# ℕ-measure for continuous state branching processes and its application

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**Abstract**

In this paper, we first give a direct construction of the ℕ-measure of a continuous state branching process. Then we prove, with the help of this ℕ-measure, that any continuous state branching process with immigration can be constructed as the independent sum of a continuous state branching process (without immigration), and two immigration parts (jump immigration and continuum immigration). As an application of this construction of a continuous state branching process with immigration, we give a proof of a necessary and sufficient condition, first stated without proof in [9], for a continuous state branching process with immigration to a proper almost sure limit. As another application of the ℕ-measure, we give a "conceptual" proof of an $L \log L$ criterion for a continuous state branching process without immigration to have an $L^1$-limit first proved in [2].

## 1 ℕ-measure for continuous state branching processes

The spine decomposition is an important probabilistic tool in branching processes, multi-type branching processes, branching Hunt processes and superprocesses. Using the spine decomposition, many classical results on these processes can be proved more directly, see, for example, [5], [6], [7] and [8]. In the spine decomposition for a superprocess under a martingale change of measure, the ℕ-measure, defined by Dynkin-Kuznetsov, is a key ingredient (see [5]). It is natural to ask if it is possible to describe the spine decomposition for a continuous state branching process using the ℕ-measure of the continuous state branching process.

The ℕ-measure of a continuous state branching process can be thought of as a special case of the ℕ-measure of a superprocess, constructed in [3], by taking the underlying spatial motion to a constant process. However, a superprocess is a much more complicated model than a continuous state branching process. It is desirable to have a direct construction of the ℕ-measure of a continuous state branching process, without using knowledge of superprocesses.

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In this paper, we first give a direct construction of the $\mathbb{N}$-measure of a continuous state branching process. Then in Section 2.1, we prove that any continuous state branching process with immigration can be constructed as the independent sum of a continuous state branching process (without immigration), and two immigration parts (jump immigration and continuum immigration). As an application of this construction, in Section 2.2, we give a proof of a necessary and sufficient condition, first stated without a proof in [9], for a continuous state branching process with immigration to have a proper scaling limit. As another application of the construction, we give a "conceptual" proof of an $L \log L$ criterion for the non-degeneracy of a martingale limit of a continuous state branching process first proved in [2].

Suppose that $X = (X_t : t \geq 0)$ is a continuous time and continuous state branching process. For any $x \in (0, \infty)$, we use $\mathbb{P}_x$ to denote the law of $X$ starting from $x$. We say that the process is canonical if $(\Omega, \mathcal{F})$ is the path space (that is, $\Omega$ consists of all $[0, \infty)$-valued functions $\omega$ that are right continuous with left limit on $[0, \infty)$, and $\mathcal{F}$ is generated by the sets $\{\omega : \omega(t) < c\}$ where $t \geq 0$, $c \in [0, \infty)$) and if $X_t(\omega) = \omega(t)$.

Throughout this paper we assume that for any $x > 0$,

$$\mathbb{P}_x(X_t = 0) > 0, \quad \text{for any } t > 0. \tag{1}$$

**Theorem 1.1** *Suppose for each $x \in [0, \infty)$, $X = (X_t, \mathbb{P}_x : t \geq 0)$ is a canonical continuous state branching process starting from $x$. Then for every $x \in [0, \infty)$, there exists a unique measure $\mathbb{N}_x$ on the space $\Omega$ such that:*

*1) For any integer $n \geq 1$, and $t_i, \lambda_i \geq 0$, $i = 1, \cdots n$,*

$$\mathbb{N}_x \left( 1 - \exp \left( - \sum_{i=1}^{n} \lambda_i X_{t_i} \right) \right) = - \log \mathbb{P}_x \exp \left( - \sum_{i=1}^{n} \lambda_i X_{t_i} \right). \tag{2}$$

*2) $\mathbb{N}_x(\widetilde{\Omega}) = 0$, where $\widetilde{\Omega} = \cap_{t \geq 0} \{X_t = 0\}$.*

The measure $\mathbb{N}_x$ is the "Lévy measure" of $\mathbb{P}_x$, and can be thought of as an "excursion measure" on path space. As we remarked before, Dynkin and Kuznetsov [3] first proved the counterpart of this result for superprocesses. The above theorem can be obtained from Theorem 1.1 of [3], by taking the underlying spatial motion to be a constant process. Below we give a direct proof of Theorem 1.1, without using the knowledge of superprocesses.

**Proof of Theorem 1.1** We follow the general strategy of the proof of Theorem 1.1 of [3]. For any integer $k > 0$ and any $t, \lambda \geq 0$, by the branching property of $X$, we have

$$\mathbb{P}_x \exp(-\lambda X_t) = \left( \mathbb{P}_{x/k} \exp(-\lambda X_t) \right)^k,$$

which implies that the distribution of $X_t$ is infinitely divisible. Thus by the Lévy-Khintchine formula there exists unique pair $(m, \mathcal{R}_x^{(t)})$ such that

$$\mathbb{P}_x \exp(-\lambda X_t) = \exp \left( -m\lambda - \int_0^{\infty} (1 - e^{-\lambda z}) \mathcal{R}_x^{(t)}(dz) \right),$$

where $m \geq 0$ is a constant and $\mathcal{R}_x^{(t)}$ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge z)\mathcal{R}_x^{(t)}(dz) < \infty$ (see P. 385 of [10]). Letting $\lambda \to \infty$ we see that $P_x(X_t = 0) > 0$ implies $m = 0$ and $\mathcal{R}_x^{(t)}((0,\infty)) < \infty$, and therefore

$$\mathbb{P}_x \exp(-\lambda X_t) = \exp\left(-\int_0^\infty (1 - e^{-\lambda z})\mathcal{R}_x^{(t)}(dz)\right).$$

Similarly, for any integers $n, k \geq 1$, and $t_i, \lambda_i \geq 0, i = 1, \cdots n$, we have

$$\mathbb{P}_x \exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right) = \left(\mathbb{P}_{x/k} \exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right)\right)^k.$$

Put $I = (t_1, \cdots, t_n)$. Then there exists a unique $\mathcal{R}_x^I$ such that

$$\mathbb{P}_x \exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right) = \exp\left(-\int_{[0,\infty)^{\times n}} (1 - e^{-\sum_{i=1}^n \lambda_i z_i})\mathcal{R}_x^I(dz)\right),$$

where $\mathcal{R}_x^I$ is a measure on $(0,\infty)^{\times n}$ and $z = (z_1, \cdots, z_n)$. $\mathcal{R}_x^I$ has the following properties i), ii) and iii):

i) For $I = (t_1, \cdots, t_n)$, and $t > 0$, put $t \circ I = (t, t_1, \cdots, t_n)$. We have

$$\begin{aligned} & \mathcal{R}_x^{t\circ I}\left(z_0 \neq 0, \exp\left(-\sum_{i=1}^n \lambda_i z_i\right)\right) \\ = & -\log \mathbb{P}_x\left(X_t = 0, \quad \exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right)\right) + \log \mathbb{P}_x\left(\exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right)\right). \end{aligned} \tag{3}$$

In fact,

$$\begin{aligned} & \mathcal{R}_x^{t\circ I}\left(-\exp\left(-\sum_{i=1}^n \lambda_i z_i - \lambda z_0\right) + \exp\left(-\sum_{i=1}^n \lambda_i z_i\right)\right) \\ = & \mathcal{R}_x^{t\circ I}\left(1 - \exp\left(-\sum_{i=1}^n \lambda_i z_i - \lambda z_0\right)\right) - \mathcal{R}_x^{t\circ I}\left(1 - \exp\left(-\sum_{i=1}^n \lambda_i z_i\right)\right) \\ = & -\log \mathbb{P}_x\left(\exp\left(-\sum_{i=1}^n \lambda_i X_{t_i} - \lambda X_t\right)\right) + \log \mathbb{P}_x\left(\exp\left(-\sum_{i=1}^n \lambda_i X_{t_i}\right)\right). \end{aligned}$$

Letting $\lambda \to \infty$, we get (3).

ii) For $t_1 < t_2$, from the property of branching process, it is obvious that

$$\mathbb{P}_x(X_{t_2} = 0 | X_{t_1} = 0) = 1. \tag{4}$$

Thus we have

$$\mathcal{R}_x^{(t_1,t_2)}(z_1 = 0, z_2 \neq 0) = 0. \tag{5}$$

In fact, it follows from (3) that

$$\mathcal{R}_x^{(t_1,t_2)}\left(z_2\neq 0,\exp(-\lambda z_1)\right)=-\log\mathbb{P}_x\left(X_{t_2}=0,\exp(-\lambda X_{t_1})\right)+\log\mathbb{P}_x\exp(-\lambda X_{t_1}). \qquad (6)$$

Letting $\lambda\to\infty$, we get

$$\begin{aligned}\mathcal{R}_x^{(t_1,t_2)}\left(z_2\neq 0,z_1=0\right)&=\ -\log\mathbb{P}_x\left(X_{t_2}=0,X_{t_1}=0\right)+\log\mathbb{P}_x(X_{t_1}=0)\\&=\ -\log\mathbb{P}_x(X_{t_2}=0|X_{t_1}=0)=0.\end{aligned}$$

iii) If $I=(t_1,\cdots,t_n)$ and $J=(t_1,\cdots,t_n,t_{n+1},\cdots,t_{n+m})$ with $m,n\geq 1$, $t_i\geq 0$, $i=1,\cdots,m+n$ and $m\geq 1$, then for any $t\geq 0$, we have

$$\mathcal{R}_x^{t\circ I}(z_0\neq 0,\quad (z_1,\cdots,z_n)\in B)=\mathcal{R}_x^{t\circ J}(z_0\neq 0,\quad (z_1,\cdots,z_n)\in B) \qquad (7)$$

for any Borel set $B\subset(0,\infty)^{\times n}$.

In fact, for any $\lambda_i\geq 0$, $i=1,\cdots,n$, we have from (3) that

$$\begin{aligned}&\mathcal{R}_x^{t\circ I}\left(z_0\neq 0,\quad \exp\left(-\sum_{i=1}^n\lambda_i z_i\right)\right)\\=&\ -\log\mathbb{P}_x\left(X_t=0,\quad \exp\left(-\sum_{i=1}^n\lambda_i X_{t_i}\right)\right)+\log\mathbb{P}_x\exp\left(-\sum_{i=1}^n\lambda_i X_{t_i}\right)\\=&\ \mathcal{R}_x^{t\circ J}\left(z_0\neq 0,\quad \exp\left(-\sum_{i=1}^n\lambda_i z_i\right)\right),\end{aligned}$$

which implies (7).

It follows from (3) that

$$\mathcal{R}_x^{t\circ I}(z_0\neq 0)=-\log\mathbb{P}_x(X_t=0)$$

is finite and does not depend on $I$. Let $\Omega_t=\{\omega;X(t)\neq 0\}$ and $\mathcal{F}_t=\Omega_t\cap\mathcal{F}$. By Kolmogorov's theorem, there exists a finite measure $\mathbb{N}_x^t$ on $(\Omega_t,\mathcal{F}_t)$ such that

$$\mathbb{N}_x^t\exp\left(-\sum_{i=1}^n\lambda_i X_{t_i}\right)=\mathcal{R}_x^{t\circ I}\left(z_0\neq 0,\quad \exp\left(-\sum_{i=1}^n\lambda_i z_i\right)\right). \qquad (8)$$

The measure $\mathbb{N}_x^t$ has the following properties a), b) and c):

a) For any nonnegative measurable function $F$,

$$\mathbb{N}_x^t\left(F\left(X_{t_1},\cdots,X_{t_n}\right)\right)=\mathcal{R}_x^{t\circ I}\left(z_0\neq 0,\quad F\left(z_1,\cdots,z_n\right)\right). \qquad (9)$$

b) If $t_1<t_2$, then $\Omega_{t_2}\subset\Omega_{t_1}$ and $\mathbb{N}_x^{t_1}=\mathbb{N}_x^{t_2}$ on $\Omega_{t_2}$.

c) For any $t_1,t_2\geq 0$, $\mathbb{N}_x^{t_1}=\mathbb{N}_x^{t_2}$ on $\Omega_{t_1}\cap\Omega_{t_2}$.

In fact, a) follows from (8). The first part of b) holds because it follows from (9) and (5) that

$$\mathbb{N}_x^{t_2}(X_{t_1}=0)=\mathcal{R}^{(t_1,t_2)}(z_2\neq 0,z_1=0)=0.$$

The second part of b) follows from the relation

$$\mathbb{N}_x^{t_2}\left(X_{t_2}\neq 0,\quad F\left(X_{t_3},\cdots,X_{t_{n+2}}\right)\right)=\mathbb{N}_x^{t_1}\left(X_{t_2}\neq 0,\quad F\left(X_{t_3},\cdots,X_{t_{n+2}}\right)\right)$$

with $F$ being any nonnegative measurable function and $t_{i+2}\geq 0$, $i=1,\cdots,n$. This relation comes from the observation that

$$\mathbb{N}_x^{t_1}\left(X_{t_2}\neq 0,\quad F\left(X_{t_3},\cdots,X_{t_{n+2}}\right)\right)=\mathcal{R}_x^{t_1\circ t_2\circ I}\left(z_1\neq 0,z_2\neq 0,\quad F\left(z_3,\cdots,z_{n+2}\right)\right),$$

and

$$\begin{aligned}\mathbb{N}_x^{t_2}\left(X_{t_2}\neq 0,\quad F\left(X_{t_3},\cdots,X_{t_{n+2}}\right)\right) &= \mathcal{R}_x^{t_2\circ I}\left(z_2\neq 0,\quad F\left(z_3,\cdots,z_{n+2}\right)\right)\\ &= \mathcal{R}_x^{t_1\circ t_2\circ I}\left(z_2\neq 0,\quad F\left(z_3,\cdots,z_{n+2}\right)\right)\\ &= \mathcal{R}_x^{t_1\circ t_2\circ I}\left(z_1\neq 0,z_2\neq 0,\quad F\left(z_3,\cdots,z_{n+2}\right)\right),\end{aligned}$$

where the second to the last equality follows from (7), and the last equality holds since

$$\mathcal{R}_x^{t_1\circ t_2\circ I}\left(z_1=0,z_2\neq 0\right)=\mathcal{R}^{(t_1,t_2)}\left(z_1=0,z_2\neq 0\right)=0.$$

c) holds because $\mathbb{N}_x^{t_1}=\mathbb{N}_x^{t_1\wedge t_2}$ on $\Omega_{t_1}$ and $\mathbb{N}_x^{t_2}=\mathbb{N}_x^{t_1\wedge t_2}$ on $\Omega_{t_2}$.

Define $\Omega^*=\bigcup_{t\geq 0}\Omega_t$. Then there exists a measure $\mathbb{N}_x$ on $\Omega^*$ such that

$$\mathbb{N}_x=\mathbb{N}_x^t\quad\text{ on }\Omega_t\quad\text{ for any } t>0.$$

Define $\mathbb{N}_x(\Omega\setminus\Omega^*)=0$. We claim that

$$\mathbb{N}_x\left(1-\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right)\right)=-\log\mathbb{P}_x\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right),\quad t_i\geq 0,i=1,\cdots,n.\tag{10}$$

In fact, let $t=\min\{t_1,\cdots,t_n\}$. Since for any $i=1,\cdots,n$, $\{X_t=0\}\subset\{X_{t_i}=0\}$ $\mathbb{N}_x$-a.s., we have

$$\begin{aligned}&\mathbb{N}_x\left(1-\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right)\right)\\ =&\ \mathbb{N}_x\left(X_t\neq 0,\quad 1-\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right)\right)\\ =&\ \mathbb{N}_x^t\left(1-\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right)\right).\end{aligned}\tag{11}$$

By (3) and (9), we have

$$\begin{aligned}\mathbb{N}_x^t\left(1-\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right)\right) &= \mathcal{R}_x^{t\circ I}\left(z_0\neq 0,\quad 1-\exp\left(-\sum_{i=1}^n\lambda_iz_i\right)\right)\\ &= -\log\mathbb{P}_x\exp\left(-\sum_{i=1}^n\lambda_iX_{t_i}\right).\end{aligned}\tag{12}$$

Combining (11) and (12), we get (10).

□

# 2 Applications

## 2.1 Construction of a continuous state branching process with immigration

Suppose that $(\mathcal{Z}, \mathbb{P}_x) = (\mathcal{Z}_t, \mathbb{P}_x : t \geq 0)$ is a supercritical continuous state branching process with immigration starting from $x \geq 0$. Suppose that the branching mechanism $\psi$ and immigration mechanism $\varphi$ are given as follows:

$$\psi(\lambda) = \beta\lambda + \alpha\lambda^2 + \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x I_{\{x<1\}}\right) \Pi(\mathrm{d}x),$$

$$\varphi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right) n(\mathrm{d}x),$$

where $\beta \in R$, $\alpha \geq 0$, $b \geq 0$, and $\Pi$ and $n$ are nonnegative measures on $(0, \infty)$ such that

$$\int_0^\infty \left(1 \wedge x^2\right) \Pi(\mathrm{d}x) < \infty, \qquad \int_0^\infty (1 \wedge x) n(\mathrm{d}x) < \infty. \tag{13}$$

The Laplace transform of $\mathcal{Z}$ is given by

$$\mathbb{E}_x e^{-\lambda \mathcal{Z}_t} = \exp\left\{-x u_t(\lambda) - \int_0^t \varphi(u_s(\lambda))\mathrm{d}s\right\}, \qquad t \geq 0, \quad \lambda \geq 0, \quad x \geq 0, \tag{14}$$

where $u_t(\lambda)$ satisfies

$$u_0(\lambda) = \lambda, \qquad \frac{\partial}{\partial t} u_t(\lambda) + \psi(u_t(\lambda)) = 0. \tag{15}$$

$\mathcal{Z} = (\mathcal{Z}_t, t \geq 0)$ is usually called a CBI$(\psi, \varphi)$. In particular, if $\varphi = 0$, CBI$(\psi, 0)$ is a continuous state branching process (without immigration), and is called a CB$(\psi)$.

In the remainder of this paper we assume that

$$\int_0^\infty (x \wedge x^2)\Pi(\mathrm{d}x) < \infty. \tag{16}$$

Then we can write $\psi$ in the following form

$$\psi(\lambda) = a\lambda + \alpha\lambda^2 + \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x\right) \Pi(\mathrm{d}x).$$

Using some ideas from [5], we can decompose the immigration of a CBI$(\psi, \varphi)$ into two parts, called jump immigration and continuum immigration respectively. And then we construct a CBI$(\psi, \varphi)$ as the independent sum of a CB$(\psi)$ and the two immigration parts. Now we construct this decomposition, which is called the spine decomposition of continuous state branching process.

Suppose that $Z = (Z_t : t \geq 0)$ is a CB$(\psi)$ starting from $x$ defined on some probability space $(\Omega^{(0)}, \mathcal{F}^{(0)}, \mathbf{P}_x^{(0)})$. Condition (16) implies that $\mathbf{E}_x^{(0)} Z_t = x e^{-\psi'(0+)t} < \infty$.

Put

$$\varphi_1(\lambda) = \int_0^\infty \left(1 - e^{-\lambda x}\right) n(\mathrm{d}x), \tag{17}$$

and

$$\varphi_2(\lambda) = b\lambda. \tag{18}$$

Suppose that $S = (S_t : t \geq 0)$, defined on a probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbf{P}^{(1)})$, is a pure jump subordinator with Laplace exponent $\varphi_1$, and $\mathbf{J}$ is the Poisson random measure associated with the jumps of $S$. That is $S_t = \int_0^t \int_0^\infty x\mathbf{J}(\mathrm{d}s\mathrm{d}x)$. For each $(s, x)$ in the support of $\mathbf{J}$, let $Z^x_{t-s}$ denote an independent copy of the process $(Z, \mathbb{P}^{(0)}_x)$ starting at time $s$. Define

$$\mathcal{Z}^{(1)}_t = \int_0^t \int_0^\infty Z^x_{t-s}\mathbf{J}(\mathrm{d}s\mathrm{d}x). \tag{19}$$

Assume $T_1$ is the set of jumping times of $S$, then $T_1$ is at most countable. Thus we can define $\mathcal{Z}^{(1)}_t$ in the following way:

$$\mathcal{Z}^{(1)}_t = \sum_{\sigma \in T_1 \cap [0,t]} Z^{\Delta S_\sigma}_{t-\sigma}. \tag{20}$$

For any jumping time $\sigma$ of $S$ and the corresponding jumping height $\Delta S_\sigma$, $Z^{\Delta S_\sigma}_{t-\sigma}$ satisfies

$$\mathbf{E}^{(1)} \exp\left\{-\lambda Z^{\Delta S_\sigma}_{t-\sigma} | S\right\} = \exp\left\{-\Delta S_\sigma u_{t-\sigma}(\lambda)\right\}, \tag{21}$$

and we also have

$$\mathbf{E}^{(1)} \left[Z^{\Delta S_\sigma}_{t-\sigma} | S\right] = \Delta S_\sigma e^{m(t-\sigma)}. \tag{22}$$

From these we can get

$$\mathbf{E}^{(1)} e^{-\lambda \mathcal{Z}^{(1)}_t} = \exp\left\{-\int_0^t \varphi_1(u_s(\lambda))\mathrm{d}s\right\}. \tag{23}$$

For details one can refer to Chapter 10 of [4].

The term $\varphi_2$ corresponds to a continuum immigration. According to Theorem 1.1, we know that for the canonical CBI$(\psi, 0)$, denoted as $X = (X_t, \mathbb{P}_x : t \geq 0)$, there exists a unique measure $\mathbb{N}_x$ on $(\Omega, \mathcal{F})$ satisfying

$$\mathbb{N}_x[1 - e^{-\lambda X_t}] = -\log \mathbb{P}_x[e^{-\lambda X_t}] = x u_t(\lambda). \tag{24}$$

Suppose that $\mathbf{n}$ is a Poisson point process with rate $\mathrm{d}s \times b\,\mathrm{d}\mathbb{N}_1$, defined on a probability space $(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbf{P}^{(2)})$. For each $(s, \omega)$ in the support of $\mathbf{n}$, $\mathbf{n}$ generates an independent copy of $(X, \mathbb{N}_1)$, denoted as $X^{\mathbf{n},s}$. Let $T_2$ be the set of its jumping times. Define

$$\mathcal{Z}^{(2)}_t = \int_0^t \int_\Omega X^{\mathbf{n},s}_{t-s}\mathbf{n}(\mathrm{d}s\mathrm{d}\mathbb{N}_1) = \sum_{s \in T_2 \cap [0,t]} X^{\mathbf{n},s}_{t-s} \tag{25}$$

where all the processes $\{X^{\mathbf{n},s}, s<\infty\}$ are independent. By (24), we have

$$\begin{aligned}\mathbf{E}^{(2)}e^{-\lambda\mathcal{Z}_t^{(2)}} &= \mathbf{E}^{(2)}\exp\Big\{-\lambda\sum_{s\in T_2\cap[0,t]}X_{t-s}^{\mathbf{n},s}\Big\} &(26)\\ &= \exp\left\{-b\int_0^t\int_\Omega 1-e^{-\lambda X_{t-s}}\mathrm{d}\mathbb{N}_1\mathrm{d}s\right\}\\ &= \exp\left\{-b\int_0^t u_{t-s}(\lambda)\mathrm{d}s\right\}. &(27)\end{aligned}$$

Define the process $Z+\mathcal{Z}^{(1)}+\mathcal{Z}^{(2)}$ on the product space

$$(\widetilde{\Omega},\widetilde{\mathcal{F}},\mathbf{P}_x)=(\Omega^{(0)},\mathcal{F}^{(0)},\mathbf{P}_x^{(0)})\times(\Omega^{(1)},\mathcal{F}^{(1)},\mathbf{P}^{(1)})\times(\Omega^{(2)},\mathcal{F}^{(2)},\mathbf{P}^{(2)})$$

Then $Z$, $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(2)}$ are independent, and $Z+\mathcal{Z}^{(1)}+\mathcal{Z}^{(2)}$ has the same Laplace transform as $(\mathcal{Z},\mathbb{P}_x)$, and therefore is a CBI$(\psi,\varphi)$ starting from $x$.

## 2.2 Almost sure limit of continuous state branching processes with immigration

First note that, since $Z$ is a CB$(\psi)$ with $m=-\psi'(0+)<\infty$, $e^{-mt}Z_t$ is a positive martingale. Hence $\lim_{n\to\infty}e^{-mt}Z_t$ exist a.s., denoted as $W$.

We only consider the supercritical case, i.e., $m>0$. The following result was stated in Pinsky [9] without proof. In this subsection we give a proof using the decompsoition developed in Section 2.1.

**Theorem 2.1** *Suppose that $(\mathcal{Z}_t, t\geq 0)$ is a supercritical CBI$(\psi,\varphi)$. Then as $t\to\infty$, $e^{-mt}\mathcal{Z}_t$ has a finite almost sure limit if and only if*

$$\int_1^\infty(\log x)n(\mathrm{d}x)<\infty. \tag{28}$$

**Proof:** (1) We first prove that if $\int_1^\infty(\log x)n(\mathrm{d}x)<\infty$, then $e^{-mt}\mathcal{Z}_t$ has a finite almost sure limit as $t\to\infty$.

Suppose $\mathcal{Z}=Z+\mathcal{Z}^{(1)}+\mathcal{Z}^{(2)}$ is a CBI$(\psi,\varphi)$ under $\mathbf{P}_x$, constructed in Section 2.1. Put

$$\mathcal{W}_t=e^{-mt}\mathcal{Z}_t=e^{-mt}Z_t+e^{-mt}\mathcal{Z}_t^{(1)}+e^{-mt}\mathcal{Z}_t^{(2)}. \tag{29}$$

By the martingale convergence theorem and Fatou's lemma, we have

$$W_t:=e^{-mt}Z_t\to W<\infty \qquad \mathbf{P}_x\text{-a.s.} \tag{30}$$

We write $\mathcal{W}_t$ as

$$\mathcal{W}_t = e^{-mt}\mathcal{Z}_t=W_t+e^{-mt}\left(\mathcal{Z}_t^{(1)}+\mathcal{Z}_t^{(2)}\right). \tag{31}$$

We need to prove that

$$\mathcal{W}_t\to\mathcal{W} \qquad \mathbf{P}_x\text{-a.s.} \tag{32}$$

for some finite random variable $\mathcal{W}$. Suppose $\mathcal{G}$ is the $\sigma-$field generated by $(S_s, s \geq 0)$, and $\mathcal{F}_t = \sigma(\mathcal{Z}_s, s \leq t)$. Then by Lemma 3.3 of [6], we only need to prove that $\mathcal{Z}_t$ is a $\mathbf{P}_x(\cdot|\mathcal{G})$ submartingale with respect to $(\mathcal{F}_t, t \geq 0)$, and

$$\sup_{t\geq 0} \mathbf{P}_x [\mathcal{W}_t|\mathcal{G}] < \infty. \tag{33}$$

For details one may refer to [6]. First observe that

$$\begin{aligned} \mathbf{P}_x [\mathcal{W}_t|\mathcal{F}_s \vee \mathcal{G}] &= e^{-mt}\mathbf{P}_{\mathcal{Z}_s} \left[ Z_{t-s} + \mathcal{Z}^{(1)}_{t-s} + \mathcal{Z}^{(2)}_{t-s}|\mathcal{G} \right] \\ &= e^{-ms}\mathcal{Z}_s + e^{-mt}\mathbf{P}_{\mathcal{Z}_s} \left[ \mathcal{Z}^{(1)}_{t-s} + \mathcal{Z}^{(2)}_{t-s}|\mathcal{G} \right] \geq \mathcal{W}_s. \end{aligned} \tag{34}$$

We claim that $\mathbf{P}_x[\mathcal{W}_t|\mathcal{G}] < \infty$ for any $t \geq 0$, which will be clear by (35) and (36) below, thus $\mathcal{Z}_t$ is a $P(\cdot|\mathcal{G})$ submartingale with respect to $(\mathcal{F}_t, t \geq 0)$. Given $\mathcal{G}$, $\mathcal{Z}^{(1)}$ is the sum of a sequence of independent $CB(\psi, 0)$. $Z$ and $\mathcal{Z}^{(2)}$ are independent of $\mathcal{G}$. Together with (22), we have

$$\mathbf{P}_x [\mathcal{W}_t|\mathcal{G}] = x + \int_0^t \int_0^\infty y e^{-ms}\mathbf{J}(\mathrm{d}s\mathrm{d}y) + \mathbf{P}_x \left[ e^{-mt}\mathcal{Z}^{(2)}_t \right]. \tag{35}$$

For the continuum immigration part, we have

$$\begin{aligned} \mathbf{P}_x \left[ e^{-mt}\mathcal{Z}^{(2)}_t \right] &= e^{-mt}\mathbf{P}_x \left[ \int_0^t \int_\Omega X^{\mathbf{n},s}_{t-s}\mathbf{n}(\mathrm{d}s\mathrm{d}\mathbb{N}_1) \right] \\ &= be^{-mt} \int_0^t \int_\Omega X_{t-s}\mathrm{d}\mathbb{N}_1\mathrm{d}s = be^{-mt} \int_0^t \mathbb{N}_1 X_{t-s}\mathrm{d}s \\ &= be^{-mt} \int_0^t e^{m(t-s)}\mathrm{d}s = b(1-e^{-mt})/m. \end{aligned} \tag{36}$$

Here we used the fact that $\mathbb{N}_1 X_s = \mathbb{P}_1 X_s$, which can be induced from Theorem 1.1 easily. Thus we have

$$\sup_{t\geq 0} \mathbf{P}_x \left[ e^{-mt}\mathcal{Z}^{(2)}_t \right] < \infty.$$

So we left to prove that, under condition (28),

$$\sup_{t\geq 0} \int_0^t \int_0^\infty y e^{-ms}\mathbf{J}(\mathrm{d}s\mathrm{d}y) < \infty, \qquad \mathbf{P}_x\text{-a.s.},$$

that is

$$\int_0^\infty \int_0^\infty y e^{-ms}\mathbf{J}(\mathrm{d}s\mathrm{d}y) < \infty, \qquad \mathbf{P}_x\text{-a.s.} \tag{37}$$

Recall that $T_1$ is the set of all jumping times of $S$, which is at most countable. The integral above can be written as

$$\sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma. \tag{38}$$

We divide the sum into two parts as follows:

$$\sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma = \sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma I_{\{e^{-\delta\sigma}\Delta S_\sigma\le 1\}} + \sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma I_{\{e^{-\delta\sigma}\Delta S_\sigma> 1\}},$$

where $0<\delta<m$ is a constant. Now we first estimate the second part:

$$\begin{aligned}\mathbf{P}_x\left[\sum_{\sigma\in T_1} I_{\{e^{-\delta\sigma}\Delta S_\sigma>1\}}\right] &= \mathbf{P}_x\left[\int_0^\infty\int_{\{e^{-\delta s}y>1\}}\mathbf{J}(\mathrm{d}y\mathrm{d}s)\right]\\ = \int_0^\infty\int_{\{e^{-\delta s}y>1\}} n(\mathrm{d}y)\mathrm{d}s &= \frac{1}{\delta}\int_1^\infty(\log y)n(\mathrm{d}y)<\infty. \qquad (39)\end{aligned}$$

By Borel-Cantelli Lemma, we get

$$\mathbf{P}_x\left(e^{-\delta\sigma}\Delta S_\sigma>1 \ \text{ i. o. }\right)=0,$$

and then

$$\sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma I_{\{e^{-\delta\sigma}\Delta S_\sigma>1\}}<\infty,\quad \mathbf{P}_x\text{-a.s.} \qquad (40)$$

On the other hand, for the first part, we have

$$\begin{aligned}\mathbf{P}_x\left[\sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma I_{\{e^{-\delta\sigma}\Delta S_\sigma\le 1\}}\right] &= \int_0^\infty\int_{\{e^{-\delta s}y\le 1\}} ye^{-ms}n(\mathrm{d}y)\mathrm{d}s\\ = \int_0^\infty\int_0^1 ye^{-ms}n(\mathrm{d}y)\mathrm{d}s &+ \int_0^\infty\int_1^{e^{\delta s}} ye^{-ms}n(\mathrm{d}y)\mathrm{d}s\\ = \frac{1}{m}\int_0^1 yn(\mathrm{d}y) &+ \int_0^\infty\int_1^{e^{\delta s}} ye^{-ms}n(\mathrm{d}y)\mathrm{d}s. \qquad (41)\end{aligned}$$

Since $y\le e^{\delta s}$ in the second integral, we have

$$\begin{aligned}(41) &\le \frac{1}{m}\int_0^1 yn(\mathrm{d}y)+\int_0^\infty\int_1^{e^{\delta s}} e^{-(m-\delta)s}n(\mathrm{d}y)\mathrm{d}s\\ &\le \frac{1}{m}\int_0^1 yn(\mathrm{d}y)+\int_0^\infty\int_1^{\infty} e^{-(m-\delta)s}n(\mathrm{d}y)\mathrm{d}s\\ &= \frac{1}{m}\int_0^1 yn(\mathrm{d}y)+\frac{1}{m-\delta}\int_1^\infty n(\mathrm{d}y)<\infty.\end{aligned}$$

The last inequality is due to (13). Thus we have

$$\sum_{\sigma\in T_1} e^{-m\sigma}\Delta S_\sigma I_{\{e^{-\delta\sigma\sigma}\Delta S_\sigma\le 1\}}<\infty,\quad \mathbf{P}_x\text{-a.s.} \qquad (42)$$

Combining (40) and (42), we obtain that (37) holds. Therefore we have proved (32).

(2) Next we prove that if

$$\int_1^\infty (\log x) n(\mathrm{d}x) = \infty,$$

then $\lim_{t\longrightarrow\infty} e^{-ct}\mathcal{Z}_t = \infty$.

For any constant $K > 1,\ c > 0$,

$$\begin{aligned}
\mathbf{P}_x\left[\sum_{\sigma\in T_1} I_{\{e^{-c\sigma}\Delta S_\sigma > K\}}\right] &= \mathbf{P}_x\left[\int_0^\infty \int_{\{e^{-cs}y>K\}} \mathbf{J}(\mathrm{d}s\mathrm{d}y)\right] \\
&= \int_0^\infty \int_{\{e^{-cs}y>K\}} n(\mathrm{d}y)\mathrm{d}s = \frac{1}{c}\int_K^\infty (\log y - \log K) n(\mathrm{d}y) \\
&= \infty. \qquad (43)
\end{aligned}$$

We thus have

$$\mathbf{P}_x\left(e^{-c\sigma}\Delta S_\sigma > K \ \text{ i. o. }\right) = 1,$$

which implies that

$$\limsup_{\sigma\longrightarrow\infty} e^{-c\sigma}\Delta S_\sigma > K.$$

Since K is arbitrary, we obtain that

$$\limsup_{\sigma\longrightarrow\infty} e^{-c\sigma}\Delta S_\sigma = \infty.$$

Therefore

$$\limsup_{t\longrightarrow\infty} e^{-ct}\mathcal{Z}_t \geq \limsup_{\sigma\longrightarrow\infty} e^{-c\sigma}\mathcal{Z}_\sigma^{(1)} \geq \limsup_{\sigma\longrightarrow\infty} e^{-c\sigma}\Delta S_\sigma = \infty, \quad \mathbf{P}_x\text{-a.s.}$$

## 2.3 $L\log L$ criterion for non-degeneracy of martingale limit of $CB(\psi)$

Suppose $(Z_t, t \geq 0)$ is a $CB(\psi)$ and $W_t = e^{-mt}Z_t$, which is a non-negative martingale. Using the results of Subsection 2.2, we will prove that $\int_1^\infty x\log x \Pi(\mathrm{d}x) < \infty$ is a sufficient and necessary condition for the martingale $W_t$ to have a non-degenerate limit. This result was given in [2]. We restate it as Theorem 2.2.

**Theorem 2.2** *Suppose $(Z_t, \mathbb{P}_x : t \geq 0)$ is a $CB(\psi)$ starting from $x > 0$.*

$$W_t \longrightarrow W \quad \mathbb{P}_x\text{-a.s. } \&\ L^1(\text{ as } t \to \infty) \Longleftrightarrow \int_1^\infty (x\log x)\Pi(\mathrm{d}x) < \infty. \qquad (44)$$

Before we prove Theorem 2.2, we first recall the following result (see [1] Theorem 3.4.3 for reference).

**Lemma 2.3** *Suppose $(\Omega, \mathcal{F})$ is a measurable space and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on it. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$, and $(W_t)_{t\geq 0}$ be a non-negative $\mathbb{P}$-martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then $\lim_{t\to\infty} W_t$ exists and is finite $\mathbb{P}$-a.s. Put $W := \lim_{t\to\infty} W_t$. If*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = W_t, \quad \forall t \geq 0,$$

*then the following holds:*

$$W = 0, \quad \mathbb{P}\text{-}a.s. \iff \mathbb{P} \perp \mathbb{Q} \iff W = \infty, \quad \mathbb{Q}\text{-}a.s.,$$
$$\int_\Omega W d\mathbb{P} = 1 \iff \mathbb{Q} \ll \mathbb{P} \iff W < \infty, \quad \mathbb{Q}\text{-}a.s.$$

**Proof of Theorem 2.2** For fixed $x$, define a new measure $\mathbb{Q}_x$ by the following martingale transform:

$$\left.\frac{d\mathbb{Q}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{1}{x} e^{-mt} Z_t. \tag{45}$$

Then the Laplace transform under $\mathbb{Q}_x$ is given by

$$\begin{aligned} \mathbb{Q}_x e^{-\lambda Z_t} &= \mathbb{P}_x \left[ \frac{1}{x} e^{-mt} Z_t e^{-\lambda Z_t} \right] \\ &= -\frac{\partial}{\partial \lambda} \mathbb{P}_x \left[ e^{-\lambda Z_t} \right] \frac{1}{x} e^{-mt} \\ &= \frac{\partial}{\partial \lambda} \left( u_t(\lambda) \right) \exp(-x u_t(\lambda)) e^{-mt}, \end{aligned} \tag{46}$$

where $u_t(\lambda)$ is the unique nonnegative solution of (15), which is equivalent to the following integral equation:

$$u_t(\lambda) = \lambda - \int_0^t \psi(u_s(\lambda)) ds.$$

Then we get

$$\frac{\partial}{\partial \lambda} \left( u_t(\lambda) \right) = \exp\left\{ -\int_0^t \psi'(u_s(\lambda)) ds \right\}. \tag{47}$$

Combining (46) and (47), we obtain

$$\mathbb{Q}_x e^{-\lambda Z_t} = \exp\left\{ -x u_t(\lambda) - \int_0^t \varphi(u_s(\lambda)) ds \right\}, \tag{48}$$

where $\varphi(\lambda) = \psi'(\lambda) + m = 2\alpha\lambda + \int_0^\infty x(1 - e^{-\lambda x})\Pi(dx)$, which is the Laplace exponent of a subordinator. The corresponding Lévy measure is $n(dx) = x\Pi(dx)$. Therefore under $\mathbb{Q}_x$, $(Z_t, t \geq 0)$ is a CBI$(\psi, \varphi)$. Set

$$W = \limsup_{t \to \infty} e^{-mt} Z_t.$$

Note that $\int_1^\infty (x \log x)\Pi(dx) < \infty$ is equivalent to $\int_1^\infty (\log x) n(dx) < \infty$. By Theorem 2.1, if $\int_1^\infty (x \log x)\Pi(dx) < \infty$, then $W < \infty$ $Q_x$-a.s., and thus by Lemma 2.3, we have $\mathbb{E}_x W = x$ and $W_t \to W$ in $L^1$. While if $\int_1^\infty (x \log x)\Pi(dx) = \infty$, by Theorem 2.1, we have $W = \infty$, $\mathbb{Q}_x$-a.s. and thus by Lemma 2.3 we have $W = 0$, $\mathbb{P}_x$-a.s.

Combining Theorem 2.1 and Theorem 2.2, we have

**Theorem 2.4** *Suppose $\mathcal{Z} = (\mathcal{Z}_t, t \geq 0)$ is a CBI$(\psi, \phi)$ and $m > 0$. Assume that $\int_1^\infty (x \log x)\Pi(dx) < \infty$. Then $\mathcal{W}_t = e^{-mt}\mathcal{Z}_t$ has a non-degenerate finite limit if and only if*

$$\int_1^\infty (\log x) n(dx) < \infty. \tag{49}$$

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