D\}$ is the first exit time of D and ∂ is a cemetery point. Any function f on D is automatically extended to $D \cup \{\partial\}$ by setting $f(\partial) = 0$. The reason we defined the coefficients a_{ij}, b_j and assumed the above conditions on them on the whole of $\mathbb{R}^d$ is to guarantee the existence of ξ. Since we are interested in superdiffusions with underlying motion ξ^D in this paper, what we really need is that the assumptions above on the coefficients a_{ij}, b_j are valid on D.

We will always assume that β is a bounded Borel function on D. We will use $\{P_t^D\}_{t\geq 0}$ to denote the following Feynman-Kac semigroup

$$P_t^D f(x) = \Pi_x\left(\exp\left(\int_0^t \beta\left(\xi_s^D\right)\mathrm{d}s\right) f\left(\xi_t^D\right)\right), \quad x \in D.$$

It is easy to show (see, for instance, the arguments in [5, Sect. 3.2] and [24, Sect. 2]) that the semigroup $\{P_t^D\}_{t\geq 0}$ is strongly continuous in $L^2(D)$ and, for any $t > 0$, P_t^D has a bounded, continuous and strictly positive density $p^D(t, x, y)$.

Let $\{\widehat{P}_t^D\}_{t\geq 0}$ be the dual semigroup of $\{P_t^D\}_{t\geq 0}$ defined by

$$\widehat{P}_t^D f(x) = \int_D p^D(t, y, x) f(y)\mathrm{d}y, \quad x \in D.$$

It is well known (see, for instance, [8, p. 8]) that $\{\widehat{P}_t^D\}_{t\geq 0}$ is also strongly continuous in $L^2(D)$.

Let $\mathbf{A}$ and $\widehat{\mathbf{A}}$ be the generators of the semigroups $\{P_t^D\}_{t\geq 0}$ and $\{\widehat{P}_t^D\}_{t\geq 0}$ in $L^2(D)$ respectively. Let $\sigma(\mathbf{A})$ ($\sigma(\widehat{\mathbf{A}})$ resp.) denote the spectrum of $\mathbf{A}$ ($\widehat{\mathbf{A}}$, resp.). It follows from Jentzsch's theorem ([23, Theorem V.6.6, p. 337]) and the strong continuity of $\{P_t^D\}_{t\geq 0}$ and $\{\widehat{P}_t^D\}_{t\geq 0}$ that the common value $\lambda_1 := \sup \mathrm{Re}(\sigma(\mathbf{A})) = \sup \mathrm{Re}(\sigma(\widehat{\mathbf{A}}))$ is an eigenvalue of multiplicity 1 for both $\mathbf{A}$ and $\widehat{\mathbf{A}}$, and that an eigenfunction ϕ of $\mathbf{A}$ associated with λ_1 can be chosen to be strictly positive a.e. on D and an eigenfunction $\widehat{\phi}$ of $\widehat{\mathbf{A}}$ associated with λ_1 can be chosen to be strictly positive a.e. on D. By [16, Proposition 2.3] we know that ϕ and $\widehat{\phi}$ are bounded and continuous on D, and so they are in fact strictly positive everywhere on D. We choose ϕ and $\widehat{\phi}$ so that $\int_D \phi(x)\widehat{\phi}(x)\mathrm{d}x = 1$.

Throughout this paper we assume the following

Assumption 1 The semigroups $\{P_t^D\}_{t\geq 0}$ and $\{\widehat{P}_t^D\}_{t\geq 0}$ are intrinsically ultracontractive, that is, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p^D(t, x, y) \leq c_t \phi(x)\widehat{\phi}(y), \quad \text{for all } (x, y) \in D \times D.$$

Intrinsic ultracontractivity for non-symmetric semigroups was defined for semigroups on $L^2(E, m)$, where E is a locally compact separable metric space and m is a finite measure on E. This is the reason that we assume that D is of finite Lebesgue measure since we are dealing with semigroups on $L^2(D)$ with respect to the Lebesgue measure. Assumption 1 is a very weak regularity assumption on D. It follows from [16, 17] that Assumption 1 is satisfied when D is a bounded Lipschitz domain. For other, more general, examples of domains D for which Assumption 1 is satisfied, we refer our readers to [17] and the references therein.

Define the ground state transform of p^D by

$$q^D(t, x, y) = \frac{e^{-\lambda_1 t}}{\phi(x)}\, p^D(t, x, y)\, \phi(y). \tag{1.1}$$

Then it follows from [16, Theorem 2.7] that if Assumption 1 holds, then for any $\sigma > 0$ there are positive constants $C(\sigma)$ and ν such that

$$\begin{aligned}\left|q^D(t,x,y) - \phi(y)\widehat{\phi}(y)\right| &= \left|\frac{e^{-\lambda_1 t} p^D(t,x,y)\phi(y)}{\phi(x)} - \phi(y)\widehat{\phi}(y)\right| \\ &\le C(\sigma)e^{-\nu t}\phi(y)\widehat{\phi}(y), \quad x, y \in D, t > \sigma. \end{aligned} \tag{1.2}$$

By the definition of ϕ and $\widehat{\phi}$, it is easy to check that, for any $t > 0$, $q^D(t,\cdot,\cdot)$ is a probability density and that $\phi\widehat{\phi}$ is its unique invariant probability density. The above display shows that $q^D(t,\cdot,x)$ converges to $\phi(x)\widehat{\phi}(x)$ uniformly with exponential rate. Denote by Q_t^D the semigroup with density $q^D(t,\cdot,\cdot)$. For any measure μ on D, Π_μ^D denotes the probability generated by $(Q_t^D)_{t\ge 0}$ with initial distribution μ. When $\mu = \delta_x$, Π_μ^D will be written as Π_x^D, and when $\mu(\mathrm{d}x) = u(x)\mathrm{d}x$ for some function u on D, Π_μ^D will be written as Π_u^D. Then $(\xi^D, \Pi_{\phi\widehat{\phi}}^D)$ is a diffusion with initial distribution $\phi(x)\widehat{\phi}(x)\mathrm{d}x$.

The superdiffusion $(X, \mathbb{P}_\mu)$, $\mu \in \mathcal{M}_F(D)$, we are going to study is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-super-process, which is a measure-valued Markov process with underlying spatial motion ξ^D, branching rate $\mathrm{d}t$ and branching mechanism $\psi(\lambda) - \beta\lambda$, where

$$\psi(x,\lambda) = \frac{1}{2}\alpha(x)\lambda^2 + \int_0^\infty \left(e^{-r\lambda} - 1 + \lambda r\right)n(x,\mathrm{d}r), \quad \lambda > 0,$$

for some nonnegative bounded measurable function α on D and for some σ-finite kernel n from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, that is, $n(x,\mathrm{d}r)$ is a σ-finite measure on $\mathbb{R}_+$ for each fixed x, and $n(\cdot, B)$ is a measurable function for each Borel set $B \subset \mathbb{R}_+$. The measure μ here is the initial value of X. In this paper we will always assume that

$$\sup_{x\in D}\int_0^\infty \left(r \wedge r^2\right)n(x,\mathrm{d}r) < \infty. \tag{1.3}$$

Note that this assumption implies, for any fixed $\lambda > 0$, $\psi(\cdot,\lambda)$ is bounded on D. Define a new kernel $n^\phi(x,\mathrm{d}r)$ from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that for any nonnegative measurable function f on $\mathbb{R}_+$,

$$\int_0^\infty f(r)n^\phi(x,\mathrm{d}r) = \int_0^\infty f\left(r\phi(x)\right)n(x,\mathrm{d}r), \quad x \in D. \tag{1.4}$$

Then, by (1.3) and the boundedness of ϕ, n^ϕ satisfies

$$\sup_{x\in D}\int_0^\infty \left(r \wedge r^2\right)n^\phi(x,\mathrm{d}r) < \infty. \tag{1.5}$$

1.3 Stochastic Integral Representation and Main Result

Let $(\Omega, \mathcal{F}, \mathbb{P}_\mu, \mu \in \mathcal{M}_F(D))$ be the underlying probability space equipped with the filtration $(\mathcal{F}_t)$, which is generated by X and is completed as usual with the $\mathcal{F}_\infty$-measurable and $\mathbb{P}_\mu$-negligible sets for every $\mu \in \mathcal{M}_F(D)$. Without loss of generality we can assume that $(\Omega, \mathcal{F})$ is the space of all the cadlag functions from $[0,\infty)$ to $\mathcal{M}_F(D)$ equipped with its Borel σ-field.

Set $M_t(\phi) := e^{-\lambda_1 t}\langle\phi, X_t\rangle$. Then $M_t(\phi)$, $t \ge 0$, is a nonnegative right continuous martingale with left limits, see (1.18) below. Denote by $M_\infty(\phi)$ the almost sure limit of $M_t(\phi)$ as

$t \to \infty$. All the martingales in this paper are right continuous with left limits. We will not mention this explicitly.

The main goal of this paper is to establish the following almost sure convergence result.

Theorem 1.1 *Suppose Assumption 1 holds, $\lambda_1 > 0$ and that X is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion. Then there exists $\Omega_0 \subset \Omega$ of probability one (that is, $\mathbb{P}_\mu(\Omega_0) = 1$ for every $\mu \in \mathcal{M}_F(D)$) such that, for every $\omega \in \Omega_0$ and for every nonnegative bounded Borel measurable function f on D such that $f \le c\phi$ for some $c > 0$ and that the set of discontinuous points of f has zero Lebesgue measure, we have*

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle f, X_t\rangle(\omega) = M_\infty(\phi)(\omega)\int_D \widehat{\phi}(y) f(y)\mathrm{d}y. \tag{1.6}$$

As a consequence of this theorem, we have the following result.

Theorem 1.2 *Suppose Assumption 1 holds, $\lambda_1 > 0$ and that X is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion. Then there exists $\Omega_0 \subset \Omega$ of probability one (that is, $\mathbb{P}_\mu(\Omega_0) = 1$ for every $\mu \in \mathcal{M}_F(D)^0$) such that, for every $\omega \in \Omega_0$ and for every nontrivial nonnegative bounded Borel function f on D such that $f \le c\phi$ for some constant $c > 0$ and that the set of discontinuous points of f has zero Lebesgue measure, we have*

$$\lim_{t\to\infty} \frac{\langle f, X_t\rangle(\omega)}{\mathbb{P}_\mu\langle f, X_t\rangle} = \frac{M_\infty(\phi)(\omega)}{\langle \phi, \mu\rangle}. \tag{1.7}$$

Note that the above result says that, on the set $\{M_\infty(\phi)(\omega) > 0\}$, the quantity $\langle f, X_t\rangle(\omega)$ grows like its expectation. So this result can be regarded as a strong law of large numbers.

As a special case of this theorem we immediately get the following

Corollary 1.3 *Suppose Assumption 1 holds, $\lambda_1 > 0$ and that X is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion. Then there exists $\Omega_0 \subset \Omega$ of probability one (that is, $\mathbb{P}_\mu(\Omega_0) = 1$ for every $\mu \in \mathcal{M}_F(D)^0$) such that, for every $\omega \in \Omega_0$ and every relatively compact Borel subset B of D of positive Lebesgue measure whose boundary is of Lebesgue measure zero, we have*

$$\lim_{t\to\infty} \frac{X_t(B)(\omega)}{\mathbb{P}_\mu[X_t(B)]} = \frac{M_\infty(\phi)(\omega)}{\langle \phi, \mu\rangle}.$$

Remark 1.4

(i) The general strategy, to be presented in Sect. 2, for proving our main result is similar to that of [3]. However, since our process ξ is not symmetric in general, one of the key steps in [3], the proof of [3, Lemma 3.5], does not go through. We have to find a way to get around this difficulty, see the proof of Theorem 2.1 below.

(ii) In [3, 9, 10, 12, 26], the branching mechanism is assumed to be binary, while in the present paper we deal with a general branching mechanism. The paper [11] considers a general branching mechanism under a $(1+\theta)$-moment condition, $\theta > 0$, while in the present paper, we only assume a $L \log L$ condition. In [3] the underlying motion is assumed to be a symmetric Hunt process, while in the present paper, our underlying process need not be symmetric.

(iii) Although our Assumption 1 on the linear semi-group P_t^D is mild, it does exclude some interesting cases. For example, the superprocess analogues of [10, Examples 10 and 11] do not satisfy Assumption 1. So it is worthwhile to relax this assumption.

(iv) Our Assumption 1 is similar to condition (M) in [1], which is called a positive regular property in [2]. We prefer to use Assumption 1 which is stated in terms of intrinsic ultracontractivity because there are many (easy to check) sufficient conditions for intrinsic ultracontractivity in the literature.

Note that the quantity $M_\infty(\phi)$ in Theorems 1.1–1.2 and Corollary 1.3 may be zero almost surely. If $M_\infty(\phi)=0$ a.s., then (1.6) does not give the exact growth rate of $\langle f, X_t\rangle$ as t goes to infinity. It is a very interesting problem to find a function $s(t)$ such that $s(t)e^{-\lambda_1 t}\langle f, X_t\rangle$ has a non-degenerate limit as t tends to infinity. This is beyond the reach of this paper, and we intend to deal with this in a future project.

In [19], we studied the relationship between the degeneracy property of $M_\infty(\phi)$ and the following function l:

$$l(y) := \int_1^\infty r \ln r n^\phi(y, \mathrm{d}r), \tag{1.8}$$

and established an $L\log L$ criterion (see Theorem 1.5 below) in the case when $\alpha = 0$. To extend this criterion to the case $\alpha \geq 0$, we will need the integral representation of superdiffusions.

We will use the standard notation $\Delta X_s = X_s - X_{s-}$ for the jump of X at time s. It is known (cf. [6, Sect. 6.1]) that the superdiffusion X is a solution to the following martingale problem: for any $\varphi \in C_0^2(D)$ and $h \in C_b^2(\mathbb{R})$,

$$\begin{aligned} & h\big(\langle\varphi, X_t\rangle\big) - h\big(\langle\varphi, \mu\rangle\big) - \int_0^t h'\big(\langle\varphi, X_s\rangle\big)\langle \mathbf{A}\varphi, X_s\rangle \mathrm{d}s - \frac{1}{2}\int_0^t h''\big(\langle\varphi, X_s\rangle\big)\big\langle\alpha\varphi^2, X_s\big\rangle \mathrm{d}s \\ & \quad - \int_0^t \int_D \int_{(0,\infty)} \big(h\big(\langle\varphi, X_s\rangle + r\varphi(x)\big) - h\big(\langle\varphi, X_s\rangle\big) - h'\big(\langle\varphi, X_s\rangle\big) r\varphi(x)\big) n(x, \mathrm{d}r) X_s(\mathrm{d}x)\mathrm{d}s \end{aligned} \tag{1.9}$$

is a martingale. Let J denote the set of all jump times of X and δ denote the Dirac measure. From the last part of martingale problem (1.9), one infers that the only possible jumps of X are point measures of the form $r\delta_x$ for some $r>0$ and $x\in D$, see [18, Sect. 2.3]. Thus the compensator of the random measure (for the definition of the compensator of a random measure, see, for instance, [6, p. 107])

$$N := \sum_{s\in J} \delta_{(s,\Delta X_s)}$$

is a random measure $\widehat{N}$ on $\mathbb{R}_+ \times \mathcal{M}_F(D)$ such that for any nonnegative predictable function F on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)$,

$$\int_0^\infty \int_{\mathcal{M}_F(D)} F(s,\omega,\nu)\widehat{N}(\mathrm{d}s, \mathrm{d}\nu) = \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_0^\infty F(s,\omega, r\delta_x) n(x, \mathrm{d}r), \tag{1.10}$$

where $n(x, \mathrm{d}r)$ is the kernel of the branching mechanism ψ. Therefore we have

$$\mathbb{P}_\mu\Big[\sum_{s\in J} F(s,\omega,\Delta X_s)\Big] = \mathbb{P}_\mu \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_0^\infty F(s,\omega,r\delta_x) n(x,\mathrm{d}r). \tag{1.11}$$

See [6, p. 111]. Let F be a predictable function on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)$ satisfying

$$\mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} F(s,\Delta X_s)^2\Big)^{1/2}\Big] < \infty, \quad \text{for all } \mu \in \mathcal{M}_F(D).$$

Then the stochastic integral of F with respect to the compensated random measure $N - \widehat{N}$

$$\int_0^t \int_{\mathcal{M}_F(D)} F(s,\nu)(N-\widehat{N})(\mathrm{d}s,\mathrm{d}\nu),$$

can be defined (cf. [18] and the reference therein) as the unique purely discontinuous martingale (vanishing at time 0) whose jumps are indistinguishable from $I_J(s)F(s,\Delta X_s)$. Here and throughout this paper, for any set A, I_A stands for the indicator function of A.

Suppose that φ is a measurable function on $\mathbb{R}_+ \times D$. Define

$$F_\varphi(s,\nu) := \int_D \varphi(s,x)\nu(\mathrm{d}x), \quad \nu \in \mathcal{M}_F(D) \tag{1.12}$$

whenever the integral above makes sense. We write

$$S_t^J(\varphi) = \int_0^t \int_D \varphi(s,x) S^J(\mathrm{d}s,\mathrm{d}x) := \int_0^t \int_{\mathcal{M}_F(D)} F_\varphi(s,\nu)(N-\widehat{N})(\mathrm{d}s,\mathrm{d}\nu), \tag{1.13}$$

whenever the right hand side of (1.13) makes sense. If φ is bounded on $\mathbb{R}_+ \times D$, then $S_t^J(\varphi)$ is well defined. Indeed, we only need to check that

$$\mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} F_\varphi(s,\Delta X_s)^2\Big)^{1/2}\Big] < \infty, \quad \text{for all } \mu \in \mathcal{M}_F(D). \tag{1.14}$$

Note that, for any $\mu \in \mathcal{M}_F(D)$,

$$\begin{aligned}
&\mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} F_\varphi(s,\Delta X_s)^2\Big)^{1/2}\Big] \\
&\quad = \mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} \Big(\int \varphi(s,x)(\Delta X_s)(\mathrm{d}x)\Big)^2\Big)^{1/2}\Big] \\
&\quad \le \|\varphi\|_\infty \mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} \langle 1, \Delta X_s\rangle^2 I_{\{\langle 1,\Delta X_s\rangle \le 1\}} + \sum_{s\in[0,t], s\in J} \langle 1, \Delta X_s\rangle^2 I_{\{\langle 1,\Delta X_s\rangle > 1\}}\Big)^{1/2}\Big] \\
&\quad \le \|\varphi\|_\infty \mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} \langle 1, \Delta X_s\rangle^2 I_{\{\langle 1,\Delta X_s\rangle \le 1\}}\Big)^{1/2}\Big] \\
&\qquad + \|\varphi\|_\infty \mathbb{P}_\mu\Big[\Big(\sum_{s\in[0,t], s\in J} \langle 1, \Delta X_s\rangle^2 I_{\{\langle 1,\Delta X_s\rangle > 1\}}\Big)^{1/2}\Big].
\end{aligned}$$

Using the first two displays on [18, p. 203], we get (1.14). Thus for any bounded function φ on $\mathbb{R}_+ \times D$, $(S_t^J(\varphi))_{t\geq 0}$ is a martingale.

For any $\varphi \in C_0^2(D)$ and $\mu \in \mathcal{M}_F(D)$,

$$\langle \varphi, X_t\rangle = \langle \varphi, \mu\rangle + S_t^J(\varphi) + S_t^C(\varphi) + \int_0^t \langle \mathbf{A}\varphi, X_s\rangle \mathrm{d}s, \tag{1.15}$$

where $S_t^C(\varphi)$ is a continuous local martingale with quadratic variation

$$\left\langle S^C(\varphi)\right\rangle_t = \int_0^t \left\langle \alpha\varphi^2, X_s\right\rangle \mathrm{d}s. \tag{1.16}$$

In fact, according to [13, 14], the above is still valid when $\mathbf{A}$ is replaced by $\mathbf{L} + \beta$, where $\mathbf{L}$ is the weak generator of ξ^D in the sense of [13, Sect. 4]. Using this, [14, Corollary 2.18] and applying a limit argument, one can show that for any bounded function g on D,

$$\langle g, X_t\rangle = \left\langle P_t^D g, \mu\right\rangle + \int_0^t \int_D P_{t-s}^D g(x) S^J(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_D P_{t-s}^D g(x) S^C(\mathrm{d}s, \mathrm{d}x), \tag{1.17}$$

where $S^J(\mathrm{d}s, \mathrm{d}x)$ is defined by (1.13) and $S^C(\mathrm{d}s, \mathrm{d}x)$ is a martingale measure in the sense of Walsh [25] (see [14] or [21] for the precise definition). In particular, taking $g = \phi$ in (1.17), where ϕ is the positive eigenfunction of $\mathbf{A}$ defined in Sect. 1.1, we get that

$$e^{-\lambda_1 t}\langle \phi, X_t\rangle = \langle \phi, \mu\rangle + \int_0^t e^{-\lambda_1 s}\int_D \phi(x) S^J(\mathrm{d}s, \mathrm{d}x) + \int_0^t e^{-\lambda_1 s}\int_D \phi(x) S^C(\mathrm{d}s, \mathrm{d}x). \tag{1.18}$$

The following result is the $L\log L$ criterion mentioned above. The condition in the first part of the theorem below says that the kernel n^ϕ satisfies an $L\log L$ integrability condition.

Theorem 1.5 [19, *Theorem* 1.1] *Suppose that Assumption* 1 *holds*, $\lambda_1 > 0$ *and that* X *is a* $(\xi^D, \psi(\lambda) - \beta\lambda)$*-superdiffusion. Then the following assertions hold*:

(1) *If* $\int_D l(y)\widehat{\phi}(y)\mathrm{d}y < \infty$, *then* $M_\infty(\phi)$ *is non-degenerate under* $\mathbb{P}_\mu$ *for any* $\mu \in \mathcal{M}_F(D)^0$, *and* $M_\infty(\phi)$ *is also the* $L^1(\mathbb{P}_\mu)$ *limit of* $M_t(\phi)$.
(2) *If* $\int_D l(y)\widehat{\phi}(y)\mathrm{d}y = \infty$, *then* $M_\infty(\phi) = 0$, $\mathbb{P}_\mu$-a.s. *for any* $\mu \in \mathcal{M}_F(D)^0$.

Remark 1.6 In [19, Theorem 1.1], we only stated that in case (1) under the extra assumption $\alpha \equiv 0$, $M_\infty(\phi)$ is non-degenerate under $\mathbb{P}_\mu$ for any $\mu \in \mathcal{M}_F(D)^0$. But actually in this case we have $\mathbb{P}_\mu M_\infty(\phi) = \mathbb{P}_\mu M_0(\phi)$ (see [19, Lemma 3.4]), and therefore $M_t(\phi)$ converges to $M_\infty(\phi)$ in $L^1(\mathbb{P}_\mu)$.

For general $\alpha \geq 0$, by the L^2 maximum inequality (see [7, Theorem 4.4.3]), and using the fact that α and ϕ are bounded in D, we have

$$\begin{aligned}
&\mathbb{P}_\mu\left[\sup_{t\geq 0}\left(\int_0^t e^{-\lambda_1 s}\int_D \phi(x) S^C(\mathrm{d}s, \mathrm{d}x)\right)^2\right] \\
&\leq 4\sup_{t\geq 0}\mathbb{P}_\mu\left(\int_0^t e^{-\lambda_1 s}\int_D \phi(x) S^C(\mathrm{d}s, \mathrm{d}x)\right)^2 \\
&= 4\mathbb{P}_\mu \int_0^\infty e^{-2\lambda_1 s}\mathrm{d}s \int_D \alpha(x)\phi^2(x) X_s(\mathrm{d}x)
\end{aligned}$$

$$= 4\int_0^\infty e^{-\lambda_1 s}\mathrm{d}s \int_D \phi(y)\mu(\mathrm{d}y)\int_D q^D(s,y,x)\alpha(x)\phi(x)\mathrm{d}x$$
$$< \infty.$$

Thus the martingale $(\int_0^t e^{-\lambda_1 s}\int_D \phi(x)S^C(\mathrm{d}s,\mathrm{d}x))_{t\geq 0}$ converges almost surely and in $L^1(\mathbb{P}_\mu)$. Denote the limit by $\int_0^\infty e^{-\lambda_1 s}\int_D \phi(x)S^C(\mathrm{d}s,\mathrm{d}x)$. Furthermore, when $\lambda_1 > 0$ and $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$, the martingale $\int_0^t e^{-\lambda_1 s}\int_D \phi(x)S^J(\mathrm{d}s,\mathrm{d}x)$ converges almost surely and in $L^1(\mathbb{P}_\mu)$ as well. Denote the limit by $\int_0^\infty e^{-\lambda_1 s}\int_D \phi(x)S^J(\mathrm{d}s,\mathrm{d}x)$. Thus it follows from (1.18) that $M_t(\phi)$ converges to a non-degenerate $M_\infty(\phi)$ $\mathbb{P}_\mu$-almost surely and in $L^1(\mathbb{P}_\mu)$ for every $\mu \in \mathcal{M}_F(D)^0$.

2 Proof of Main Results

In this section we will give the proofs of our main results, Theorems 1.1–1.2. These proofs will be based on Theorem 2.1 below. The proof of Theorem 2.1 is pretty long and contains most of technical contributions of this paper. For the benefit of our readers, the proof of Theorem 2.1 will be postponed until the last section.

Let $\{U^q; q>0\}$ be the resolvent operators associated with the semigroup $\{Q_t^D; t\geq 0\}$, i.e., for any $f \in \mathcal{B}_b(D)$,

$$U^q f(x) = \int_0^\infty e^{-qt} Q_t^D f(x)\mathrm{d}t, \quad x \in D. \tag{2.1}$$

In particular, if $f = I_A(x)$ for some Borel measurable set $A \subset D$, $U^q I_A(x)$ will be denoted by $U^q(x,A)$:

$$U^q(x,A) = U^q I_A(x) = \int_0^\infty e^{-qt}\Pi_x^D(\xi_t \in A)\mathrm{d}t, \quad x \in D. \tag{2.2}$$

Here is the statement of our main technical result. This result constitutes the major ingredient in the proofs of our main results.

Theorem 2.1 *Suppose Assumption 1 holds, $\lambda_1 > 0$ and that X is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion. Then for any $f \in \mathcal{B}_b^+(D)$, $q > 0$, and $\mu \in \mathcal{M}_F(D)$,*

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle \phi q U^q f, X_t\rangle = M_\infty(\phi)\int_D \widehat{\phi}(x)\phi(x)f(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.} \tag{2.3}$$

Moreover, when $\int_D \widehat{\phi}(x)l(x)\mathrm{d}x < \infty$, the above limit holds in the $L^1(\mathbb{P}_\mu)$ sense as well.

The main goal of this paper is to prove (2.4) below. For this we want to use a technique which was first used in [1] and later in [10] for branching diffusions and in [4] for more general branching Markov processes. The technique consists of first obtaining the almost sure limit result at discrete times and then extending it to all times. However the transition from discrete time to continuous time is pretty difficult for superdiffusions. In [3], the symmetry of the underlying Markov process played an essential role. Without the symmetry assumption, one of the key steps in [3], the proof of [3, Lemma 3.5], does not go through. Our strategy is to extend the discrete time limit result with I_A replaced by $U^q f$ first and then approach I_A by functions of the form $U^q f$.

Theorem 2.2 *Suppose Assumption 1 holds, $\lambda_1 > 0$ and that X is a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion. Then for any $\mu \in \mathcal{M}_F(D)$ and any relatively compact open subset A of D with $|\partial A| = 0$,*

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle = M_\infty(\phi)\int_A \widehat{\phi}(x)\phi(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.} \tag{2.4}$$

Proof Recall the definition of $U^q(x, A)$, $x \in D$, given by (2.2). Define the first hitting time $\sigma_A := \inf\{t > 0; \xi_t \in A\}$. Then for any $x \in D$,

$$qU^q(x, A) \le \int_0^\infty qe^{-qt}\Pi_x^D(\sigma_A \le t)\mathrm{d}t = \Pi_x^D\int_{\sigma_A}^\infty qe^{-qt}\mathrm{d}t = \Pi_x^D\big(\exp\{-q\sigma_A\}\big),$$

while for any $x \notin A$ and any closed subset A' of D,

$$U^q\big(x, A'\big) \ge \Pi_x^D\int_{\sigma_A}^\infty e^{-qt}I_{A'}(\xi_t)\mathrm{d}t = \Pi_x^D\big(e^{-q\sigma_A}U^q\big(\xi_{\sigma_A}, A'\big)\big).$$

Since ξ is a diffusion, we have $\xi_{\sigma_A} \in \partial A$ when it starts from $x \notin A$. Thus for $x \notin A$,

$$\Pi_x^D\big(e^{-q\sigma_A}\big) \le \Big(\inf_{y\in\partial A} U^q\big(y, A'\big)\Big)^{-1} U^q\big(x, A'\big). \tag{2.5}$$

Define $A_\varepsilon = \{x \in A, \mathrm{dist}(x, \partial A) \ge \varepsilon\}$, where $\varepsilon > 0$ is chosen such that $A_\varepsilon \ne \emptyset$. Then

$$\begin{aligned} e^{-\lambda_1 t}\big\langle \phi q U^q I_{A_\varepsilon}, X_t\big\rangle &\le e^{-\lambda_1 t}\big\langle \phi \Pi_\cdot^D e^{-q\sigma_{A_\varepsilon}}, X_t\big\rangle \\ &\le e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle + e^{-\lambda_1 t}\big\langle I_{D\setminus A}\phi\Pi_\cdot^D e^{-q\sigma_{A_\varepsilon}}, X_t\big\rangle. \end{aligned} \tag{2.6}$$

Recall that Π_x is the probability of ξ with infinitesimal generator L on $\mathbb{R}^d$ and that τ is the first exit time of ξ from D. According to the definition of Π_x^D, $x \in D$, for any $F \in \mathcal{G}_t := \sigma(\xi_s; s \le t)$,

$$\Pi_y^D(F) = \phi(y)^{-1}e^{-\lambda_1 t}\Pi_y\bigg(\exp\bigg\{\int_0^t \beta(\xi_s)\mathrm{d}s\bigg\}\phi(\xi_t); F, \tau > t\bigg). \tag{2.7}$$

Set $A'_\varepsilon = \{x \in A; \mathrm{dist}(x, \partial A_\varepsilon) \le \varepsilon/2\}$. When $\xi_0 \in \partial A_\varepsilon$, $\{\sup_{0\le s\le t/q}|\xi_s - \xi_0| \le \varepsilon/2\} \subset \{\xi_{t/q} \in A'_\varepsilon\}$. Moreover these two events belong to $\mathcal{G}_{t/q}$, thus for any $y \in \partial A_\varepsilon$,

$$\begin{aligned} &\Pi_y^D\big(\xi_{t/q} \in A'_\varepsilon\big) \\ &\quad\ge \Pi_y^D\Big(\sup_{0\le s\le t/q}|\xi_s - y| \le \varepsilon/2\Big) \\ &\quad= \phi(y)^{-1}e^{-\lambda_1 t/q}\Pi_y\bigg(\exp\bigg\{\int_0^{t/q}\beta(\xi_s)\mathrm{d}s\bigg\}\phi(\xi_{t/q});\ \sup_{0\le s\le t/q}|\xi_s - y| \le \varepsilon/2, \tau > t/q\bigg) \\ &\quad\ge \Big(\sup_{x\in\partial A_\varepsilon}\phi(x)\Big)^{-1}e^{-(\lambda_1+\|\beta\|_\infty)t/q}\Pi_y\Big(\inf_{x\in A'_\varepsilon}\phi(x);\ \sup_{0\le s\le t/q}|\xi_s - y| \le \varepsilon/2, \tau > t/q\Big). \end{aligned}$$

Since $\partial A_\varepsilon, A'_\varepsilon \subset A$, and $\{\sup_{0\le s\le t/q}|\xi_s - y| \le \varepsilon/2\} \subset \{\tau > t/q\}$ Π_y-a.s., we have

$$\Pi_y^D\big(\xi_{t/q} \in A'_\varepsilon\big) \ge \frac{\inf_{x\in A}\phi(x)}{\sup_{x\in A}\phi(x)}e^{-(\lambda_1+\|\beta\|_\infty)t/q}\Pi_y\Big(\sup_{0\le s\le t/q}|\xi_s - y| \le \varepsilon/2\Big).$$

Denote $\inf_{x\in A}\phi(x)/\sup_{x\in A}\phi(x)$ by $c(A,\phi)$. Then for any $y\in\partial A_\varepsilon$ and any fixed $T>0$,

$$
\begin{aligned}
qU^q\big(y,A'_\varepsilon\big) &= \int_0^\infty e^{-t}\,\Pi_y^D\big(\xi_{t/q}\in A'_\varepsilon\big)\mathrm{d}t\\
&\ge c(A,\phi)\int_0^T e^{-(1+(\lambda_1+\|\beta\|_\infty)/q)t}\,\Pi_y\Big(\sup_{0\le s\le t/q}|\xi_s-y|\le\varepsilon/2\Big)\mathrm{d}t\\
&\ge c(A,\phi)\int_0^T e^{-(1+(\lambda_1+\|\beta\|_\infty)/q)t}\,\mathrm{d}t\,\Pi_y\Big(\sup_{0\le s\le T/q}|\xi_s-y|\le\varepsilon/2\Big).
\end{aligned}
\tag{2.8}
$$

The operator L satisfies the assumptions in [22, Theorem 2.2.2], so when q is chosen to be large enough so that $\varepsilon>4(\|\nabla a\|_\infty+\|b\|_\infty)T/q$,

$$
\Pi_y\Big(\sup_{0\le s\le T/q}|\xi_s-y|\le\varepsilon/2\Big)\ge 1-2d\exp\left\{\frac{-q\varepsilon^2}{32dCT}\right\},
\tag{2.9}
$$

where

$$
C=\sup_{x\in D}\sup_{|v|=1}\big\langle v,a(x)v\big\rangle.
$$

Therefore, for q large enough,

$$
\begin{aligned}
&\inf_{y\in\partial A_\varepsilon} qU^q\big(y,A'_\varepsilon\big)\\
&\quad\ge \frac{c(A,\phi)(1-\exp\{-(1+(\lambda_1+\|\beta\|_\infty)/q)T\})}{1+(\lambda_1+\|\beta\|_\infty)/q}\left(1-2d\exp\left\{\frac{-q\varepsilon^2}{32dCT}\right\}\right).
\end{aligned}
\tag{2.10}
$$

Denote the right hand side of the above display by $V(q,T)$. It is obvious that $\lim_{T\to\infty}\lim_{q\to\infty}V(q,T)=c(A,\phi)$. Using (2.5) (applied to A_ε) and (2.10), we get that for any fixed $T>0$, and sufficiently large $q>0$,

$$
\begin{aligned}
&\limsup_{t\to\infty} e^{-\lambda_1 t}\big\langle I_{D\setminus A}\phi\Pi_\cdot^D e^{-q\sigma_{A_\varepsilon}},X_t\big\rangle\\
&\quad\le \frac{1}{V(q,T)}\limsup_{t\to\infty} e^{-\lambda_1 t}\big\langle \phi qU^q I_{A'_\varepsilon},X_t\big\rangle\\
&\quad= \frac{1}{V(q,T)}M_\infty(\phi)\int_{A'_\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x,\quad \mathbb{P}_\mu\text{-a.s.}
\end{aligned}
\tag{2.11}
$$

where in the last equality, we used Theorem 2.1. Letting $t\to\infty$ on both sides of (2.6), we get from Theorem 2.1 and (2.11) that

$$
\begin{aligned}
M_\infty(\phi)\int_{A_\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x &= \liminf_{t\to\infty} e^{-\lambda_1 t}\big\langle\phi qU^q I_{A_\varepsilon},X_t\big\rangle\\
&\le \liminf_{t\to\infty} e^{-\lambda_1 t}\langle\phi I_A,X_t\rangle+\limsup_{t\to\infty} e^{-\lambda_1 t}\big\langle I_{D\setminus A}\phi\Pi_\cdot^D e^{-q\sigma_{A_\varepsilon}},X_t\big\rangle\\
&\le \liminf_{t\to\infty} e^{-\lambda_1 t}\langle\phi I_A,X_t\rangle+\frac{1}{V(T,q)}M_\infty(\phi)\int_{A'_\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x,\quad \mathbb{P}_\mu\text{-a.s.}
\end{aligned}
$$

Now letting $q \to \infty$, and then $T \to \infty$, we get

$$M_\infty(\phi)\int_{A_\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x \le \liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle + \frac{M_\infty(\phi)}{c(A,\phi)}\int_{A'_\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.}$$

Finally letting ε tend to 0, we obtain

$$M_\infty(\phi)\int_{A}\widehat{\phi}(x)\phi(x)\mathrm{d}x \le \liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle, \quad \mathbb{P}_\mu\text{-a.s.} \tag{2.12}$$

Now we define the set $A^\varepsilon := \{x \in D; \operatorname{dist}(x, A) \le \varepsilon/2\}$, where $\varepsilon > 0$ is small enough so that $\overline{A^\varepsilon} \Subset D$. Applying (2.9) and using a similar argument as for A'_ε, we get that for any $T > 0$, $q > 0$ and $x \in A$,

$$\begin{aligned}
qU^q\big(x, A^\varepsilon\big) &= \int_0^\infty e^{-t}\Pi_x^D\big(\xi_{t/q} \in A^\varepsilon\big)\mathrm{d}t \\
&\ge \frac{\inf_{|y-x|<\varepsilon/2}\phi(y)}{\phi(x)}\int_0^T e^{-(1+(\lambda_1+\|\beta\|_\infty)/q)t}\mathrm{d}t\,\Pi_x\Big(\sup_{0\le s\le T/q}|\xi_s - x| \le \varepsilon/2\Big) \\
&\ge \frac{\inf_{|y-x|<\varepsilon/2}\phi(y)(1-e^{-(1+(\lambda_1+\|\beta\|_\infty)/q)T})}{\phi(x)((\lambda_1+\|\beta\|_\infty)/q+1)}\left(1 - 2d\exp\left\{\frac{-q\varepsilon^2}{32dCT}\right\}\right) \\
&= \frac{\inf_{|y-x|<\varepsilon/2}\phi(y)V(q,T)}{\phi(x)c(A,\phi)}.
\end{aligned} \tag{2.13}$$

Since ϕ is a positive continuous function in D and $\overline{A^\varepsilon} \Subset D$, ϕ is uniformly continuous and has a positive lower bound in A^ε. Thus for any $\kappa \in (0,1)$, we can choose ε small enough such that $\inf_{|y-x|<\varepsilon/2}\phi(y) > \kappa\phi(x)$ for any $x \in A$. (We can choose ε so that $\varepsilon \to 0$ as $\kappa \to 1$.) In this case, $I_A(x) \le c(A,\phi)(\kappa V(q,T))^{-1}qU^q(x, A^\varepsilon)$. Thus,

$$\limsup_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle \le c(A,\phi)\big(\kappa V(q,T)\big)^{-1}\limsup_{t\to\infty} e^{-\lambda_1 t}\big\langle \phi qU^q\big(\cdot, A^\varepsilon\big), X_t\big\rangle.$$

Letting $q \to \infty$ and then $T \to \infty$, using Theorem 2.1, we get

$$\limsup_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle \le \kappa^{-1}M_\infty(\phi)\int_{A^\varepsilon}\widehat{\phi}(x)\phi(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.}$$

Finally letting $\kappa \to 1$ (which implies $\varepsilon \to 0$), we obtain

$$\limsup_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle \le M_\infty(\phi)\int_{A}\widehat{\phi}(x)\phi(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.}$$

The proof is now complete. □

Theorem 1.1 strengthens Theorem 2.2 in the sense that the exceptional set does not depend on f and μ.

Proof of Theorem 1.1 Note that there exists a countable base $\mathcal{U}$ of open subsets $\{U_k, k \ge 1\}$ of D so that $\mathcal{U}$ is closed under finite unions and each open set in $\mathcal{U}$ is a relatively compact set whose boundary has zero Lebesgue measure. Define

$$\Omega_0 := \left\{\omega \in \Omega : \lim_{t\to\infty} e^{-\lambda_1 t}\langle I_{U_k}\phi, X_t\rangle(\omega) = M_\infty(\phi)(\omega)\int_{U_k}\widehat{\phi}(y)\phi(y)\mathrm{d}y \text{ for every } k \ge 1\right\}.$$

By Theorem 2.2, $\mathbb{P}_\mu(\Omega_0) = 1$ for any $\mu \in \mathcal{M}_F(D)$.

We first consider (1.6) on $\{M_\infty(\phi) > 0\}$. For each $\omega \in \Omega_0 \cap \{M_\infty(\phi) > 0\}$ and $t \geq 0$, we define two probability measures ν_t and ν on D respectively by

$$\nu_t(A)(\omega) = \frac{e^{-\lambda_1 t}\langle I_A\phi, X_t\rangle(\omega)}{M_t(\phi)(\omega)}, \quad \text{and} \quad \nu(A) = \int_A \widehat{\phi}(y)\phi(y)\mathrm{d}y, \quad A \in \mathcal{B}(D).$$

Note that the measure ν_t is well-defined for every $t \geq 0$. By the definition of Ω_0 we know that ν_t converges vaguely to ν as $t \to \infty$. Since ν is a probability measure, ν_t actually converges weakly to ν as $t \to \infty$. Using the fact that ϕ is strictly positive and continuous on D, we know that if f is a nonnegative function on D such that $f \leq c\phi$ for some $c > 0$ and that the discontinuity set of f has zero Lebesgue-measure (equivalently zero ν-measure), $g := f/\phi$ is a nonnegative bounded function with the same set of discontinuity. We thus have

$$\lim_{t\to\infty} \int_D g(x)\nu_t(\mathrm{d}x) = \int_D g(x)\nu(\mathrm{d}x),$$

which is equivalent to saying

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle f, X_t\rangle(\omega) = M_\infty(\phi)(\omega)\int_D \widehat{\phi}(y)f(y)\mathrm{d}y,$$
$$\text{for every } \omega \in \Omega_0 \cap \left\{M_\infty(\phi) > 0\right\}. \tag{2.14}$$

If $f \leq c\phi$ for some positive constant $c > 0$, (1.6) holds automatically on $\{M_\infty(\phi) = 0\}$. This completes the proof of the theorem. □

It is well known that for any $g \in \mathcal{B}^+(D)$,

$$\mathbb{P}_\mu\langle g, X_t\rangle = \left\langle P_t^D g, \mu\right\rangle. \tag{2.15}$$

The above formula is the super-process counterpart of the so-called 'many-to-one' formula in branching particle systems, see [15] for example. The formula (2.15) will be used quite a few times later in this paper.

Proof of Theorem 1.2 It follows from (2.15) that

$$\begin{aligned} e^{-\lambda_1 t}\mathbb{P}_\mu\langle f, X_t\rangle &= \int_D \mu(\mathrm{d}x)e^{-\lambda_1 t}P_t^D f(x) \\ &= \int_D \mu(\mathrm{d}x)\int_D e^{-\lambda_1 t}p^D(t,x,y)f(y)\mathrm{d}y \\ &= \int_D \mu(\mathrm{d}x)\phi(x)\int_D q^D(t,x,y)\frac{f(y)}{\phi(y)}\mathrm{d}y. \end{aligned}$$

Using (1.2) and the dominated convergence theorem, we get

$$\lim_{t\to\infty} e^{-\lambda_1 t}\mathbb{P}_\mu\langle f, X_t\rangle = \langle\phi, \mu\rangle\int_D f(y)\widehat{\phi}(y)\mathrm{d}y.$$

Theorem 1.2 is simply a combination of this and Theorem 1.1. □

3 Proof of Theorem 2.1

In this section we will give the proof of Theorem 2.1. We will first prove the discrete-time version, Theorem 3.5, for which we do not need to use the resolvent operators. However, a substantial amount of work is needed to go from discrete time to continuous time. For this we need to use the resolvent operators U^q.

According to Theorem 1.5(2), when $\int_D \widehat{\phi}(x)l(x)\mathrm{d}x = \infty$, we have

$$M_\infty(\phi) = \lim_{t\to\infty} e^{-\lambda_1 t}\langle \phi, X_t\rangle = 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

For any $f \in \mathcal{B}_b^+(D)$,

$$\limsup_{t\to\infty} e^{-\lambda_1 t}\langle \phi f, X_t\rangle \le \|f\|_\infty \limsup_{t\to\infty} e^{-\lambda_1 t}\langle \phi, X_t\rangle = 0 \quad \mathbb{P}_\mu\text{-a.s.}$$

and (2.3) follows immediately from the nonnegativity of f. Thus we only need to deal with the case when $\int_D \widehat{\phi}(x)l(x)\mathrm{d}x < \infty$. In this case, $e^{-\lambda_1 t}\langle \phi q U^q f, X_t\rangle$ is controlled by a constant multiple of $M_t(\phi)$, which is uniformly integrable by Theorem 1.5(1), thus the L^1 limit result is an immediate consequence of the almost sure limit result. So we will only need to prove the almost sure limit result.

In the remainder of this section, we assume that the assumptions of Theorem 2.1 hold and that $f \in \mathcal{B}_b^+(D)$ is fixed. Define

$$S(\mathrm{d}s, \mathrm{d}x) = S^J(\mathrm{d}s, \mathrm{d}x) + S^C(\mathrm{d}s, \mathrm{d}x).$$

As mentioned in Sect. 2, to prove Theorem 2.1, we will first prove the almost sure limit result at discrete times, see Theorem 3.5 below. The steps are similar to that of [1]. Since we are considering superdiffusions here, we will use stochastic integrals with respect to continuous random measures and jump random measures. For the jump part, we also need to handle ‘small jumps’ and ‘large jumps’ separately. Now let us give the precise definition of ‘small jumps’ and ‘large jumps’. A jump at time s is called ‘small’ if $0 < \Delta X_s(\phi) < e^{\lambda_1 s}$, and ‘large’ if $\Delta X_s(\phi) \ge e^{\lambda_1 s}$, here $\Delta X_s(\phi) = r\phi(x)$ when $\Delta X_s = r\delta_x$ with $r > 0$ and $x \in D$.

Define

$$N_\phi^{(1)} := \sum_{0<\Delta X_s(\phi)<e^{\lambda_1 s}} \delta_{(s,\,\Delta X_s)} \quad \text{and} \quad N_\phi^{(2)} := \sum_{\Delta X_s(\phi)\ge e^{\lambda_1 s}} \delta_{(s,\,\Delta X_s)},$$

and denote the compensators of $N_\phi^{(1)}$ and $N_\phi^{(2)}$ by $\widehat{N}_\phi^{(1)}$ and $\widehat{N}_\phi^{(1)}$ respectively. Then for any nonnegative predictable function F on $\mathbb{R}_+ \times \Omega \times \mathcal{M}_F(D)$,

$$\int_0^\infty \int_{\mathcal{M}_F(D)} F(s,\nu)\widehat{N}_\phi^{(1)}(\mathrm{d}s,\mathrm{d}\nu) = \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_0^{e^{\lambda_1 s}} F\big(s, r\phi(x)^{-1}\delta_x\big) n^\phi(x,\mathrm{d}r), \tag{3.1}$$

and

$$\int_0^\infty \int_{\mathcal{M}_F(D)} F(s,\nu)\widehat{N}_\phi^{(2)}(\mathrm{d}s,\mathrm{d}\nu) = \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_{e^{\lambda_1 s}}^\infty F\big(s, r\phi(x)^{-1}\delta_x\big) n^\phi(x,\mathrm{d}r), \tag{3.2}$$

where n^{ϕ} was defined in (1.4). Let $J_{\phi}^{(1)}$ denote the set of jump times of $N_{\phi}^{(1)}$, and $J_{\phi}^{(2)}$ the set of jump times of $N_{\phi}^{(2)}$. Then

$$\int_0^\infty \int_{\mathcal{M}_F(D)} F(s,\nu) N_{\phi}^{(1)}(\mathrm{d}s,\mathrm{d}\nu) = \sum_{s\in J_{\phi}^{(1)}} F(s,\omega,\Delta X_s), \tag{3.3}$$

$$\int_0^\infty \int_{\mathcal{M}_F(D)} F(s,\nu) N_{\phi}^{(2)}(\mathrm{d}s,\mathrm{d}\nu) = \sum_{s\in J_{\phi}^{(2)}} F(s,\omega,\Delta X_s), \tag{3.4}$$

$$\mathbb{P}_\mu\Big[\sum_{s\in J_{\phi}^{(1)}} F(s,\omega,\Delta X_s)\Big] = \mathbb{P}_\mu \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_0^{e^{\lambda_1 s}} F\big(s,\omega,r\phi(x)^{-1}\delta_x\big) n^{\phi}(x,\mathrm{d}r), \tag{3.5}$$

and

$$\mathbb{P}_\mu\Big[\sum_{s\in J_{\phi}^{(2)}} F(s,\omega,\Delta X_s)\Big] = \mathbb{P}_\mu \int_0^\infty \mathrm{d}s \int_D X_s(\mathrm{d}x) \int_{e^{\lambda_1 s}}^\infty F\big(s,\omega,r\phi(x)^{-1}\delta_x\big) n^{\phi}(x,\mathrm{d}r). \tag{3.6}$$

We construct two martingale measures $S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)$ and $S^{J,(2)}(\mathrm{d}s,\mathrm{d}x)$ respectively from $N_{\phi}^{(1)}(\mathrm{d}s,\mathrm{d}\nu)$ and $N_{\phi}^{(2)}(\mathrm{d}s,\mathrm{d}\nu)$, similar to the way we constructed $S^J(\mathrm{d}s,\mathrm{d}x)$ from $N(\mathrm{d}s,\mathrm{d}\nu)$. Then for any bounded measurable function g on $\mathbb{R}_+\times D$,

$$S_t^{J,(1)}(g) = \int_0^t \int_D g(s,x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x) = \int_0^t \int_{\mathcal{M}_F(D)} F_g(s,\nu)\big(N_{\phi}^{(1)} - \widehat{N}_{\phi}^{(1)}\big)(\mathrm{d}s,\mathrm{d}\nu), \tag{3.7}$$

and

$$S_t^{J,(2)}(g) = \int_0^t \int_D g(s,x) S^{J,(2)}(\mathrm{d}s,\mathrm{d}x) = \int_0^t \int_{\mathcal{M}_F(D)} F_g(s,\nu)\big(N_{\phi}^{(2)} - \widehat{N}_{\phi}^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu), \tag{3.8}$$

where $F_g(s,\nu) = \int_D g(s,x)\nu(\mathrm{d}x)$.

For any $m,n\in\mathbb{N}$, $\sigma>0$ and $f\in\mathcal{B}_b^+(D)$, define

$$H_{(n+m)\sigma}(f) := e^{-\lambda_1(n+m)\sigma} \int_0^{(n+m)\sigma} \int_D P_{(n+m)\sigma-s}^D(\phi f)(x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)$$

and

$$L_{(n+m)\sigma}(f) := e^{-\lambda_1(n+m)\sigma} \int_0^{(n+m)\sigma} \int_D P_{(n+m)\sigma-s}^D(\phi f)(x) S^{J,(2)}(\mathrm{d}s,\mathrm{d}x).$$

Lemma 3.1 *If $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$, then for any $m\in\mathbb{N}$, $\sigma>0$, $\mu\in\mathcal{M}_F(D)$ and $f\in\mathcal{B}_b^+(D)$,*

$$\sum_{n=1}^\infty \mathbb{P}_\mu\big[H_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big(H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\big]^2 < \infty \tag{3.9}$$

and

$$\lim_{n\to\infty}\big(H_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\big) = 0, \quad \textit{in } L^2(\mathbb{P}_\mu) \textit{ and } \mathbb{P}_\mu\text{-a.s.} \tag{3.10}$$

Proof Since $P_t^D(\phi f)$ is bounded in $[0,T]\times D$ for any $T>0$, the process

$$H_t(f) := e^{-\lambda_1(n+m)\sigma}\int_0^t\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x),\quad t\in\big[0,(n+m)\sigma\big]$$

is a martingale with respect to $(\mathcal{F}_t)_{t\le(n+m)\sigma}$. Thus

$$\mathbb{P}_\mu\big(H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)=e^{-\lambda_1(n+m)\sigma}\int_0^{n\sigma}\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x),$$

and hence

$$\begin{aligned}&H_{(n+m)\sigma}(f)-\mathbb{P}_\mu\big(H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\\&\quad=e^{-\lambda_1(n+m)\sigma}\int_{n\sigma}^{(n+m)\sigma}\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x).\end{aligned}\tag{3.11}$$

Since

$$\begin{aligned}M_t &:= e^{-\lambda_1(n+m)\sigma}\int_{n\sigma}^t\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\\&=\int_{n\sigma}^t\int_{\mathcal{M}_F(D)}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\nu)\big(N_\phi^{(1)}-\widehat{N}_\phi^{(1)}\big)(\mathrm{d}s,\mathrm{d}\nu),\quad t\in\big[n\sigma,(n+m)\sigma\big]\end{aligned}$$

is a martingale with quadratic variation

$$\int_{n\sigma}^t\int_{\mathcal{M}_F(D)}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\nu)^2\widehat{N}_\phi^{(1)}(\mathrm{d}s,\mathrm{d}\nu),$$

we have

$$\begin{aligned}&\mathbb{P}_\mu\big[H_{(n+m)\sigma}(f)-\mathbb{P}_\mu\big(H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\big]^2\\&\quad=\mathbb{P}_\mu\int_{n\sigma}^{(n+m)\sigma}\int_{\mathcal{M}_F(D)}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\nu)^2\widehat{N}_\phi^{(1)}(\mathrm{d}s,\mathrm{d}\nu)\\&\quad=\mathbb{P}_\mu\Big[\sum_{s\in\widetilde{J}^{(1)}_{n,m}}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\Delta X_s)^2\Big],\end{aligned}\tag{3.12}$$

where $\widetilde{J}^{(1)}_{n,m}=J^{(1)}_\phi\cap\,[n\sigma,(n+m)\sigma]$. Note that for any $f\in\mathcal{B}_b(D)$, $\|Q_t^D f\|_\infty\le\|f\|_\infty$ for all $t\ge 0$, which is equivalent to

$$P_t^D(\phi f)(y)\le\|f\|_\infty e^{\lambda_1 t}\phi(y),\quad\forall t\ge 0,\ y\in D.\tag{3.13}$$

Using (3.1) and (3.5), we obtain

$$\begin{aligned}&\mathbb{P}_\mu\Big[\sum_{s\in\widetilde{J}^{(1)}_{n,m}}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\Delta X_s)^2\Big]\\&\quad=\mathbb{P}_\mu\int_{n\sigma}^{(n+m)\sigma}\mathrm{d}s\int_D X_s(\mathrm{d}x)\int_0^{e^{\lambda_1 s}}F^2_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}\big(s,r\phi(x)^{-1}\delta_x\big)n^\phi(x,\mathrm{d}r)\end{aligned}$$

$$
\begin{aligned}
&= e^{-2\lambda_1(n+m)\sigma}\int_{n\sigma}^{(n+m)\sigma}\mathrm{d}s\int_D \mu(\mathrm{d}y)\\
&\quad\times\int_D p^D(s,y,x)\mathrm{d}x\int_0^{e^{\lambda_1 s}}\left[P^D_{(n+m)\sigma-s}(\phi f)(x)\phi(x)^{-1}\right]^2 r^2 n^{\phi}(x,\mathrm{d}r)\\
&\le \|f\|_\infty^2\int_{n\sigma}^{(n+m)\sigma}e^{-2\lambda_1 s}\mathrm{d}s\int_D\mu(\mathrm{d}y)\int_D p^D(s,y,x)\mathrm{d}x\int_0^{e^{\lambda_1 s}} r^2 n^{\phi}(x,\mathrm{d}r),
\end{aligned}
$$

where in the second equality we used the fact that

$$
F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}\left(s, r\phi(x)^{-1}\delta_x\right) = re^{-\lambda_1(n+m)\sigma}\phi^{-1}(x)P^D_{(n+m)\sigma-s}(\phi f)(x) \tag{3.14}
$$

and in the last inequality we used (3.13). It follows from (1.2) that there is a constant $C>0$ such that

$$
p^D(s,y,x)\le Ce^{\lambda_1 s}\phi(y)\widehat{\phi}(x),\quad \forall s>\sigma,\ x,y\in D. \tag{3.15}
$$

Thus

$$
\begin{aligned}
&\mathbb{P}_\mu\left[\sum_{s\in\widetilde{J}^{(1)}_{n,m}}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\Delta X_s)^2\right]\\
&\quad\le C\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_{n\sigma}^{\infty}e^{-\lambda_1 s}\mathrm{d}s\int_0^{e^{\lambda_1 s}}r^2 n^{\phi}(x,\mathrm{d}r).
\end{aligned}
$$

Summing over n, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}_\mu\left[\sum_{s\in\widetilde{J}^{(1)}_{n,m}}F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)}(s,\Delta X_s)^2\right]\\
&\le\sum_{n=1}^{\infty}C\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_{n\sigma}^{\infty}e^{-\lambda_1 s}\mathrm{d}s\int_0^{e^{\lambda_1 s}}r^2 n^{\phi}(x,\mathrm{d}r)\\
&\le C\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_0^{\infty}\mathrm{d}t\int_{t\sigma}^{\infty}e^{-\lambda_1 s}\mathrm{d}s\int_0^{e^{\lambda_1 s}}r^2 n^{\phi}(x,\mathrm{d}r)\\
&=\frac{C}{\sigma}\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_0^{\infty}se^{-\lambda_1 s}\mathrm{d}s\int_0^{e^{\lambda_1 s}}r^2 n^{\phi}(x,\mathrm{d}r)\\
&\le\frac{C}{\sigma}\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_1^{\infty}r^2 n^{\phi}(x,\mathrm{d}r)\int_{\lambda_1^{-1}\ln r}^{\infty}se^{-\lambda_1 s}\mathrm{d}s\\
&\quad+\frac{C}{\sigma}\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_0^{1}r^2 n^{\phi}(x,\mathrm{d}r)\int_0^{\infty}se^{-\lambda_1 s}\mathrm{d}s\\
&=: I+II.
\end{aligned} \tag{3.16}
$$

Using (1.5) we immediately get that $II<\infty$. On the other hand,

$$
I=\frac{C}{\lambda_1^2\sigma}\|f\|_\infty^2\langle\phi,\mu\rangle\int_D\widehat{\phi}(x)\mathrm{d}x\int_1^{\infty}r(\ln r+1)n^{\phi}(x,\mathrm{d}r).
$$

Now we can use $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$ and (1.3) to get that $I < \infty$. The proof of (3.9) is now complete. For any $\varepsilon > 0$, using (3.9) and Chebyshev's inequality we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}_\mu\Big(\big|H_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\big| > \varepsilon\Big)\\
&\quad\le \varepsilon^{-2}\sum_{n=1}^{\infty}\mathbb{P}_\mu\big[H_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big(H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\big]^2\\
&\quad< \infty.
\end{aligned}$$

Then (3.10) follows easily from the Borel-Cantelli lemma. □

Lemma 3.2 *If $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$, then for any $m \in \mathbb{N}$, $\sigma > 0$, $\mu \in \mathcal{M}_F(D)$ and $f \in \mathcal{B}_b^+(D)$ we have*

$$\lim_{n\to\infty}\big(L_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[L_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\big) = 0, \quad \textit{in } L^1(\mathbb{P}_\mu) \textit{ and } \mathbb{P}_\mu\text{-a.s.} \tag{3.17}$$

Proof It is easy to see that

$$\mathbb{P}_\mu\big[L_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big] = e^{-\lambda_1(n+m)\sigma}\int_0^{n\sigma}\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(2)}(\mathrm{d}s,\mathrm{d}x).$$

Therefore,

$$\begin{aligned}
&\big|L_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[L_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\big|\\
&\quad= \left|e^{-\lambda_1(n+m)\sigma}\int_{n\sigma}^{(n+m)\sigma}\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S^{J,(2)}(\mathrm{d}s,\mathrm{d}x)\right|\\
&\quad= \left|\int_{n\sigma}^{(n+m)\sigma}\int_{\mathcal{M}_F(D)} F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)(\cdot)}(s,\nu)\big(N_\phi^{(2)} - \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu)\right|\\
&\quad\le \int_{n\sigma}^{(n+m)\sigma}\int_{\mathcal{M}_F(D)} F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)(\cdot)}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu).
\end{aligned} \tag{3.18}$$

Using (3.13) we get,

$$\begin{aligned}
&\int_{n\sigma}^{(n+m)\sigma}\int_{\mathcal{M}_F(D)} F_{e^{-\lambda_1(n+m)\sigma}P^D_{(n+m)\sigma-\cdot}(\phi f)(\cdot)}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu)\\
&\quad\le \int_{n\sigma}^{\infty}\int_{\mathcal{M}_F(D)} F_{\|f\|_\infty e^{-\lambda_1\cdot}\phi}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu).
\end{aligned}$$

Using (3.2) and (3.15) we get

$$\begin{aligned}
&\mathbb{P}_\mu\int_{n\sigma}^{\infty}\int_{\mathcal{M}_F(D)} F_{\|f\|_\infty e^{-\lambda_1\cdot}\phi}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu)\\
&\quad= 2\|f\|_\infty\mathbb{P}_\mu\left[\int_{n\sigma}^{\infty}e^{-\lambda_1 s}\mathrm{d}s\int_D X_s(\mathrm{d}x)\int_{e^{\lambda_1 s}}^{\infty} rn^\phi(x,\mathrm{d}r)\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\|f\|_\infty \int_{n\sigma}^\infty e^{-\lambda_1 s}\mathrm{d}s \int_D \mu(\mathrm{d}y) \int_D p^D(s,y,x)\mathrm{d}x \int_{e^{\lambda_1 s}}^\infty r n^\phi(x,\mathrm{d}r) \\
&\le 2C\|f\|_\infty \langle \phi,\mu\rangle \int_{n\sigma}^\infty \mathrm{d}s \int_D \widehat{\phi}(x)\mathrm{d}x \int_{e^{\lambda_1 s}}^\infty r n^\phi(x,\mathrm{d}r) \\
&\le 2C\|f\|_\infty \langle \phi,\mu\rangle \int_D \widehat{\phi}(x)\mathrm{d}x \int_{e^{\lambda_1 n\sigma}}^\infty r n^\phi(x,\mathrm{d}r) \int_0^{\lambda_1^{-1}\ln r} \mathrm{d}s \\
&= \frac{2C\|f\|_\infty}{\lambda_1} \langle \phi,\mu\rangle \int_D \widehat{\phi}(x)\mathrm{d}x \int_{e^{\lambda_1 n\sigma}}^\infty r\ln r n^\phi(x,\mathrm{d}r).
\end{aligned}
$$

Note that $\int_D \widehat{\phi}(x)\mathrm{d}x \int_{e^{\lambda_1 n\sigma}}^\infty r\ln r n^\phi(x,\mathrm{d}r) \le \int_D \widehat{\phi}(x) l(x)\mathrm{d}x$. Applying the dominated convergence theorem and using the fact that

$$
\int_{n\sigma}^\infty \int_{\mathcal{M}_F(D)} F_{\|f\|_\infty e^{-\lambda_1 \cdot}\phi}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu)
$$

is decreasing in n, we obtain that, when $\int_D \widehat{\phi}(x) l(x)\mathrm{d}x < \infty$,

$$
\lim_{n\to\infty} \int_{n\sigma}^\infty \int_{\mathcal{M}_F(D)} F_{\|f\|_\infty e^{-\lambda_1 \cdot}\phi}(s,\nu)\big(N_\phi^{(2)} + \widehat{N}_\phi^{(2)}\big)(\mathrm{d}s,\mathrm{d}\nu) = 0, \quad \text{in } L^1(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.} \tag{3.19}
$$

Therefore by (3.18), we have (3.17). The proof is complete. □

For any $m, n \in \mathbb{N}$, $\sigma > 0$, set

$$
C_{(n+m)\sigma}(f) := e^{-\lambda_1 (n+m)\sigma} \int_0^{(n+m)\sigma} \int_D \big(P^D_{(n+m)\sigma - s}\phi f\big)(x) S^C(\mathrm{d}s,\mathrm{d}x), \quad f \in \mathcal{B}_b^+(D).
$$

Then

$$
\mathbb{P}_\mu\big(C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big) = e^{-\lambda_1 (n+m)\sigma} \int_0^{n\sigma} \int_D \big(P^D_{(n+m)\sigma - s}\phi f\big)(x) S^C(\mathrm{d}s,\mathrm{d}x).
$$

Lemma 3.3 *For any $m \in \mathbb{N}$, $\sigma > 0$, $\mu \in \mathcal{M}_F(D)$ and $f \in \mathcal{B}_b^+(D)$ we have*

$$
\lim_{n\to\infty} \big(C_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\big) = 0, \quad \textit{in } L^2(\mathbb{P}_\mu) \textit{ and } \mathbb{P}_\mu\text{-a.s.} \tag{3.20}
$$

Proof Note that

$$
C_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big] = e^{-\lambda_1 (n+m)\sigma} \int_{n\sigma}^{(n+m)\sigma} \int_D \big(P^D_{(n+m)\sigma - s}\phi f\big)(x) S^C(\mathrm{d}s,\mathrm{d}x). \tag{3.21}
$$

From the quadratic variation formula (1.16) and the definition of Q_t^D,

$$
\begin{aligned}
&\mathbb{P}_\mu\big[C_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big(C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\big]^2 \\
&= \int_{n\sigma}^{(n+m)\sigma} e^{-2\lambda_1 (n+m)\sigma}\mathrm{d}s \int_D \mu(\mathrm{d}x) \int_D p^D(s,x,y)\big(P^D_{(n+m)\sigma - s}\phi f\big)^2(y)\alpha(y)\mathrm{d}y \\
&\le \int_{n\sigma}^{(n+m)\sigma} e^{-2\lambda_1 s}\mathrm{d}s \int_D \mu(\mathrm{d}x) \int_D p^D(s,x,y)\phi^2(y)\big(Q^D_{(n+m)\sigma - s} f\big)^2(y)\alpha(y)\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&\le \|f\|_\infty^2 \int_{n\sigma}^{(n+m)\sigma} e^{-\lambda_1 s}\mathrm{d}s \int_D \phi(x) Q_s^D(\alpha\phi)(x)\mu(\mathrm{d}x)\\
&\le \frac{1}{\lambda_1}\|\phi\alpha\|_\infty \|f\|_\infty^2 \langle\phi,\mu\rangle e^{-\lambda_1 n\sigma}.
\end{aligned}
\tag{3.22}
$$

Therefore, we have

$$
\sum_{n=1}^{\infty} \mathbb{P}_\mu\big[C_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big(C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big)\big]^2 < \infty.
\tag{3.23}
$$

By the Borel-Cantelli lemma, we get (3.20). □

Combining the three lemmas above, we have the following result. The idea for proving the next result comes from [1].

Lemma 3.4 *If $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$, then for any $m \in \mathbb{N}$, $\sigma > 0$, $\mu \in \mathcal{M}_F(D)$ and $f \in \mathcal{B}_b^+(D)$ we have*

$$
\begin{aligned}
&\lim_{n\to\infty} e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle - \mathbb{P}_\mu\big[e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle|\mathcal{F}_{n\sigma}\big] = 0,\\
&\quad \textit{in } L^1(\mathbb{P}_\mu) \textit{ and } \mathbb{P}_\mu\text{-a.s.}
\end{aligned}
\tag{3.24}
$$

Proof From (1.17), we know that $e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle$ can be decomposed into three parts:

$$
\begin{aligned}
&e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle\\
&\quad = e^{-\lambda_1(n+m)\sigma}\big\langle P^D_{(n+m)\sigma}(\phi f),\mu\big\rangle + e^{-\lambda_1(n+m)\sigma}\int_0^{(n+m)\sigma}\int_D P^D_{(n+m)\sigma-s}(\phi f)(x)S(\mathrm{d}s,\mathrm{d}x)\\
&\quad = e^{-\lambda_1(n+m)\sigma}\big\langle P^D_{(n+m)\sigma}(\phi f),\mu\big\rangle + H_{(n+m)\sigma}(f) + L_{(n+m)\sigma}(f) + C_{(n+m)\sigma}(f).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle - \mathbb{P}_\mu\big[e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle|\mathcal{F}_{n\sigma}\big]\\
&\quad = H_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[H_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big] + L_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[L_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big]\\
&\qquad + C_{(n+m)\sigma}(f) - \mathbb{P}_\mu\big[C_{(n+m)\sigma}(f)|\mathcal{F}_{n\sigma}\big].
\end{aligned}
$$

Now the conclusion of this lemma follows immediately from Lemmas 3.1–3.3. □

Theorem 3.5 *If $\int_D l(x)\widehat{\phi}(x)\mathrm{d}x < \infty$, then for any $\sigma > 0$, $\mu \in \mathcal{M}_F(D)$ and $f \in \mathcal{B}_b^+(D)$ we have*

$$
\lim_{n\to\infty} e^{-\lambda_1 n\sigma}\langle\phi f, X_{n\sigma}\rangle = M_\infty(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z, \quad \textit{in } L^1(\mathbb{P}_\mu) \textit{ and } \mathbb{P}_\mu\text{-a.s.}
$$

Proof By (2.15) and the Markov property of super-processes, we have

$$
\mathbb{P}_\mu\big[e^{-\lambda_1(n+m)\sigma}\langle\phi f, X_{(n+m)\sigma}\rangle|\mathcal{F}_{n\sigma}\big] = e^{-\lambda_1 n\sigma}\big\langle e^{-\lambda_1 m\sigma}P^D_{m\sigma}(\phi f), X_{n\sigma}\big\rangle.
\tag{3.25}
$$

Note that it follows from (1.2) that there exist constants $c > 0$ and $\nu > 0$ such that

$$\left|\frac{e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f)(x)}{\phi(x)} - \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z\right| \le ce^{-\nu m\sigma}\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z,$$
$$\text{for every } x \in D,$$

which is equivalent to

$$\left|\frac{e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f)(x)}{\phi(x)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z} - 1\right| \le ce^{-\nu m\sigma}, \quad \text{for every } x \in D.$$

Thus there exist positive constants $k_m \le 1$ and $K_m \ge 1$ such that for any $x \in D$,

$$k_m\phi(x)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z \le e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f)(x) \le K_m\phi(x)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z,$$

and that $\lim_{m\to\infty} k_m = \lim_{m\to\infty} K_m = 1$. Hence,

$$\begin{aligned} e^{-\lambda_1 n\sigma}\big\langle e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f), X_{n\sigma}\big\rangle &\ge k_m e^{-\lambda_1 n\sigma}\langle \phi, X_{n\sigma}\rangle \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z \\ &= k_m M_{n\sigma}(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z, \quad \mathbb{P}_\mu\text{-a.s.} \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} e^{-\lambda_1 n\sigma}\big\langle e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f), X_{n\sigma}\big\rangle &\le K_m e^{-\lambda_1 n\sigma}\langle \phi, X_{n\sigma}\rangle \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z \\ &= K_m M_{n\sigma}(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z, \quad \mathbb{P}_\mu\text{-a.s.} \end{aligned} \tag{3.27}$$

Using (3.25), Lemma 3.4 and (3.27) we get that for any $m \in \mathbb{N}$,

$$\begin{aligned} \limsup_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle &= \limsup_{n\to\infty} e^{-\lambda_1 (n+m)\sigma}\langle \phi f, X_{(n+m)\sigma}\rangle \\ &= \limsup_{n\to\infty} e^{-\lambda_1 n\sigma}\big\langle e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f), X_{n\sigma}\big\rangle \\ &\le \limsup_{n\to\infty} K_m M_{n\sigma}(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z \\ &= K_m M_\infty(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z. \end{aligned}$$

Letting $m \to \infty$, we get

$$\limsup_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle \le M_\infty(\phi)\int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z. \tag{3.28}$$

Similarly, using (3.25), Lemma 3.4 and (3.26) we get that for any $m \in \mathbb{N}$,

$$\begin{aligned} \liminf_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle &= \liminf_{n\to\infty} e^{-\lambda_1 (n+m)\sigma}\langle \phi f, X_{(n+m)\sigma}\rangle \\ &= \liminf_{n\to\infty} e^{-\lambda_1 n\sigma}\big\langle e^{-\lambda_1 m\sigma} P^D_{m\sigma}(\phi f), X_{n\sigma}\big\rangle \end{aligned}$$

$$\geq \liminf_{n\to\infty} k_m M_{n\sigma}(\phi) \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z$$
$$= k_m M_\infty(\phi) \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z.$$

Letting $m \to \infty$, we get

$$\liminf_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle \geq M_\infty(\phi) \int_D \widehat{\phi}(z)\phi(z)f(z)\mathrm{d}z. \tag{3.29}$$

Combining (3.28) and (3.29) we arrive at the almost sure assertion of the theorem. Since $e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle$ is controlled by a constant multiple of $M_{n\sigma}(\phi)$, which is uniformly integrable by Theorem 1.5(1), the L^1 assertion now follows immediately from the almost sure assertion. □

We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1 First note that for $t \in [n\sigma, (n+1)\sigma]$,

$$|e^{-\lambda_1 t}\langle \phi Q^D_{(n+1)\sigma-t} qU^q f, X_t\rangle - e^{-\lambda_1 t}\langle \phi qU^q f, X_t\rangle| \leq M_t(\phi)\Delta_\sigma(f), \tag{3.30}$$

where $\Delta_\sigma(f) := \sup_{0\leq t\leq\sigma} \|q(Q^D_t U^q f - U^q f)\|_\infty$. Since for any $s > 0$, we have by the definition of the resolvent U^q

$$\left|q\left(Q^D_s U^q f(x) - U^q f(x)\right)\right| = \left|\left(e^{qs}-1\right)\int_s^\infty qe^{-qt}Q^D_t f(x)\mathrm{d}t - \int_0^s qe^{-qt}Q^D_t f(x)\mathrm{d}t\right|$$
$$\leq 2\|f\|_\infty\left(1-e^{-qs}\right),$$

we have $\lim_{\sigma\to 0}\Delta_\sigma(f) = 0$. Thus, for any $q > 0$,

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]}\left|e^{-\lambda_1 t}\langle \phi Q^D_{(n+1)\sigma-t} qU^q f, X_t\rangle - e^{-\lambda_1 t}\langle \phi qU^q f, X_t\rangle\right| = 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

Since $\widehat{\phi}\phi$ is the invariant probability density of the semigroup (Q^D_t), we have that

$$\int_D \widehat{\phi}(x)\phi(x)qU^q f(x)\mathrm{d}x = \int_D \widehat{\phi}(x)\phi(x)f(x)\mathrm{d}x.$$

Therefore, to obtain (2.3), we only need to prove for $f \in \mathcal{B}^+_b(D)$, the following holds,

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]} e^{-\lambda_1 t}\langle \phi Q^D_{(n+1)\sigma-t} f, X_t\rangle = M_\infty(\phi)\int_D \widehat{\phi}(x)\phi(x)f(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.31}$$

For any $n \in \mathbb{N}$ and $\sigma > 0$, $(X_t, t \in [n\sigma, (n+1)\sigma], \mathbb{P}_\mu(\cdot|\mathcal{F}_{n\sigma}))$ can be regarded as a $(\xi^D, \psi(\lambda) - \beta\lambda)$-superdiffusion with initial value $X_{n\sigma}$. Thus, for arbitrary $g \in \mathcal{B}^+_b(D)$, we have by (1.17)

$$e^{-\lambda_1 t}\langle \phi g, X_t\rangle = e^{-\lambda_1 t}\langle P^D_{t-n\sigma}(\phi g), X_{n\sigma}\rangle + e^{-\lambda_1 t}\int_{n\sigma}^t \int_D P^D_{t-s}(\phi g)(x)S(\mathrm{d}s, \mathrm{d}x),$$
$$t \in \left[n\sigma, (n+1)\sigma\right].$$

Taking $g(x)=Q^D_{(n+1)\sigma-t}f(x)$ in the above identity and using (1.1), we get

$$e^{-\lambda_1 t}\langle \phi Q^D_{(n+1)\sigma-t}f, X_t\rangle = e^{-\lambda_1 n\sigma}\langle \phi Q^D_{\sigma}f, X_{n\sigma}\rangle + \int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S(\mathrm{d}s,\mathrm{d}x). \tag{3.32}$$

Since $\widehat{\phi}\phi$ is the invariant probability density of the semigroup (Q^D_t), we have by Theorem 3.5,

$$\begin{aligned}\lim_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi Q^D_{\sigma}f, X_{n\sigma}\rangle &= M_\infty(\phi)\int_D \widehat{\phi}(x)\phi(x)Q^D_\sigma f(x)\mathrm{d}x\\ &= M_\infty(\phi)\int_D \widehat{\phi}(x)\phi(x)f(x)\mathrm{d}x, \quad \mathbb{P}_\mu\text{-a.s.}\end{aligned} \tag{3.33}$$

Hence, by (3.32) and (3.33), to prove (3.31) it suffices to show that

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]}\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S(\mathrm{d}s,\mathrm{d}x)=0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.34}$$

Since $S(\mathrm{d}s,\mathrm{d}x)=S^J(\mathrm{d}s,\mathrm{d}x)+S^C(\mathrm{d}s,\mathrm{d}x)=S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)+S^{J,(2)}(\mathrm{d}s,\mathrm{d}x)+S^C(\mathrm{d}s,\mathrm{d}x)$, we have

$$\begin{aligned}&\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S(\mathrm{d}s,\mathrm{d}x)\\ &\quad=\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\\ &\qquad+\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S^{J,(2)}(\mathrm{d}s,\mathrm{d}x)\\ &\qquad+\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S^{C}(\mathrm{d}s,\mathrm{d}x)\\ &\quad=: H^\sigma_{n,t}(f)+L^\sigma_{n,t}(f)+C^\sigma_{n,t}(f).\end{aligned}$$

Thus we only need to prove that

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]} H^\sigma_{n,t}(f)=0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.35}$$

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]} L^\sigma_{n,t}(f)=0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.36}$$

and

$$\lim_{\sigma\to 0}\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]} C^\sigma_{n,t}(f)=0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.37}$$

It follows from Chebyshev's inequality that, for any $\varepsilon>0$, we have

$$\begin{aligned}&\mathbb{P}_\mu\Big(\sup_{t\in[n\sigma,(n+1)\sigma]}\big|H^\sigma_{n,t}(f)\big|>\varepsilon\Big)\\ &\quad\le \frac{1}{\varepsilon^2}\mathbb{P}_\mu\Big(\sup_{t\in[n\sigma,(n+1)\sigma]}\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \big(\phi Q^D_{(n+1)\sigma-s}f\big)(x)S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\Big)^2.\end{aligned} \tag{3.38}$$

Since the process $(H^{\sigma}_{n,t}(f); t \in [n\sigma, (n+1)\sigma])$ is a martingale with respect to $(\mathcal{F}_t)_{t\in[n\sigma,(n+1)\sigma]}$, applying the Burkholder-Davis-Gundy inequality to $H^{\sigma}_{n,t}(f)$, we obtain

$$\begin{aligned}
&\mathbb{P}_\mu\left(\sup_{t\in[n\sigma,(n+1)\sigma]}\int_{n\sigma}^{t} e^{-\lambda_1 s}\int_D \left(\phi Q^D_{(n+1)\sigma-s} f\right)(x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\right)^2\\
&\quad\le C_1\mathbb{P}_\mu\left(\int_{n\sigma}^{(n+1)\sigma} e^{-\lambda_1 s}\int_D \phi(x) Q^D_{(n+1)\sigma-s} f(x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\right)^2\\
&\quad\le C_1 e^{\sigma}\mathbb{P}_\mu\left(e^{-(n+1)\sigma}\int_{n\sigma}^{(n+1)\sigma}\int_D P^D_{(n+1)\sigma-s}(\phi f)(x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\right)^2,
\end{aligned} \tag{3.39}$$

where C_1 is a positive constant independent of n. Using Lemma 3.1 with $m = 1$ and the identity (3.11), we obtain

$$\sum_{n=1}^{\infty}\mathbb{P}_\mu\left(e^{-(n+1)\sigma}\int_{n\sigma}^{(n+1)\sigma}\int_D P^D_{(n+1)\sigma-s}(\phi f)(x) S^{J,(1)}(\mathrm{d}s,\mathrm{d}x)\right)^2 < \infty. \tag{3.40}$$

Combining (3.38)–(3.40), we get that, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbb{P}_\mu\left(\sup_{t\in[n\sigma,(n+1)\sigma]}\left|H^{\sigma}_{n,t}(f)\right| > \varepsilon\right) < \infty. \tag{3.41}$$

Thus by the Borel-Cantelli lemma we have, for any $\sigma > 0$,

$$\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]} H^{\sigma}_{n,t}(f) = 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

Therefore (3.35) is valid.

Similarly, we can prove that

$$\lim_{n\to\infty}\sup_{t\in[n\sigma,(n+1)\sigma]}\left|C^{\sigma}_{n,t}(f)\right| = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.42}$$

and thus (3.37) holds. We omit the details here.

Using an argument similar to (3.18) we see that

$$\begin{aligned}
\left|L^{\sigma}_{n,t}(f)\right| &= \left|\int_{n\sigma}^{t}\int_{\mathcal{M}_F(D)} F_{e^{-\lambda_1\cdot} Q^D_{(n+1)\sigma-\cdot} f}(s,\nu)\left(N^{(2)}_\phi - \widehat{N}^{(2)}_\phi\right)(\mathrm{d}s,\mathrm{d}\nu)\right|\\
&\le \int_{n\sigma}^{\infty}\int_{\mathcal{M}_F(D)} F_{\|f\|_\infty e^{-\lambda_1\cdot}\phi}(s,\nu)\left(N^{(2)}_\phi + \widehat{N}^{(2)}_\phi\right)(\mathrm{d}s,\mathrm{d}\nu).
\end{aligned}$$

Now using (3.19), we get (3.36) holds. The proof is now complete. □

Acknowledgements We thank the referees for very helpful comments on the first version of this paper.

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