

A large deviation for occupation time of critical branching α -stable process

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Received January 28, 2010; accepted January ??, 2011

Abstract In this paper we establish a large deviation principle for the occupation times of critical branching α -stable processes for large dimensions $d > 2\alpha$, by investigating two related nonlinear differential equations. Our result is an extension of Cox and Griffeath's (Ann Probab, 1985, 13: 1108-1132) for branching Brownian motion for $d > 4$.

Keywords large deviation, critical branching α -stable process, occupation time

MSC(2000): 60F10, 60J80

Citation: LI Q Y, Ren Y X. A large deviation for occupation time of critical branching α -stable process. Sci China Math, 2011, 54(2): 00–00, doi: 10.1007/s11425-009-0000-0

1 Introduction and main results

In this paper we will always assume that $\alpha \in (0, 2]$ and that $\xi = \{\xi_t, t \geq 0\}$ is a symmetric α -stable process in \mathbf{R}^d . The law of the symmetric α -stable process starting at $x \in \mathbf{R}^d$ will be denoted by Π_x . We will use $P_\alpha(t, x, y) = P_\alpha(t, x - y)$ to denote the transition density of ξ with respect to the Lebesgue measure and Δ_α to denote the infinitesimal generator of ξ . The domain of the generator Δ_α will be denoted by $\mathcal{D}(\Delta_\alpha)$.

In this paper we will consider a particle system in \mathbf{R}^d satisfying the following conditions:

- i) At time 0 there are finitely many particles in the system;
- ii) Each particle undergoes an independent motion according to a symmetric α -stable process during its lifetime and the lifetime of each particle is an exponential random variable with mean 1. At the end of each particle's lifetime, it produces, at its death site, 2 particles or 0 particle with equal probability;
- iii) Each new particle evolves in the same manner. All particles' movements, lifetimes and offspring numbers are independent.

And we also consider another system which keep the conditions ii) and iii), but replace i) by i)' below:

- i)' At time 0 particles are distributed according to a random point measure on \mathbf{R}^d with intensity λ , the Lebesgue measure on \mathbf{R}^d .

For any Borel subset of \mathbf{R}^d , we will use $N_t(B)$ to denote the number of particles in B at time t . Then $\{N_t : t \geq 0\}$ is a Markov process on the space of point measures on \mathbf{R}^d . This process is called a branching

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symmetric α -stable process. When the initial distribution of N is δ_x for some $x \in \mathbf{R}^d$, that is, initially there is only one particle in the system and the particle is located at x , we will denote the law of the branching α -stable process by P_x . When the initial distribution of the branching symmetric α -stable process is an independent Poisson random measure on \mathbf{R}^d with intensity λ , we will denote the law of the branching α -stable process by P . Expectation with respect to the measure P will be denoted by E . The occupation time process of the branching symmetric α -stable process is defined as

$$\int_0^t N_s ds, \quad t \geq 0.$$

It is well-known that λ is invariant for the symmetric α -stable process, which implies in particular that $EN_t = \lambda$ for all t . The purpose of this paper is to establish a large deviation principle for the occupation time process of the branching α -stable process under the measure P .

Denote by $\mathcal{S}(\mathbf{R}^d)_+$ the space of positive C^∞ functions on \mathbf{R}^d such that $\sup_{x \in \mathbf{R}^d} f(x)|x|^r < \infty$ with $r > d$ and $r < d + \alpha$ in case $\alpha < 2$. The Laplace functional of the occupation time process under P is given by

$$\begin{aligned} E \exp \left[- \int_0^t \langle N_s, \varphi \rangle ds \right] &= E [\exp (\langle \log(1 - \hat{v}(t)), \eta \rangle)] \\ &= \exp \langle -\hat{v}(t), \lambda \rangle, \quad \varphi(x) \in \mathcal{S}(\mathbf{R}^d)_+, t \geq 0, \end{aligned} \quad (1.1)$$

where η stands for a Poisson random measure on \mathbf{R}^d with intensity λ and

$$v(t, x) =: 1 - \hat{v}(t, x) = E_x \exp \left[- \int_0^t \langle N_s, \varphi \rangle ds \right]$$

satisfies the equation

$$\begin{aligned} &v(t, x) \\ &= \Pi_x \left[e^{-t} \exp \left(- \int_0^t \varphi(\xi_s) ds \right) + \int_0^t e^{-s} \exp \left(- \int_0^s \varphi(\xi_r) dr \right) \frac{1}{2} (1 + v^2(t-s, \xi_s)) ds \right]. \end{aligned} \quad (1.2)$$

Using the Feynman-Kac formula (see [1], for instance), it follows that $\hat{v}(t, x)$ is the mild solution of the non-linear equation:

$$\begin{cases} \frac{\partial}{\partial t} \hat{v}(t, x) = \Delta_\alpha \hat{v}(t, x) - \frac{1}{2} \hat{v}^2(t, x) + \varphi(x)(1 - \hat{v}(t, x)), \\ \hat{v}(0, x) = 0. \end{cases} \quad (1.3)$$

Large deviations of occupation times of critical branching Brownian motions have been studied in several papers. Cox and Griffeath [2] established a large deviation principle for the occupation times of critical branching Brownian motions when the dimension $d \geq 3$. However, the large deviation results of [2] are not entirely satisfactory in dimensions $d = 3, 4$. For dimension $d = 3$, only a large deviation result in a neighborhood of 1 was established in [2], and for dimension $d = 4$, the large deviation result of [2] was in a weak form. Iscoe and Lee [3] improved the results of [2] in dimensions $d = 3$ and $d = 4$.

Large deviations for occupation times of super-Brownian motions have also been the object of several papers, for instance, see [3], [4], [5] and [6].

Recently occupation time processes of branching α -stable processes have been studied by quite a few authors. In [7] and [8], Bojdecki, Gorostiza and Talarczyk established functional limit theorems for the occupation time fluctuations of a critical branching α -stable process. These papers treated the rescaled occupation time process in three different cases: the intermediate dimensions ($\alpha < d < 2\alpha$), the critical dimension ($d = 2\alpha$) and the large dimensions ($d > 2\alpha$), and they showed that the limiting behaviors are different in these 3 cases. Many interesting results were also obtained for the subcritical branching α -stable processes (see, for example, [9], [10] and [11]). Hong and Li [12] and Miłoś [13] obtained large deviation and moderate deviation results for occupation times of subcritical branching processes with immigration.

In this paper, we will deal with the large dimensions $d > 2\alpha$ and prove a large deviation principle for occupation times of critical branching α -stable processes under P .

Before we state the main result of this paper, we introduce some notations: $\mathcal{B}(\mathbf{R}^d)$ will stand for the space of Borel function in \mathbf{R}^d , $C_c(\mathbf{R}^d)(C_c^+(\mathbf{R}^d))$ will stand for the space of (positive) continuous functions with compact support in \mathbf{R}^d , and $H(\mathbf{R}^d)$ will stand for the space of Hölder continuous functions in \mathbf{R}^d . Define

$$\mathbf{A} = \left\{ V(x) : V(x) \in H(\mathbf{R}^d) \cap C_c^+(\mathbf{R}^d), \int_{\mathbf{R}^d} V(x) dx = 1 \right\}.$$

For any $V(x) \in \mathbf{A}$, define

$$\mathbf{W}_{V,T} = \frac{1}{T} \int_0^T \int_{\mathbf{R}^d} V(x) N_t(dx) dt.$$

$T\mathbf{W}_{V,T}$ is called the occupation time of the $T\mathbf{W}_{V,T}$. To discuss the large deviation for $T\mathbf{W}_{V,T}$, the occupation time, we first need to find the cumulant generating function $\ln E[\exp(\theta T\mathbf{W}_{V,T})]$. The following lemma relates it to a nonlinear differential equation.

Lemma 1.1. *Let $d > 2\alpha$. Then there exists $\theta_0 > 0$, for every $V \in \mathbf{A}$ and $|\theta| \leq \theta_0$,*

$$E[\exp(\theta t \mathbf{W}_{V,t})] = \exp \left[\int_{\mathbf{R}^d} w(t, x, \theta V) dx \right], \quad t \geq 0,$$

where

$$w(t, x, \theta V) = E_x \exp[\theta t \mathbf{W}_{V,t}] - 1 \quad (1.4)$$

is the unique bounded mild solution of the following problem:

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta_\alpha w + \frac{1}{2} w^2 + \theta V w + \theta V, & (t, x) \in (0, \infty) \times \mathbf{R}^d, \\ w(0, x) = 0. \end{cases} \quad (1.5)$$

Proof. Replacing $-\varphi(x)$ by $\theta V(x)$ and $-\hat{v}(t, x)$ by $w(t, x, \theta V)$ in (1.1) and (1.3) we can see that the conclusion of the lemma is in fact valid for all $\theta < 0$. Observe that, for any real θ ,

$$\begin{aligned} E[\exp(\theta t \mathbf{W}_{V,t})] &= E \exp \left[\theta \int_0^t \langle N_s, V \rangle ds \right] \\ &= E \exp(\log v(t), \eta) \\ &= \exp\langle v(t) - 1, \lambda \rangle, \quad t \geq 0, \end{aligned} \quad (1.6)$$

where η is a Poisson random measure on \mathbf{R}^d with intensity λ and

$$v(t, x) = E_x \exp(\theta t \mathbf{W}_{V,t}) = E_x \exp \left[\theta \int_0^t \langle N_s, V \rangle ds \right] = w(t, x, \theta V) + 1. \quad (1.7)$$

We need to prove that there is $\theta_0 > 0$ such that when $|\theta| \leq \theta_0$, $E_x[\exp(\theta t \mathbf{W}_{V,t})] < \infty$ for any $x \in \mathbf{R}^d$ and $E[\exp(\theta t \mathbf{W}_{V,t})] < \infty$.

Define

$$\mathcal{E}(\theta, t) = \log E[\exp(\theta t \mathbf{W}_{V,t})].$$

It follows from (1.6) that

$$\mathcal{E}(\theta, t) = \int_{\mathbf{R}^d} [E_x(\exp(\theta t \mathbf{W}_{V,t})) - 1] dx.$$

Thus when the expansion of $\mathcal{E}(\theta, t)$ in terms of its cumulants

$$\mathcal{E}(\theta, t) = \sum_{n=1}^{\infty} m_n(t) \frac{\theta^n}{n!} \quad (1.8)$$

converges absolutely, we have

$$m_n(t) = \int_{\mathbf{R}^d} m_n(t, x) dx, \quad (1.9)$$

with

$$m_n(t, x) = E_x[(t\mathbf{W}_{V,t})^n]. \quad (1.10)$$

As in the proof of (2.2) in [2], $m_n(t, x)$ is given recursively by

$$\begin{aligned} m_1(t, x) &= \int_0^t \int_{\mathbf{R}^d} P_\alpha(t-s, x-y) V(y) dy ds, \quad t \geq 0, \quad x \in \mathbf{R}^d, \\ m_n(t, x) &= \int_0^t \int_{\mathbf{R}^d} P_\alpha(t-s, x-y) \\ &\quad \cdot \left[nm_{n-1}(s, y) V(y) + \frac{1}{2} \sum_{j=1}^{n-1} C_n^j m_j(s, y) m_{n-j}(s, y) \right] dy ds, \\ &\quad n \geq 2, \quad t \geq 0, \quad x \in \mathbf{R}^d, \end{aligned} \quad (1.11)$$

where $C_n^j = \frac{n!}{j!(n-j)!}$. It is well known that the transition density $P_\alpha(t, x)$ is smooth and symmetric in x , and that it satisfies the scaling property $P_\alpha(t, x) = t^{-d/\alpha} P_\alpha(1, t^{-1/\alpha} x)$ (see [1], for instance). Hence we can follow the proof of (3.3) in [2] to get the following:

$$m_n(t, x) \leq 3C(n)\tilde{H}(t)^{n-1}, \quad m_n(t) \leq tC(n)\tilde{H}(t)^{n-1}, \quad t \geq 1, \quad (1.12)$$

where

$$C(n) = n!4^{n-1}(\|V\| \vee 1)^n, \quad \tilde{H}(t) = 12 + 2\bar{H}(2t),$$

and

$$\bar{H}(t) = \int_1^t u P_\alpha(u, 0) du = \frac{\alpha P_\alpha(1, 0)}{d - 2\alpha} (1 - t^{2-\frac{d}{\alpha}}).$$

Therefore there exist a $\theta_0 > 0$, $M > 0$ and $T_0 > 1$ such that

$$w(t, x, \theta V) = \sum_{n=1}^{\infty} \theta^n \frac{m_n(t, x)}{n!} = E_x[\exp(\theta t \mathbf{W}_{V,t})] - 1 < \infty, \quad \theta \leq \theta_0$$

and that

$$\sup_{0 \leq t \leq T, \theta \leq \theta_0} \int_{\mathbf{R}^d} w(t, x, \theta V) dx < MT, \quad \text{for all } T \geq T_0.$$

Using the recursive formula (1.11), we get

$$w(t, x, \theta V) = \int_0^t \int_{\mathbf{R}^d} P_\alpha(t-s, x-y) \left[\theta V(y)(1 + w(s, y, \theta V)) + \frac{1}{2} w(s, y, \theta V)^2 \right] dy ds,$$

thus $w(t, x, \theta V)$ is the mild solution of equation (1.5). We will revisit the existence of $w(t, x, \theta V)$ in Lemma 2.6. For the uniqueness of the bounded solution of (1.5), see Lemma 2.6. \square

To investigate the large deviation for $t\mathbf{W}_{V,t}$, we also need to investigate the limit $w(x, \theta V) =: \lim_{t \rightarrow \infty} w(t, x, \theta V)$. We will show in Theorem 2.5 below that it is a solution of the following non-linear equation:

$$\Delta_\alpha w + \frac{1}{2} w^2 + \theta V w + \theta V = 0, \quad x \in \mathbf{R}^d. \quad (1.13)$$

The main result of this paper is the following

Theorem 1.2. Assume $d > 2\alpha$. For every $V \in \mathbf{A}$, there exists a positive constant $\bar{\theta}$ and a neighborhood O of 1 such that if $U \subset O$ is open and $C \subset O$ is closed, then

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P(\mathbf{W}_{V,T} \in U) \geq - \inf_{\sigma \in U} \sup_{-\bar{\theta} \leq \theta \leq \bar{\theta}} [\sigma \cdot \theta - \Lambda(\theta)],$$

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P(\mathbf{W}_{V,T} \in C) \leq - \inf_{\sigma \in C} \sup_{-\bar{\theta} \leq \theta \leq \bar{\theta}} [\sigma \cdot \theta - \Lambda(\theta)],$$

where

$$\Lambda(\theta) =: \int_{\mathbf{R}^d} \left[\frac{1}{2} w(x, \theta V)^2 + (1 + w(x, \theta V)) \theta V(x) \right] dx,$$

and $w(x, \theta V)$ is the unique solutions of (1.13).

Remark 1.3. In particular, when $\alpha = 2$, we get a large deviation principle for the occupation times of critical branching Brownian motions when $d > 4$. So, our result is an extension of Theorem 5 in [2].

In [6], we proved a large deviation principle for occupation times of critical super α -stable processes by considering the following two nonlinear differential equations:

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta_\alpha w + w^2 + V, & (t, x) \in (0, \infty) \times \mathbf{R}^d, \\ w(0, x) = 0, \end{cases} \quad (1.14)$$

and

$$\Delta_\alpha w + w^2 + V = 0, \quad x \in \mathbf{R}^d. \quad (1.15)$$

Comparing with (1.14) and (1.15), there is an extra term $\theta V w$ in the corresponding equations (1.5) and (1.13) for critical branching α -stable processes. Dealing with this extra term $\theta V w$ is the main difficulty of this paper.

Throughout this paper C denotes a constant which may change values from line to line.

2 Nonlinear Differential Equations

When $\alpha < d$, the process ξ is transient and its potential density $G(x, y) = G(x - y)$ is given by

$$G(x, y) = \int_0^\infty P_\alpha(t, x, y) dt = \mathcal{A}_1(d, \alpha) |x - y|^{\alpha-d},$$

where $\mathcal{A}_1(d, \alpha) = 2^{-\alpha} \pi^{-\frac{\alpha}{2}} \Gamma(\frac{d-\alpha}{2}) \Gamma(\frac{\alpha}{2})^{-1}$. For any function V , we define

$$GV(x) = \int_{\mathbf{R}^d} G(x - y) V(y) dy.$$

Definition 2.1. (1) A function $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is said to be in the Kato class $K_{d,\alpha}$, if

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbf{R}^d} \int_{|x-y| < r} \frac{|f(y)|}{|x-y|^{d-\alpha}} dy = 0.$$

(2) A function $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is said to be in $K_{d,\alpha}^\infty$ if it is $K_{d,\alpha}$ and if for every $\epsilon > 0$, there exists a compact set K such that

$$\sup_{x \in \mathbf{R}^d} \int_{K^c} \frac{|f(y)|}{|x-y|^{d-\alpha}} dy < \epsilon.$$

The following two lemmas have been proved in [6].

Lemma 2.2. Suppose $d > \alpha$. For any $\rho > \alpha$, $\{V(x) \in K_{d,\alpha} : V(x) = O(|x|^{-\rho}) \text{ as } |x| \rightarrow \infty\} \subseteq K_{d,\alpha}^\infty$.

Lemma 2.3. Suppose $d > \alpha$, $f \in K_{d,\alpha}$ and that $|f| = O(|x|^{-\rho})$ with $\rho > \alpha$. Then the function Gf is a bounded continuous solution of $\Delta_\alpha u = -f$. Conversely, if u is a bounded continuous solution of $\Delta_\alpha u = -f$, then $u = Gf + C$ for some constant C .

The following lemma is a modification of Lemma 3.1 in [6], and it will play an important role in dealing with the nonlinear differential equations (1.5) and (1.13).

Lemma 2.4. Suppose $d > 2\alpha$. There exists a constant $M > 0$ such that the function $\varphi(x) = M(1 + |x|)^{2(\alpha-d)}$ on \mathbf{R}^d satisfies

$$G\varphi(x) < \frac{1}{4} \left(\varphi(x)^{\frac{1}{2}} \wedge 1 \right), \quad x \in \mathbf{R}^d.$$

Proof. Since $G(x) \leq C|x|^{\alpha-d}$, we have

$$\frac{G\varphi(x)}{\varphi^{\frac{1}{2}}(x)} = \frac{\int_{\mathbf{R}^d} G(x-y)\varphi(y)dy}{\varphi^{\frac{1}{2}}(x)} \leq CM^{(1-1/2)} \int_{\mathbf{R}^d} \left(\frac{1+|x|}{|x-y|} \right)^{d-\alpha} \frac{dy}{(1+|y|)^{(d-\alpha)2}}.$$

It is not hard to check that $\int_{\mathbf{R}^d} \left(\frac{1+|x|}{|x-y|} \right)^{d-\alpha} \frac{dy}{(1+|y|)^{(d-\alpha)2}}$ is bounded in \mathbf{R}^d (for details see the proof of Lemma 3.1 in [6]). Then there is $M > 0$ such that $G\varphi(x) < \frac{1}{4}\varphi(x)^{\frac{1}{2}}$. Since $G\varphi(x)$ is bounded in \mathbf{R}^d , we can choose M small enough such that $\|G\varphi\|_\infty < 1/4$. \square

Now we are ready to discuss (1.5) and (1.13). The goal of this section is to prove the following theorem:

Theorem 2.5. Suppose $d > 2\alpha$ and that $V \in \mathbf{A}$. Define

$$\bar{\theta} = \min \left\{ \theta_0, \min_{x \in \text{supp} V} \frac{\varphi(x)}{V(x)} \right\} (> 0), \quad (2.1)$$

where θ_0 is the same as in Lemma 1.1 and $\text{supp} V$ is the support of V . Then for every $\theta \in [-\bar{\theta}, \bar{\theta}]$, Equation (1.5) has a unique bounded solution $w(t, x, \theta V)$; furthermore, the limit

$$w(x, \theta V) =: \lim_{t \rightarrow \infty} w(t, x, \theta V)$$

exists for every $x \in \mathbf{R}^d$ and $w(x, \theta V)$ is a bounded $K_{d,\alpha}^\infty$ solution of (1.13).

Let S_t^α denote the transition semigroup of ξ . To prove this theorem, we first consider, as in [6], solutions of the following integral equations:

$$w(t, x, \theta V) = \int_0^t S_{t-s}^\alpha \left[\frac{1}{2} w^2(s, \cdot, \theta V) + \theta(1 + w(s, \cdot, \theta V))V \right] (x) ds, \quad (2.2)$$

and

$$w(x, \theta V) = G \left[\frac{1}{2} w^2(\cdot, \theta V) + \theta(1 + w(\cdot, \theta V))V \right] (x), \quad x \in \mathbf{R}^d. \quad (2.3)$$

$w(t, x, \theta V)$ and $w(x, \theta V)$ are called mild solutions of (1.5) and (1.13) respectively. Then we prove that the mild solutions are classical solutions under some conditions.

Also, we say $\underline{w}(t, x)$ is a mild subsolution of Equation (2.2) and $\bar{w}(t, x)$ is a mild supersolution of Equation (2.2), if they satisfy

$$\underline{w}(t, x) \leq \int_0^t S_{t-s}^\alpha \left[\frac{1}{2} \underline{w}^2(s, \cdot) + \theta(1 + \underline{w}(s, \cdot))V \right] (x) ds,$$

$$\bar{w}(t, x) \geq \int_0^t S_{t-s}^\alpha \left[\frac{1}{2} \bar{w}^2(s, \cdot) + \theta(1 + \bar{w}(s, \cdot))V \right] (x) ds,$$

respectively.

Lemma 2.6. Suppose $d > 2\alpha$, $V \in \mathbf{A}$ and that $\theta \in [-\bar{\theta}, \bar{\theta}]$. The integral Equation (2.2) has a unique bounded solution $w(t, x, \theta V)$.

Proof. For $\theta \in [-\bar{\theta}, \bar{\theta}]$, to show the existence of a solution of Equation (2.2), we first find its mild supersolution and mild subsolution. Set

$$\underline{w}(t, x) = -8G\varphi(x); \quad \bar{w}(t, x) = 8G\varphi(x).$$

Note that $\theta \in [-\bar{\theta}, \bar{\theta}]$ implies $-\varphi \leq \theta V \leq \varphi$. Then using Lemma 2.4, we get

$$\begin{aligned} & \int_0^t S_{t-s}^\alpha \left(\frac{1}{2} \underline{w}^2(s, \cdot) + \theta(1 + \underline{w}(s, \cdot))V \right) (x) ds \\ & \geq \int_0^t S_{t-s}^\alpha (-\varphi - 8\varphi G\varphi) (x) ds \\ & \geq \int_0^t S_{t-s}^\alpha (-\varphi - 2\varphi) (x) ds \\ & \geq -G(3\varphi)(x) \geq \underline{w}(t, x), \end{aligned}$$

and

$$\begin{aligned} & \int_0^t S_{t-s}^\alpha \left(\frac{1}{2} \bar{w}^2(s, \cdot) + \theta(1 + \bar{w}(s, \cdot))V \right) (x) ds \\ & \leq \int_0^t S_{t-s}^\alpha \left(\frac{1}{2} (8G\varphi)^2 + \varphi + 8\varphi G\varphi \right) (x) ds \\ & \leq \int_0^t S_{t-s}^\alpha (2\varphi + \varphi + 2\varphi) (x) ds \\ & \leq G(5\varphi)(x) \leq \bar{w}(t, x). \end{aligned}$$

Therefore, $\underline{w}(t, x)$ and $\bar{w}(t, x)$ are mild subsolution and mild supersolution of Equation (2.2), respectively. So by a standard iteration argument, a mild solution $w(t, x; \theta V)$ of Equation (2.2) exists and satisfies

$$\underline{w}(t, x) \leq w(t, x; \theta V) \leq \bar{w}(t, x).$$

Indeed, define

$$\left\{ \begin{array}{l} w_0(t, x, \theta V) = 0, \\ w_1(t, x, \theta V) = \int_0^t S_{t-s}^\alpha (\theta V)(x) ds, \\ \dots \\ w_{n+1}(t, x, \theta V) = \int_0^t S_{t-s}^\alpha \left(\frac{1}{2} w_n^2(s, \cdot, \theta V) + \theta(1 + w_n(s, \cdot, \theta V))V \right) (x) ds, \\ \dots \end{array} \right.$$

It can be checked by induction that $\underline{w}(t, x) \leq w_n(t, x, \theta V) \leq \bar{w}(t, x)$ for all n , t , x and θ , and then $|w_n(t, x, \theta V)| \leq 2$, $(t, x) \in [0, \infty) \times \mathbf{R}^d$. To prove the limit function $w(t, x, \theta V) = \lim_{n \rightarrow \infty} w_n(t, x, \theta V)$ exists, we only need to prove that, for any fixed $t \geq 0$ and any $x \in \mathbf{R}^d$, the series

$$\sum_{n=0}^{\infty} [w_{n+1}(t, x, \theta V) - w_n(t, x, \theta V)] \quad (2.4)$$

is convergent. For any function $h(t, x)$ defined on $[0, \infty) \times \mathbf{R}^d$, we use $\|h(t)\|_x$ to denote the sup norm of $h(t, x)$ with respect to x , that is, $\|h(t)\|_x = \sup_{x \in \mathbf{R}^d} |h(t, x)|$. Recall that $\varphi(x) = M(1 + |x|)^{2(\alpha-d)} \leq M$ for any $x \in \mathbf{R}^d$. We claim that

$$\|w_{n+1}(t) - w_n(t)\|_x \leq \frac{[(2+M)t]^{n+1}}{(n+1)!}. \quad (2.5)$$

In fact, for $n = 0$, we have

$$\|w_1(t) - w_0(t)\|_x = \left\| \int_0^t S_{t-s}^\alpha (\theta V)(x) ds \right\|_x \leq \|\varphi\|_\infty t \leq (2 + M)t.$$

Now suppose that the claim (2.5) holds for $n = k - 1$, we have

$$\begin{aligned} \|w_{k+1}(t) - w_k(t)\|_x &= \left\| \int_0^t S_{t-s}^\alpha \left[\frac{1}{2}(w_k^2 - w_{k-1}^2) + \theta(w_k - w_{k-1})V \right] (x) ds \right\|_x \\ &\leq \left\| \frac{1}{2} \int_0^t S_{t-s}^\alpha |w_k + w_{k-1}| |w_k - w_{k-1}|(x) ds + \int_0^t S_{t-s}^\alpha |\theta V| |w_k - w_{k-1}|(x) ds \right\|_x \\ &\leq (2 + M) \int_0^t \|w_k(s) - w_{k-1}(s)\|_x ds \\ &\leq (2 + M) \int_0^t \frac{[(2 + M)s]^k}{k!} ds = \frac{[(2 + M)t]^{k+1}}{(k + 1)!}. \end{aligned}$$

That is (2.5) also holds in the case when $n = k$. Therefore the claim above is valid. It follows from the claim above that, for any $t \geq 0$, the series $\sum_{n=0}^\infty [w_{n+1}(t, x, \theta V) - w_n(t, x, \theta V)]$ is convergent in the sup norm $\|\cdot\|_x$. Notice $\bar{w}(t, x)$ is bounded in x and by the bounded convergence theorem, $w(t, x, \theta V)$ is a bounded solution of Equation (2.2).

We now prove the uniqueness of the solution of Equation (2.2). Suppose $w_1(t, x, \theta V)$ and $w_2(t, x, \theta V)$ are two bounded solutions of Equation (2.2). Then we have

$$\begin{aligned} &\|w_1(t) - w_2(t)\|_x \\ &= \left\| \int_0^t S_{t-s}^\alpha \left[\frac{1}{2} (w_1^2(s, \cdot, \theta V) - w_2^2(s, \cdot, \theta V)) + \theta V(w_1(s, \cdot, \theta V) - w_2(s, \cdot, \theta V)) \right] (x) ds \right\|_x \\ &\leq C \int_0^t \|w_1(s) - w_2(s)\|_x ds. \end{aligned}$$

By Gronwall's inequality, we have $\|w_1(t) - w_2(t)\|_x = 0$. So $w_1(t, x, \theta V) = w_2(t, x, \theta V)$ for any $t \geq 0$ and any $x \in \mathbf{R}^d$. \square

The proof of the next result is similar to that of Lemma 3.4 in [6]. We omit the details.

Lemma 2.7. Suppose $d > 2\alpha$, $V \in \mathbf{A}$ and that $\theta \in [-\bar{\theta}, \bar{\theta}]$. w is a solution of the integral Equation (2.2) if and only if it is a solution of Equation (1.5).

Proof of Theorem 2.5. It follows from Lemma 2.6 and Lemma 2.7 that Equation (1.5) has a unique bounded solution $w(t, x, \theta V)$. It is easy to see from the definition of $w(t, x, \theta V)$ that for $0 \leq \theta \leq \bar{\theta}$, $w(t, x, \theta V)$ is increasing in t and for $-\bar{\theta} \leq \theta < 0$, $w(t, x, \theta V)$ is decreasing in t . Thus the limit $w(x, \theta V) \equiv \lim_{t \rightarrow \infty} w(t, x, \theta V)$ exists. By the proof of Lemma 2.6, we know that $|w(t, x, \theta V)| \leq 8G\varphi$, and hence

$$|w(x, \theta V)| \leq 8G\varphi(x). \quad (2.6)$$

By Lemma 2.4 and the dominated convergence theorem, we have

$$w(x, \theta V) = \int_0^\infty S_s^\alpha \left[\frac{1}{2} w^2(\cdot, \theta V) + \theta(1 + w(\cdot, \theta V))V \right] (x) ds, \quad x \in \mathbf{R}^d,$$

which can be written as

$$w(x, \theta V) = G \left[\frac{1}{2} w^2(\cdot, \theta V) + \theta(1 + w(\cdot, \theta V))V \right] (x), \quad x \in \mathbf{R}^d.$$

Then Lemma 2.3 implies that $w(x, \theta V)$ satisfies

$$\Delta_\alpha w(x, \theta V) + \frac{1}{2} w^2(x, \theta V) + \theta(1 + w(x, \theta V))V(x) = 0, \quad x \in \mathbf{R}^d.$$

Now we prove that $w(x, \theta V) \in K_{d, \alpha}^\infty$. Using Lemma 2.4 and (2.6), we get $|w(x, \theta V)| \leq 8G\varphi(x) \leq 2\varphi^{\frac{1}{2}}(x) = O(|x|^{\alpha-d})$ as $x \rightarrow \infty$. The assumption $d > 2\alpha$ implies that $\alpha - d < -\alpha$. Thus we have $w(x, \theta V) \in K_{d, \alpha}^\infty$ by Lemma 2.2. \square

Lemma 2.8. Suppose $d > 2\alpha$, $V \in \mathbf{A}$ and that $\theta \in [-\bar{\theta}, \bar{\theta}]$. The differential equation

$$\begin{cases} \Delta_\alpha(f-1) + (w(x, \theta V) + \theta V)f = 0, & x \in \mathbf{R}^d, \\ f > 0, & f(x) \rightarrow 1, \text{ as } x \rightarrow \infty \end{cases} \quad (2.7)$$

has a unique solution which can be written as $f(x, \theta V)$, where $w(x, \theta V)$ is the bounded solution of (1.13) constructed in Theorem 2.5.

Proof. Let $c(x) = w(x, \theta V) + \theta V$. It is easy to see that $|\theta V| \leq \varphi = O(|x|^{2(\alpha-d)})$. Together with the fact that $w(x, \theta V) \in K_{d, \alpha}^\infty$ proved in the proof of Theorem 2.5, we get $c(x) \in K_{d, \alpha}^\infty$.

Put $U(x) = w(x, \bar{\theta}V) - w(x, \theta V)$, $\theta \in [-\bar{\theta}, \bar{\theta}]$. It is easy to see that $U > 0$ and

$$\left(\Delta_\alpha + \frac{1}{2} \frac{w(x, \bar{\theta}V)^2 - w(x, \theta V)^2}{w(x, \bar{\theta}V) - w(x, \theta V)} + \frac{(\bar{\theta} - \theta)V(x)}{w(x, \bar{\theta}V) - w(x, \theta V)} + \frac{\bar{\theta}V(x)w(x, \bar{\theta}V) - \theta V(x)w(x, \theta V)}{w(x, \bar{\theta}V) - w(x, \theta V)} \right) U = 0.$$

We denote the above equation simply by $(\Delta_\alpha + c + q)U = 0$, where

$$q = \frac{1}{2} (w(x, \bar{\theta}V) - w(x, \theta V)) + \frac{(\bar{\theta} - \theta)V(x)}{w(x, \bar{\theta}V) - w(x, \theta V)} + \frac{(\bar{\theta} - \theta)V(x)w(x, \bar{\theta}V)}{w(x, \bar{\theta}V) - w(x, \theta V)}.$$

Obviously $q \geq 0$. Thus the Schrödinger equation $(\Delta_\alpha + c + q)u = 0$ has a bounded solution U . This implies that $\Delta_\alpha + c$ is subcritical by Proposition 2.5 in [6]. By Proposition 2.6 in [6], $w_0(x) = E_x [\exp(\int_0^\infty c(\xi_s)ds)]$ is a bounded c -harmonic function.

Now we show that $w_0(x) \rightarrow 1$, as $|x| \rightarrow \infty$. Using the Markov property of $\{\xi_s, s \geq 0\}$, it can be checked that

$$w_0(x) - 1 = G(cw_0)(x) = \int_{\mathbf{R}^d} G(x, z)c(z)w_0(z)dz$$

(see the argument in the proof of (32) in [14]). Since $c(x) \in K_{d, \alpha}^\infty$, we have the family $\{\int_{\mathbf{R}^d} G(x, z)c(z)dz, x \in \mathbf{R}^d\}$ is uniformly integrable. Using the fact that $w_0(x)$ is bounded, we get

$$\lim_{|x| \rightarrow \infty} \int_{\mathbf{R}^d} G(x, z)c(z)w_0(z)dz = 0.$$

Therefore, $\lim_{|x| \rightarrow \infty} w_0(x) = 1$. By Lemma 2.3, $w_0(x)$ satisfies $\Delta_\alpha(w_0(x) - 1) = -c(x)w_0(x)$ in \mathbf{R}^d , which means that $w_0(x)$ is a solution of (2.7).

We now prove the uniqueness. Assume that $w(x)$ is a solution of $\Delta_\alpha w(x) + c(x)w(x) = 0$ and satisfies $\lim_{|x| \rightarrow \infty} w(x) = 0$. It suffices to prove that $w(x) \equiv 0$. Note that $\Delta_\alpha w(x) = -c(x)w(x)$. By Lemma 2.3, $w(x) = G(cw)(x) + C$ for some constant C . Since $\lim_{|x| \rightarrow \infty} G(cw)(x) = 0$, $C = 0$ and then we have $w(x) = G(cw)(x)$, which implies that $w(x) \equiv 0$ by using the iteration method. \square

3 Proof of the main result

Recall that

$$\Lambda(\theta) = \int_{\mathbf{R}^d} \left[\frac{1}{2} w(x, \theta V)^2 + (1 + w(x, \theta V))\theta V(x) \right] dx, \quad (3.1)$$

where $w(x, \theta V)$ is the unique solutions of (1.13).

Lemma 3.1. Suppose $d > 2\alpha$, $V \in \mathbf{A}$ and that $\theta \in [-\bar{\theta}, \bar{\theta}]$, where $\bar{\theta}$ is defined by (2.1). We have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} w(t, x, \theta V) dx = \Lambda(\theta), \quad (3.2)$$

where $w(t, x, \theta V)$ is the unique solutions of (1.5).

Proof.

$$\begin{aligned} & \frac{1}{t} \int_{\mathbf{R}^d} w(t, x, \theta V) dx \\ &= \frac{1}{t} \int_0^t \int_{\mathbf{R}^d} S_{t-s}^\alpha \left[\frac{1}{2} w^2(s, \cdot, \theta V) + \theta(1 + w(s, \cdot, \theta V))V(\cdot) \right] (x) dx ds \\ &= \frac{1}{t} \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} P_\alpha(t-s, x-y) \left[\frac{1}{2} w^2(s, y, \theta V) + \theta(1 + w(s, y, \theta V))V(y) \right] dx dy ds \\ &= \frac{1}{t} \int_0^t \int_{\mathbf{R}^d} \left[\frac{1}{2} w^2(s, y, \theta V) + \theta(1 + w(s, y, \theta V))V(y) \right] dy ds. \end{aligned} \quad (3.3)$$

By the proof of Lemma 2.6, we have $|w(t, x, \theta V)| \leq 8G\varphi$, and then using Lemma 2.4, we get $w(t, x, \theta V)^2 \leq 4\varphi(x)$ for $\theta \in [-\bar{\theta}, \bar{\theta}]$. Thus by the dominated convergence theorem, we have

$$\lim_{t \rightarrow \infty} \int_{\mathbf{R}^d} \left(\frac{1}{2} w(t, y, \theta V)^2 + \theta(1 + w(t, y, \theta V))V(y) \right) dy = \int_{\mathbf{R}^d} \left(\frac{1}{2} w(y, \theta V)^2 + \theta(1 + w(y, \theta V))V(y) \right) dy.$$

Hence, from (3.3) we see that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} w(t, x, \theta V) dx &= \lim_{t \rightarrow \infty} \int_{\mathbf{R}^d} \left(\frac{1}{2} w(t, y, \theta V)^2 + \theta(1 + w(t, y, \theta V))V(y) \right) dy \\ &= \int_{\mathbf{R}^d} \left(\frac{1}{2} w(y, \theta V)^2 + \theta(1 + w(y, \theta V))V(y) \right) dy. \end{aligned}$$

□

Lemma 3.2. Assume $V \in \mathbf{A}$. Then $\Lambda(\theta)$ defined in (3.1) is strictly convex, continuously differentiable on $[-\bar{\theta}, \bar{\theta}]$ and $\Lambda'(0) = 1$.

Proof. From the definition of $w(t, x, \theta V)$ given in (1.4), we know that $w(t, x, \theta V)$ is increasing and convex in $\theta \in [-\bar{\theta}, \bar{\theta}]$. Thus $w(x, \theta V) = \lim_{t \rightarrow \infty} w(t, x, \theta V)$ is increasing and convex in $\theta \in [-\bar{\theta}, \bar{\theta}]$. Using the fact that

$$w(x, \theta V) \begin{cases} \geq 0, & \text{if } \theta \geq 0, \\ \leq 0, & \text{if } \theta < 0, \end{cases}$$

we can easily get $\theta w(x, \theta V)$ is also convex in $\theta \in [-\bar{\theta}, \bar{\theta}]$. The difficult part in proving the strict convexity of $\Lambda(\theta)$ is the term $w^2(x, \theta V)$. We can not get the convexity of $w^2(x, \theta V)$ directly from the convexity of $w(x, \theta V)$ since $w(x, \theta V)$ can take negative values. We overcome this difficulty by giving another representation of the function $\Lambda(\theta)$.

Consider the nonnegative function

$$g(t, x, \theta V) := w(t, x, \theta V) - w(t, x, -\bar{\theta}V). \quad (3.4)$$

$g(t, x, \theta V)$ satisfies

$$\begin{cases} \frac{\partial g(t, x, \theta V)}{\partial t} = [\Delta_\alpha + w(t, x, -\bar{\theta}V) - \bar{\theta}V] g + h(g(t, x, \theta V), t, x) + k(\theta, t, x), & t > 0, x \in \mathbf{R}^d \\ g(0, x) = 0, & x \in \mathbf{R}^d, \end{cases}$$

where $h(g, t, x) = \frac{1}{2} [g(t, x, \theta V) + w(t, x, -\bar{\theta}V)]^2 - \frac{1}{2} w^2(t, x, -\bar{\theta}V) - [w(t, x, -\bar{\theta}V) - \bar{\theta}V] g(t, x, \theta V)$, and $k(\theta, t, x) = \theta V w(t, x, \theta V) + \bar{\theta}V w(t, x, -\bar{\theta}V) + (\theta + \bar{\theta})V$. It is easy to see that $h(g, t, x)$ is strictly convex in g for each fixed t and x , and $k(\theta, t, x)$ is convex in θ for each fixed t and x . Note that $h(0, t, x) = 0$ and $w(t, x, -\bar{\theta}V) \leq 0$ for $t \geq 0$, $x \in \mathbf{R}^d$. Then

$$\begin{aligned}\Lambda(\theta) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} w(t, x, \theta V) dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} w(t, x, -\bar{\theta}V) dx + \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} g(t, x, \theta V) dx \\ &= \Lambda(-\bar{\theta}) + \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} g(t, x, \theta V) dx.\end{aligned}$$

Recall that, by Lemma 2.8,

$$f(x, -\bar{\theta}V) = E_x \left[\exp \left(\int_0^\infty (w(\xi_s, -\bar{\theta}V) - \bar{\theta}V(\xi_s)) ds \right) \right] \quad (3.5)$$

is the unique solution of the following equation

$$\begin{cases} \Delta_\alpha(f - 1) + (w(x, -\bar{\theta}V) - \bar{\theta}V)f = 0, & x \in \mathbf{R}^d, \\ f > 0, & f(x) \rightarrow 1, \text{ as } x \rightarrow \infty. \end{cases}$$

We claim that

$$\Lambda(\theta) = \Lambda(-\bar{\theta}) + \int_{\mathbf{R}^d} [h(g(x, \theta V), x) + k(\theta, x)] f(x, -\bar{\theta}V) dx, \quad (3.6)$$

where

$$g(x, \theta V) = \lim_{t \rightarrow \infty} g(t, x, \theta V) = w(x, \theta V) - w(x, -\bar{\theta}V),$$

$$k(\theta, x) = \lim_{t \rightarrow \infty} k(\theta, t, x) = \theta V w(x, \theta V) + \bar{\theta}V w(x, -\bar{\theta}V) + (\theta + \bar{\theta})V,$$

and

$$h(g(x, \theta V), x) = \lim_{t \rightarrow \infty} h(g, t, x) = \frac{1}{2} [g(x, \theta V) + w(x, -\bar{\theta}V)]^2 - \frac{1}{2} w^2(x, -\bar{\theta}V) - [w(x, -\bar{\theta}V) - \bar{\theta}V] g.$$

Noticing that the function $u \mapsto u^2$ is strictly convex and $g \geq 0$, together with the convexity of $k(\theta, x)$ for fixed x , we can get the strict convexity of $\Lambda(\theta)$. Now we prove (3.6). We denote by Q_t the Schrödinger semigroup corresponding to the operator $\Delta_\alpha + w(t, x, -\bar{\theta}V) - \bar{\theta}V$, and by $q(t, x, y)$ its density function. Then by (3.9) in [16], we have $Q_t f(x) = E_x[\exp(\int_0^t (w(s, \xi_s, -\bar{\theta}V) - \bar{\theta}V(\xi_s)) ds) f(\xi_t)]$. Then

$$\begin{aligned}& \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} g(t, x, \theta V) dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} \int_0^t Q_{t-s}(h(g(t, \cdot, \theta V), t, \cdot) + k(t, \theta, \cdot))(x) ds dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} q(t-s, x, y) (h(g(t, y, \theta V), t, y) + k(t, \theta, y)) dy ds dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} q(t-s, x, y) dx (h(g(t, y, \theta V), t, y) + k(t, \theta, y)) dy ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_{\mathbf{R}^d} (Q_{t-s} 1)(y) (h(g(t, \cdot, \theta V), t, y) + k(t, \theta, y)) dy,\end{aligned}$$

where in the last equality we used the self-adjointness of Δ_α . Using Cesaro's theorem and the dominated

convergence theorem, we get

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbf{R}^d} g(t, x, \theta V) dx \\ &= \lim_{t \rightarrow \infty} \int_{\mathbf{R}^d} (Q_{t-s} 1)(y) (h(g(t, \cdot, \theta V), t, y) + k(t, \theta, y)) dy \\ &= \int_{\mathbf{R}^d} E_y \left[\exp \left(\int_0^\infty (w(\xi_s, -\bar{\theta} V) - \bar{\theta} V(\xi_s)) ds \right) \right] (h(g(\cdot, \theta V), y) + k(\theta, y)) dy \\ &= \int_{\mathbf{R}^d} [h(g(x, \theta V), x) + k(\theta, x)] f(x, -\bar{\theta} V) dx. \end{aligned}$$

Therefore, (3.6) holds.

Next, we prove that $\Lambda'(\theta)$ exists and is continuous. We simply denote $w(x, \theta V)$ by $w(x, \theta)$. Define

$$q(\epsilon, x) = \frac{w(x, \theta + \epsilon) - w(x, \theta)}{\epsilon},$$

$$\beta(\epsilon, x) = \frac{1}{2} (w(x, \theta + \epsilon) + w(x, \theta)) + \theta V.$$

Then the function $q(\epsilon, x)$ satisfies the linear elliptic PDE,

$$[\Delta_\alpha + \beta(\epsilon, x)]q(\epsilon, x) + [1 + w(x, \theta + \epsilon)]V(x) = 0.$$

We claim that the Feynman-Kac representation holds:

$$q(\epsilon, x) = \Pi_x \left\{ \int_0^\infty [1 + w(\xi_t, \theta + \epsilon)] V(\xi_t) \exp \left(\int_0^t \beta(\epsilon, \xi_s) ds \right) dt \right\}.$$

Since $V \in K_{d, \alpha}^\infty$, the dominated convergence theorem implies that

$$\frac{\partial w}{\partial \theta}(x, \theta) = \lim_{\epsilon \rightarrow 0} q(\epsilon, x) = \Pi_x \left\{ \int_0^\infty [1 + w(\xi_t, \theta)] V(\xi_t) \exp \left(\int_0^t (w(\xi_s, \theta) + \theta V(\xi_s)) ds \right) dt \right\}. \quad (3.7)$$

Using (3.7), (3.1), and the dominated convergence theorem, we get

$$\Lambda'(\theta) = \int_{\mathbf{R}^d} \left[(w(x, \theta) + \theta V(x)) \frac{\partial w}{\partial \theta}(x, \theta) + V(x) w(x, \theta) + V(x) \right] dx,$$

which is continuous for $\theta \in [-\bar{\theta}, \bar{\theta}]$. It is easy to see that $\Lambda'(0) = \int_{\mathbf{R}^d} V(x) dx = 1$. \square

Proof of Theorem 1.1. The argument is similar to that of the proof of Theorem 1.1 in [15]. Here we only give an outline of the proof. Let $\bar{\theta}$ be defined by (2.1). From Lemmas 1.1 and 3.1, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log E \{ \exp(T\theta \cdot \mathbf{W}_{V,T}) \} = \Lambda(\theta)$$

for $-\bar{\theta} \leq \theta \leq \bar{\theta}$. A general large deviation result (see [17], for instance) ensures two estimates:

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P(\mathbf{W}_{V,T} \in U) \geq - \inf_{\sigma \in U} \sup_{-\bar{\theta} \leq \theta \leq \bar{\theta}} [\sigma \cdot \theta - \Lambda(\theta)],$$

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P(\mathbf{W}_{V,T} \in C) \leq - \inf_{\sigma \in C} \sup_{-\bar{\theta} \leq \theta \leq \bar{\theta}} [\sigma \cdot \theta - \Lambda(\theta)].$$

The theorem is proved. \square

Acknowledgements. We thank the referees for the careful reading of the first version of this paper and for useful suggestions for improving the quality of the paper. The research of the first author is supported by National Natural Science Foundation of China (Grant No. 10971003 and 10926110) and Chinese Universities Scientific Fund (2009-2-05). The research of the second author is supported by National Natural Science Foundation of China (Grant No. 10871103 and 10971003) and Specialized Research Fund for the Doctoral Program of Higher Education.

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