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Limit theorem for derivative martingale at criticality w.r.t branching Brownian motion[☆]

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ABSTRACT

We consider a branching Brownian motion on $\mathbb{R}$ in which one particle splits into $1+X$ children. There exists a critical value $\underline{\lambda}$ in the sense that $\underline{\lambda}$ is the lowest velocity such that a traveling wave solution to the corresponding Kolmogorov–Petrovskii–Piskunov equation exists. It is also known that the traveling wave solution with velocity $\underline{\lambda}$ is closely connected with the rescaled Laplace transform of the limit of the so-called derivative martingale $\partial W_t(\underline{\lambda})$. Thus special interest is put on the property of its limit $\partial W(\underline{\lambda})$. Kyprianou [Kyprianou, A.E., 2004. Traveling wave solutions to the K–P–P equation: alternatives to Simon Harris' probability analysis. Ann. Inst. H. Poincaré 40, 53–72.] proved that, $\partial W(\underline{\lambda}) > 0$ if $EX(\log^+ X)^{2+\delta} < +\infty$ for some $\delta > 0$ while $\partial W(\underline{\lambda}) = 0$ if $EX(\log^+ X)^{2-\delta} = +\infty$. It is conjectured that $\partial W(\underline{\lambda})$ is non-degenerate if and only if $EX(\log^+ X)^2 < +\infty$. The purpose of this article is to prove this conjecture.

1. Introduction

We consider a branching Brownian motion: an initial particle commences a standard Brownian motion on $\mathbb{R}$. After a lifetime η which is exponentially distributed with parameter $\beta > 0$, it splits into $1+X$ particles where X is distributed according to $\{p_k: k \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}\}$ and $m := \sum_k kp_k < +\infty$. These children, starting from their point of creation, will move and reproduce in the same way as their ancestor. P is implicitly understood as the law of such a process.

To avoid ambiguity, we use the same notation as used in Kyprianou (2004). We give a short description here. The sample path of the branching Brownian motion is a marked Galton–Watson tree τ. Let $\mathcal{T}$ denote the space of all marked G–W trees. All particles in τ are labeled according to the Ulam–Harris convention, for example, Ø231 or 231 is the first child of the third child of the second child of the initial ancestor Ø. Besides, each particle $u \in \tau$ has a mark (Ξ_u, σ_u, X_u) where $\Xi_u : [b_u, v_u) \to \mathbb{R}$ is the spatial location of u during its lifetime $[b_u, v_u)$ (b_u is its birth time, and v_u its death time), σ_u is the length of its life, and $1 + X_u$ is the number of its offspring. We use $u < v$ to mean that v is an ancestor of u. Let N_t denote the set of particles alive at time t and the sigma algebra $\mathcal{F}_t$ includes all the information of particles born before time t.

Since the process survives with probability 1, for each tree τ, we can choose a distinguished genealogical line of descent from the initial ancestor. Such a line is called a *spine* and denoted as $\xi = \{\xi_0 := \emptyset, \xi_1, \xi_2, \ldots\}$, where ξ_i is the label of the ith spine node. We shall use $\Xi := \{\Xi(t): t \geq 0\}$ and $n := \{n_t: t \geq 0\}$ respectively to denote the spatial path and the counting process of fission times along the spine. $\widetilde{\mathcal{G}}$ is the sigma field generated by ξ, Ξ, n and $\{X_{\xi_i}: i \geq 0\}$. $\widetilde{\mathcal{T}}$ denotes the space of Galton–Watson trees with a distinguished spine. Suppose $(P^*, \widetilde{\mathcal{T}})$ is an extension of $(P, \mathcal{T})$ under which the nth spine node

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is uniformly chosen from the children of the $(n-1)$th spine node. We refer back to Kyprianou (2004) for the definition of P^* and are not going to give details here.

For any $\lambda > 0$, let $c_\lambda := \frac{\lambda}{2} + \frac{\beta m}{\lambda}$. c_λ reaches its minimum $\underline{c} := \sqrt{2\beta m}$ when $\lambda = \underline{\lambda} := \sqrt{2\beta m}$. It is known that

$$\partial W_t(\lambda) := \sum_{u \in N_t} (\Xi_u(t) + \lambda t) \mathrm{e}^{-\lambda(\Xi_u(t) + c_\lambda t)} \tag{1}$$

is a P-martingale with respect to $\mathcal{F}_t$ which is also referred to as the *derivative martingale*.

Let $\widetilde{N}_t$ denote the set of particles in N_t whose ancestors and themselves never meet the space–time barrier $y + \underline{\lambda} t = -x$. Define

$$V_t^x(\lambda) = \sum_{u \in \widetilde{N}_t} \frac{x + \Xi_u(t) + \lambda t}{x} \mathrm{e}^{-\lambda(\Xi_u(t) + c_\lambda t)}. \tag{2}$$

Kyprianou (2004) proved that $V_t^x(\lambda)$ is a mean 1 P-martingale, and that when $\lambda \ge \underline{\lambda}$, $\partial W(\lambda) := \lim_{t \to +\infty} \partial W_t(\lambda)$ exists and equals $\lim_{t \to +\infty} x V_t^x(\lambda)$.

The importance of the limit $\partial W(\lambda)$ lies in that when $\partial W(\lambda)$ is non-degenerate, its rescaled Laplace transform provides a traveling wave solution to the K–P–P equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \beta(f(u) - u), \tag{3}$$

where $f(s) = E(s^{1+X})$. By traveling wave we mean the solutions of the form $u(t,x) = w(x - ct)$ where w is a monotone function connecting 0 at $-\infty$ to 1 at $+\infty$ and c is called the speed of the wave. Kyprianou (2004) proved the existence and uniqueness of the traveling wave at the criticality case (that is, when $c = \underline{c}$) under the assumption that $EX(\log^+ X)^{2+\delta} < +\infty$ for some $\delta > 0$. Our Corollary 1 shows that the same result holds under a looser condition $EX(\log^+ X)^2 < +\infty$.

2. Main result and proof

Theorem 1. *If $\lambda = \underline{\lambda}$, $\partial W(\underline{\lambda}) > 0$ P-almost surely when $EX(\log^+ X)^2 < +\infty$ and $\partial W(\underline{\lambda}) = 0$ P-almost surely when $EX(\log^+ X)^2 = +\infty$.*

Remark 1. Our Theorem 1 fills the 'gap' that appears in the necessary and sufficient conditions for $\partial W(\underline{\lambda})$ to be non-degenerate in the case of branching Brownian motion studied in Kyprianou (2004). Kyprianou et al. (2010) considered traveling waves for the corresponding nonlinear differential equation related to super-Brownian motion and prove the above result for super-Brownian motion.

Proof of Theorem 1. We use the method of measure change developed in (Kyprianou (2004) Section 6). Let Π_t^* be a probability on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}}_t)$ such that

$$\left.\frac{\mathrm{d}\Pi_t^*}{\mathrm{d}P_t^*}\right|_{\mathcal{F}_t} = V_t^x(\underline{\lambda}),$$

where $P_t^* := P^*|_{\mathcal{F}_t}$. It is known (see, for example, P. 66 in Kyprianou, 2004) that under the new probability measure Π^*, the diffusion along the spine is such that $\{x + \Xi(t) + \underline{\lambda} t : t \ge 0\}$ is a Bessel-3 process on $(0, +\infty)$ started at x (that is to say the diffusion along the spine Ξ moves away from the line $\{(t,y) : y + \underline{\lambda} t = -x,\ t \ge 0\}$ as a Bessel-3 process and never meets it); the points of fission along the spine form a Poisson process with rate $\beta(m+1)$; and the offspring number at each point of fission on the spine has the size-biased distribution $\left\{\tilde{p}_k := \frac{k+1}{m+1} p_k : k \in \mathbb{Z}_+\right\}$. Revisiting the proof of Theorem 3 in Kyprianou (2004), it is straightforward to see from the spine methodology presented there that the proof of Theorem 1 is complete as soon as we can show:

(i) when $EX(\log^+ X)^2 = +\infty$,

$$\limsup_{n \to +\infty} X_{\xi_n} (x + \Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n}) \mathrm{e}^{-\underline{\lambda}(\Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n})} = +\infty \quad \Pi^*\text{-a.s.}; \tag{4}$$

(ii) when $EX(\log^+ X)^2 < +\infty$,

$$\sum_{n=0}^{+\infty} X_{\xi_n} (x + \Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n}) \mathrm{e}^{-\underline{\lambda}(\Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n})} < +\infty \quad \Pi^*\text{-a.s.} \tag{5}$$

We first prove (i). It suffices to show that for any $M > 0$

$$\sum_{n=0}^{+\infty} 1_{\{X_{\xi_n}(x + \Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n}) \mathrm{e}^{-\underline{\lambda}(\Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n})} \ge M\}} = +\infty \quad \Pi^*\text{-a.s.} \tag{6}$$

Let $\mathcal{G}$ denote the sigma field generated by Ξ (the spatial path of the spine). For any set $B \in \mathscr{B}[0,+\infty) \times \mathscr{B}(\mathbb{Z}_+)$, define

$$\varphi(B) := \#\{n: (v_{\xi_n}, X_{\xi_n}) \in B\}, \tag{7}$$

then conditioned on $\mathcal{G}$, φ is a Poisson random measure on $[0,+\infty) \times \mathbb{Z}_+$ with intensity $\beta(m+1)\mathrm{d}t \sum_{k\in\mathbb{Z}_+} \tilde{p}_k \delta_k(\mathrm{d}y)$ (here δ denotes the delta function). Thus for any $T \in (0,+\infty)$, when $\mathcal{G}$ is given,

$$N_T := \#\{n: v_{\xi_n} \le T,\ X_{\xi_n}(x + \Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n})\mathrm{e}^{-\underline{\lambda}(\Xi(v_{\xi_n}) + \underline{\lambda} v_{\xi_n})} \ge M\}$$

is a Poisson random variable with parameter

$$\int_0^T \beta(m+1) \sum_k \tilde{p}_k 1_{\{k(x+\Xi(t)+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(\Xi(t)+\underline{\lambda}t)} \ge M\}} \mathrm{d}t.$$

Hence to prove (6), we only need to show that

$$\int_0^{+\infty} \beta(m+1) \sum_k \tilde{p}_k 1_{\{k(x+\Xi(t)+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(\Xi(t)+\underline{\lambda}t)} \ge M\}} \mathrm{d}t = +\infty \quad \Pi^*\text{-a.s.}$$

For any $c \in (0,+\infty)$, let

$$A_c := \left\{ \int_0^{+\infty} \beta(m+1) \sum_k \tilde{p}_k 1_{\{k(x+\Xi(t)+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(\Xi(t)+\underline{\lambda}t)} \ge M\}} \mathrm{d}t \le c \right\}.$$

We only need to prove that

$$\Pi^*(A_c) = 0, \quad \forall c > 0. \tag{8}$$

Note that under Π^*, $x + \Xi(t) + \underline{\lambda}t$ is a Bessel-3 process starting from x which is identically distributed to the modulus process of $W_t + \hat{x}$ where $(W_t, \mathbb{P})$ is a three-dimensional standard Brownian motion and $\hat{x}$ is a point in $\mathbb{R}^3$ with norm x. We still use A_c to denote the same set corresponding to $(W_t, \mathbb{P})$.

$$\begin{aligned} c &\ge \Pi^*\left(1_{A_c} \int_0^{+\infty} \beta(m+1) \sum_{k\in\mathbb{Z}_+} \tilde{p}_k 1_{\{k(x+\Xi(t)+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(\Xi(t)+\underline{\lambda}t)} \ge M\}} \mathrm{d}t \right) \\ &= \int_0^{+\infty} \beta(m+1) \sum_{k\in\mathbb{Z}_+} \tilde{p}_k \Pi^*\left(1_{A_c} 1_{\{(x+\Xi(t)+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(x+\Xi(t)+\underline{\lambda}t)} \ge Mk^{-1}\mathrm{e}^{-\underline{\lambda}x}\}} \right) \mathrm{d}t \\ &= \beta(m+1) \sum_{k\in\mathbb{Z}_+} \tilde{p}_k \int_0^{+\infty} \mathbb{P}(1_{A_c} 1_{\{|W_t+\hat{x}|\mathrm{e}^{-\underline{\lambda}|W(t)+\hat{x}|} \ge Mk^{-1}\mathrm{e}^{-\underline{\lambda}x}\}}) \mathrm{d}t. \end{aligned} \tag{9}$$

We claim that there exists $K_1 > 1$ such that when $k \ge K_1$

$$\left\{ y \in \mathbb{R}^3 : 1 + x \le |y| \le \frac{\log k}{2\underline{\lambda}} \right\} \subset \left\{ y \in \mathbb{R}^3 : |y+\hat{x}|\mathrm{e}^{-\underline{\lambda}|y+\hat{x}|} \ge Mk^{-1}\mathrm{e}^{-\underline{\lambda}x} \right\}. \tag{10}$$

In fact, $1 + x \le |y| \le \frac{\log k}{2\underline{\lambda}}$ implies $1 \le |y + \hat{x}| \le \frac{\log k}{2\underline{\lambda}} + x$. Consider the function $f(x) = x\mathrm{e}^{-\underline{\lambda}x}$. On the positive half line, it increases to a supremum and then decreases to 0 as x goes to infinity. Thus we can find $K_1 > 1$ large enough such that when $k \ge K_1$,

$$\begin{aligned} 1 + x \le |y| \le \frac{\log k}{2\underline{\lambda}} &\Rightarrow f(|y+\hat{x}|) \ge f\left(\frac{\log k}{2\underline{\lambda}} + x \right) \\ &\Rightarrow |y+\hat{x}|\mathrm{e}^{-\underline{\lambda}|y+\hat{x}|} \ge \left(\frac{\log k}{2\underline{\lambda}} + x \right) k^{-1/2}\mathrm{e}^{-\underline{\lambda}x}. \end{aligned}$$

Then we get (10).

We continue the estimation of (9): when $k \ge K_1$,

$$\begin{aligned} c &\ge \beta(m+1) \sum_{k:k\ge K_1} \tilde{p}_k \int_0^{+\infty} \mathbb{P}\left(1_{A_c} 1_{\{1+x\le|W_t|\le \frac{\log k}{2\underline{\lambda}}\}} \right) \mathrm{d}t \\ &= \beta(m+1) \sum_{k:k\ge K_1} \tilde{p}_k \mathbb{P}\left(1_{A_c} \int_0^{+\infty} 1_{\{1+x\le|W_t|\le \frac{\log k}{2\underline{\lambda}}\}} \mathrm{d}t \right). \end{aligned} \tag{11}$$

$(|W_t|, \mathbb{P})$ is a Bessel-3 process starting from 0. Let $\{l^a : a \geq 0\}$ be the family of its local times, then the process $\{l^a_\infty, a \geq 0\}$ is a BESQ$^2(0)$ process which implies $l^a_\infty \overset{d}{=} a l^1_\infty$ and $\mathbb{P}(l^1_\infty = 0) = 0$ (see Revuz and Yor, 1991, P. 425, Exercise 2.5).

$$\begin{aligned}\mathbb{P}\left(1_{A_c}\int_0^{+\infty} 1_{\{1+x\leq |W_t|\leq \frac{\log k}{2\underline{\lambda}}\}}\mathrm{d}t\right) &= \mathbb{P}\left(1_{A_c}\int_{1+x}^{\frac{\log k}{2\underline{\lambda}}} l^a_\infty \mathrm{d}a\right)\\ &= \mathbb{P}\left(1_{A_c}\int_{1+x}^{\frac{\log k}{2\underline{\lambda}}} a\,\mathrm{d}a\int_0^{a^{-1}l^a_\infty}\mathrm{d}u\right)\\ &= \int_{1+x}^{\frac{\log k}{2\underline{\lambda}}} a\mathrm{d}a\int_0^{+\infty}\mathbb{P}(1_{A_c}1_{\{u\leq a^{-1}l^a_\infty\}})\mathrm{d}u. \end{aligned}\tag{12}$$

Note that

$$\mathbb{P}(1_{A_c}1_{\{u\leq a^{-1}l^a_\infty\}}) \leq \left(\mathbb{P}(A_c) - \mathbb{P}(a^{-1}l^a_\infty < u)\right)^+ = \left(\mathbb{P}(A_c) - \mathbb{P}(l^1_\infty < u)\right)^+,$$

and there exists a constant $C > 0$ and $K_2 > 1$ such that for large $k \geq K_2$

$$\int_{1+x}^{\frac{\log k}{2\underline{\lambda}}} a\mathrm{d}a = \frac{1}{2}\left(\left(\frac{\log k}{2\underline{\lambda}}\right)^2 - (1+x)^2\right) \geq C(\log k)^2.$$

Then (12) implies

$$\mathbb{P}\left(1_{A_c}\int_0^{+\infty} 1_{\{1+x\leq |W_t|\leq \frac{\log k}{2\underline{\lambda}}\}}\mathrm{d}t\right) \geq C(\log k)^2\int_0^\infty \left(\mathbb{P}(A_c) - \mathbb{P}(l^1_\infty < u)\right)^+ \mathrm{d}u. \tag{13}$$

Set $K = K_1 \vee K_2$. Using (11) and (13) we get

$$\sum_{k:k\geq K}\tilde{p}_k(\log k)^2\int_0^{+\infty}\left(\mathbb{P}(A_c) - \mathbb{P}(l^1_\infty < u)\right)^+ \mathrm{d}u < +\infty. \tag{14}$$

The assumption that $EX(\log^+ X)^2 = +\infty$ is equivalent to $\sum_{k\in\mathbb{Z}_+}\tilde{p}_k(\log^+ k)^2 = +\infty$. Then by (14),

$$\int_0^{+\infty}\left(\mathbb{P}(A_c) - \mathbb{P}(l^1_\infty < u)\right)^+ \mathrm{d}u = 0.$$

Thus $\mathbb{P}(A_c) = 0$ by the property that $\mathbb{P}(l^1_\infty = 0) = 0$. Thus we proved (8), and then the second part of Theorem 1.

Next we prove (ii). Choose any $h \in (0, \underline{\lambda})$,

$$\begin{aligned}\sum_{n=0}^{+\infty}(x + \Xi(v_{\xi_n}) + \underline{\lambda}v_{\xi_n})X_{\xi_n}\mathrm{e}^{-\underline{\lambda}(\Xi(v_{\xi_n})+\underline{\lambda}v_{\xi_n})} &= \sum_{n=0}^{+\infty}(\cdots)1_{\{X_{\xi_n}\leq \mathrm{e}^{h(\Xi(v_{\xi_n})+\underline{\lambda}v_{\xi_n})}\}} + \sum_{n=0}^{+\infty}(\cdots)1_{\{X_{\xi_n}> \mathrm{e}^{h(\Xi(v_{\xi_n})+\underline{\lambda}v_{\xi_n})}\}}\\ &=: \mathrm{I} + \mathrm{II}. \end{aligned}\tag{15}$$

We will prove that both I and II are finite almost surely under Π^*.

Recall that φ is defined by (7). We can rewrite I as

$$\mathrm{I} = \int_{[0,+\infty)\times\mathbb{Z}_+}(x + \Xi(s) + \underline{\lambda}s)y\mathrm{e}^{-\underline{\lambda}(\Xi(s)+\underline{\lambda}s)}1_{\{y\leq \mathrm{e}^{h(\Xi(s)+\underline{\lambda}s)}\}}\varphi(\mathrm{d}s\times \mathrm{d}y).$$

Since $\Pi^*(\mathrm{I}) = \Pi^*(\Pi^*(\mathrm{I}|\mathcal{G}))$, by compensation formula of Poisson random measure (see, for example, Theorem 4.4 in Kyprianou, 2006), we have

$$\begin{aligned}\Pi^*(\mathrm{I}) &= \Pi^*\left(\int_0^{+\infty}\beta(m+1)(x + \Xi(s) + \underline{\lambda}s)\sum_k \tilde{p}_k k\mathrm{e}^{-\underline{\lambda}(\Xi(s)+\underline{\lambda}s)}1_{\{k\leq \mathrm{e}^{h(\Xi(s)+\underline{\lambda}s)}\}}\mathrm{d}s\right)\\ &\leq \beta(m+1)\sum_k\tilde{p}_k\int_0^{+\infty}\Pi^*\left((x + \Xi(s) + \underline{\lambda}s)\mathrm{e}^{-(\underline{\lambda}-h)(\Xi(s)+\underline{\lambda}s)}1_{\{\Xi(s)+\underline{\lambda}s\geq h^{-1}\log^+ k\}}\right)\mathrm{d}s. \end{aligned}\tag{16}$$

Hereafter, we will write "$A \lesssim B$" when there exists a constant $c > 0$, which may only depend on x, such that $A \leq cB$. Note that $x + \Xi(t) + \underline{\lambda}t$ under Π^* is a Bessel-3 process, which has the same distribution as $|W_t + \hat{x}|$ under $\mathbb{P}$ where $(W_t, \mathbb{P})$ is a three-dimensional standard Brownian motion starting from 0. Using the distribution of W_t, we can continue the above estimation to get

$$\begin{aligned}
\Pi^*(\mathrm{I}) &\lesssim \sum_k \tilde{p}_k \int_0^{+\infty} \mathbb{P}\left(|W_s+\hat{x}|\mathrm{e}^{-(\underline{\lambda}-h)|W_s+\hat{x}|} 1_{\{|W_s+\hat{x}|\ge h^{-1}\log^+ k+x\}}\right)\mathrm{d}s \\
&\lesssim \sum_k \tilde{p}_k \int_{\{|y+\hat{x}|\ge h^{-1}\log^+ k+x\}} |y+\hat{x}|\mathrm{e}^{-(\underline{\lambda}-h)|y+\hat{x}|}\mathrm{d}y \int_0^{+\infty} s^{-3/2}\mathrm{e}^{-|y|^2/2\pi s}\mathrm{d}s \\
&= \sum_k \tilde{p}_k \int_{\{|y+\hat{x}|\ge h^{-1}\log^+ k+x\}} \frac{|y+\hat{x}|}{|y|}\mathrm{e}^{-(\underline{\lambda}-h)|y+\hat{x}|}\mathrm{d}y \int_0^{+\infty} t^{-1/2}\mathrm{e}^{-t/2\pi}\mathrm{d}t \\
&\lesssim \sum_k \tilde{p}_k \int_{\{|y+\hat{x}|\ge h^{-1}\log^+ k+x\}} \frac{|y+\hat{x}|}{|y|}\mathrm{e}^{-(\underline{\lambda}-h)|y+\hat{x}|}\mathrm{d}y \\
&\le \sum_k \tilde{p}_k \int_{\{|y|\ge h^{-1}\log^+ k\}} \frac{|y|+x}{|y|}\mathrm{e}^{-(\underline{\lambda}-h)(|y|-x)}\mathrm{d}y.
\end{aligned}$$

By changing the above triple integration to integration under polar coordinates, we get

$$\begin{aligned}
\Pi^*(I) &\lesssim \sum_k \tilde{p}_k \int_{h^{-1}\log^+ k}^{+\infty} (r^2+xr)\mathrm{e}^{-(\underline{\lambda}-h)r}\mathrm{d}r \\
&< +\infty,
\end{aligned}$$

and therefore,

$$\Pi^*(I<+\infty)=1. \tag{17}$$

On the other hand, by similar calculation, we have

$$\begin{aligned}
\Pi^*\left(\sum_{n=0}^{+\infty} 1_{\{X_{\xi_n}>\mathrm{e}^{h(\Xi(v_{\xi_n})+\underline{\lambda}v_{\xi_n})}\}}\right) &= \beta(1+m)\sum_k \tilde{p}_k \int_0^{+\infty} \Pi^*\left((x+\Xi(s)+\underline{\lambda}s)<h^{-1}\log^+ k+x\right)\mathrm{d}s \\
&\lesssim \sum_k \tilde{p}_k \int_0^{+\infty} \mathbb{P}\left(|W_s+\hat{x}|<h^{-1}\log^+ k+x\right)\mathrm{d}s \\
&\lesssim \sum_k \tilde{p}_k \int_{\{|y+\hat{x}|<h^{-1}\log^+ k+x\}} \mathrm{d}y \int_0^{+\infty} s^{-3/2}\mathrm{e}^{-|y|^2/2\pi s}\mathrm{d}s \\
&\lesssim \sum_k \tilde{p}_k \int_{\{|y|<h^{-1}\log^+ k+2x\}} |y|^{-1}\mathrm{d}y \\
&\lesssim \sum_k \tilde{p}_k (h^{-1}\log^+ k+2x)^2.
\end{aligned} \tag{18}$$

The assumption that $EX(\log^+ X)^2 < +\infty$ implies that $\sum_{k\in\mathbb{Z}_+}\tilde{p}_k(\log^+ k)^2 < +\infty$, which implies that the sum in (18) is finite. Therefore $\sum_{n=0}^{+\infty} 1_{\{X_{\xi_n}>\mathrm{e}^{h(\Xi(v_{\xi_n})+\underline{\lambda}v_{\xi_n})}\}} < +\infty$, Π^*-a.s., that means, II is a sum of finite terms. Hence

$$\Pi^*(\mathrm{II}<+\infty)=1. \tag{19}$$

Combining (15), (17) and (19), we get (5). Hence we complete the proof. □

Corollary 1. *When $c=\underline{c}$ and $EX(\log^+ X)^2 < +\infty$ then there is a unique traveling wave at speed $\underline{c}$ given by*

$$\Phi_{\underline{c}}(x) = E\left(\exp\{-\mathrm{e}^{-\underline{\lambda}x}\partial W(\underline{\lambda})\}\right).$$

Proof. The proof is similar to that of Kyprianou (2004) on the existence and uniqueness of traveling wave under the condition that $EX(\log^+ X)^{2+\delta} < +\infty$ for some $\delta > 0$. We will not repeat the proof here. □

Remark 2. Obviously, if $\Phi_c(x)$ is a traveling wave then so is $\Phi_c(x+y)$ for every $y\in\mathbb{R}$. Therefore, uniqueness is established up to a spatial shift.

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References

Kyprianou, A.E., 2004. Traveling wave solutions to the K–P–P equation: alternatives to Simon Harris' probability analysis. Ann. Inst. H. Poincaré 40, 53–72.
Kyprianou, A.E., 2006. Introductory Lectures on Fluctuations of Lévy Processes with Applications. Springer.
Kyprianou, A.E., Liu, R.-L., Murillosalas, A., Ren, Y.-X., 2010. Suppercritical super-Brownian motion with a general branching mechanism and traveling waves. Preprint. Available at: http://arxiv.org/abs/1005.3659.
Revuz, D., Yor, M., 1991. Continuous Martingales and Brownian Motion. Springer-Verlag.