

$L \log L$ CRITERION FOR A CLASS OF SUPERDIFFUSIONS

RONG-LI LIU * ** AND

YAN-XIA REN, * *** *Peking University*

RENMING SONG, **** *University of Illinois*

Abstract

In Lyons, Pemantle and Peres (1995), a martingale change of measure method was developed in order to give an alternative proof of the Kesten–Stigum $L \log L$ theorem for single-type branching processes. Later, this method was extended to prove the $L \log L$ theorem for multiple- and general multiple-type branching processes in Biggins and Kyprianou (2004), Kurtz *et al.* (1997), and Lyons (1997). In this paper we extend this method to a class of superdiffusions and establish a Kesten–Stigum $L \log L$ type theorem for superdiffusions. One of our main tools is a spine decomposition of superdiffusions, which is a modification of the one in Engländer and Kyprianou (2004).

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1. Introduction and main result

Suppose that $\{Z_n, n \geq 1\}$ is a Galton–Watson branching process with each particle having probability p_n of giving birth to n children. Let L stand for a random variable with this offspring distribution. Let $m := \sum_{n=1}^{\infty} n p_n$ be the mean number of children per particle. Then Z_n/m^n is a nonnegative martingale. Let W be the limit of Z_n/m^n as $n \rightarrow \infty$. Kesten and Stigum [8] proved that if $1 < m < \infty$ (that is, in the supercritical case) then W is nondegenerate (i.e. not almost surely zero) if and only if

$$E(L \log^+ L) = \sum_{n=1}^{\infty} p_n n \log n < \infty.$$

This result is usually referred to as the Kesten–Stigum $L \log L$ theorem. In [1], Asmussen and Hering generalized this result to the case of branching Markov processes under some conditions.

Lyons *et al.* [14] developed a martingale change of measure method in order to give an alternative proof of the Kesten–Stigum $L \log L$ theorem for single-type branching processes.

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* Postal address: LMAM School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China.

** Email address: lrl@math.pku.edu.cn

Research supported by the CSC.

*** Email address: yxren@math.pku.edu.cn

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**** Postal address: Department of Mathematics, University of Illinois, Urbana, IL 61801, USA.

Email address: rsong@math.uiuc.edu

Later, this method was extended to prove the $L \log L$ theorem for multiple- and general multiple-type branching processes in [2], [12], and [13].

In this paper we will extend this method to a class of superdiffusions and establish an $L \log L$ criterion for superdiffusions. To state our main result, we need to introduce the setup we are going to work with first.

Let a_{ij} , $i, j = 1, \dots, d$, be bounded functions in $C^1(\mathbb{R}^d)$ such that all their first partial derivatives are bounded. We assume that the matrix (a_{ij}) is symmetric and satisfies

$$0 < a|v|^2 \leq \sum_{i,j} a_{ij} v_i v_j \quad \text{for all } x \in \mathbb{R}^d \text{ and } v \in \mathbb{R}^d$$

for some positive constant a . Let b_i , $i = 1, \dots, d$, be bounded Borel functions on \mathbb{R}^d .

We will use $(Y, \Pi_x, x \in \mathbb{R}^d)$ to denote a diffusion process on \mathbb{R}^d corresponding to the operator

$$L = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla.$$

In this paper we will always assume that β is a bounded Borel function on \mathbb{R}^d and that D is a bounded domain in \mathbb{R}^d . We will use $(Y^D, \Pi_x, x \in D)$ to denote the process obtained by killing Y upon exiting from D , that is,

$$Y_t^D = \begin{cases} Y_t & \text{if } t < \tau, \\ \partial & \text{if } t \geq \tau, \end{cases}$$

where $\tau = \inf\{t \geq 0: Y_t \notin D\}$ is the first exit time of D and ∂ is a cemetery point. Any function f on D is automatically extended to $D \cup \{\partial\}$ by setting $f(\partial) = 0$. For convenience, we use the following convention throughout this paper. For any probability measure P , we also use P to denote the expectation with respect to P . When there is only one probability measure involved, we sometimes also use E to denote the expectation with respect to that measure.

We will use $\{P_t\}_{t \geq 0}$ to denote the following Feynman–Kac semigroup:

$$P_t f(x) = \Pi_x \left(\exp \left\{ \int_0^t \beta(Y_s^D) ds \right\} f(Y_t^D) \right), \quad x \in D.$$

It is well known that the semigroup $\{P_t\}_{t \geq 0}$ is strongly continuous in $L^2(D)$ and, for any $t > 0$, P_t has a bounded, continuous, and strictly positive density $p(t, x, y)$.

Let $\{\hat{P}_t\}_{t \geq 0}$ be the dual semigroup of $\{P_t\}_{t \geq 0}$ defined by

$$\hat{P}_t f(x) = \int_D p(t, y, x) f(y) dy, \quad x \in D.$$

It is well known that $\{\hat{P}_t\}_{t \geq 0}$ is also strongly continuous on $L^2(D)$.

Let A and \hat{A} be the generators of the semigroups $\{P_t\}_{t \geq 0}$ and $\{\hat{P}_t\}_{t \geq 0}$ on $L^2(D)$, respectively. We can formally write A as $L|_D + \beta$, where $L|_D$ is the restriction of L to D with Dirichlet boundary condition. Let $\sigma(A)$ and $\sigma(\hat{A})$ respectively denote the spectrum of A and \hat{A} . It follows from Jentzsch's theorem [16, Theorem V.6.6, p. 337] and the strong continuity of $\{P_t\}_{t \geq 0}$ and $\{\hat{P}_t\}_{t \geq 0}$ that the common value $\lambda_1 := \sup \operatorname{Re}(\sigma(A)) = \sup \operatorname{Re}(\sigma(\hat{A}))$ is an eigenvalue of multiplicity 1 for both A and \hat{A} , and that an eigenfunction ϕ of A associated with λ_1 can be chosen to be strictly positive almost everywhere (a.e.) on D and an eigenfunction $\tilde{\phi}$ of \hat{A} associated with λ_1 can be chosen to be strictly positive a.e. on D . We assume that ϕ and $\tilde{\phi}$

are strictly positive a.e. on D . By [9, Proposition 2.3] we know that ϕ and $\tilde{\phi}$ are bounded and continuous on D , and they are in fact strictly positive everywhere on D . We choose ϕ and $\tilde{\phi}$ so that $\int_D \phi(x) \tilde{\phi}(x) dx = 1$.

Throughout this paper, we make the following assumptions.

Assumption 1.1. $\lambda_1 > 0$.

Assumption 1.2. The semigroups $\{P_t\}_{t \geq 0}$ and $\{\tilde{P}_t\}_{t \geq 0}$ are intrinsic ultracontractive, that is, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p(t, x, y) \leq c_t \phi(x) \tilde{\phi}(y) \quad \text{for all } (x, y) \in D \times D.$$

Assumption 1.2 is a very weak regularity assumption on D . It follows from [9] and [10] that Assumption 1.2 is satisfied when D is a bounded Lipschitz domain. For other, more general, examples of domain D for which Assumption 1.2 is satisfied, we refer the reader to [10] and the references therein.

Let $\mathcal{E}_t = \sigma(Y_s^D, s \leq t)$. For any $x \in D$, we define a probability measure Π_x^ϕ by the martingale change of measure:

$$\left. \frac{d\Pi_x^\phi}{d\Pi_x} \right|_{\mathcal{E}_t} = \frac{\phi(Y_t^D)}{\phi(x)} \exp \left\{ - \int_0^{t \wedge \tau} (\lambda_1 - \beta(Y_s)) ds \right\}.$$

The process (Y^D, Π_x^ϕ) is an ergodic Markov process and its transition density is given by

$$p^\phi(t, x, y) = \frac{\exp\{-\lambda_1 t\}}{\phi(x)} p(t, x, y) \phi(y).$$

The function $\phi \tilde{\phi}$ is the unique invariant density for the process (Y^D, Π_x^ϕ) .

By our choices for ϕ and $\tilde{\phi}$, $\int_D \phi(x) \tilde{\phi}(x) dx = 1$. Thus, it follows from [9, Theorem 2.8] that

$$\left| \frac{\exp\{-\lambda_1 t\} p(t, x, y)}{\phi(x) \tilde{\phi}(y)} - 1 \right| \leq c e^{-\nu t}, \quad x \in D,$$

for some positive constants c and ν , which is equivalent to

$$\sup_{x \in D} \left| \frac{p^\phi(t, x, y)}{\phi(y) \tilde{\phi}(y)} - 1 \right| \leq c e^{-\nu t}.$$

Thus, for any $f \in L_+^\infty(D)$, we have

$$\sup_{x \in D} \left| \int_D p^\phi(t, x, y) f(y) dy - \int_D \phi(y) \tilde{\phi}(y) f(y) dy \right| \leq c e^{-\nu t} \int_D \phi(y) \tilde{\phi}(y) f(y) dy.$$

Consequently, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sup_{x \in D} \sup_{f \in L_+^\infty(D)} \left(\int_D \phi(y) \tilde{\phi}(y) f(y) dy \right)^{-1} \\ & \times \left| \int_D p^\phi(t, x, y) f(y) dy - \int_D \phi(y) \tilde{\phi}(y) f(y) dy \right| \\ & = 0, \end{aligned}$$

which is equivalent to

$$\lim_{t \rightarrow \infty} \frac{\int_D p^\phi(t, x, y) f(y) dy}{\int_D \phi(y) \tilde{\phi}(y) f(y) dy} = 1 \quad \text{uniformly for } f \in L^\infty(D)_+ \text{ and } x \in D. \quad (1.1)$$

For any finite Borel measure μ on D , we define a probability measure $\Pi_{\phi\mu}^\phi$ as follows:

$$\Pi_{\phi\mu}^\phi = \int_D \mu(dx) \frac{\phi(x)}{\langle \phi, \mu \rangle} \Pi_x^\phi.$$

Note that, for any $A \in \mathcal{E}_t$,

$$\Pi_{\phi\mu}^\phi(A) = \frac{1}{\langle \phi, \mu \rangle} \Pi_\mu \left(\phi(Y_t^D) \exp \left\{ - \int_0^{t \wedge \tau} (\lambda_1 - \beta(Y_s)) ds \right\} \mathbf{1}_A \right).$$

The superdiffusion X we are going to study is a $(Y, \psi(\lambda) - \beta\lambda)$ -superprocess, which is a measure-valued Markov process with underlying spatial motion Y , branching rate dt , and branching mechanism $\psi(\lambda) - \beta\lambda$, where

$$\psi(x, \lambda) = \int_0^\infty (e^{-r\lambda} - 1 + \lambda r) n(x, dr)$$

for some σ -finite kernel n from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, that is, $n(x, dr)$ is a σ -finite measure on \mathbb{R}_+ for each fixed x , and $n(\cdot, A)$ is a measurable function for each Borel set $A \subset \mathbb{R}_+$. In this paper we will always assume that $\sup_{x \in D} \int_0^\infty (r \wedge r^2) n(x, dr) < \infty$. Note that this assumption implies that, for fixed $\lambda > 0$, $\psi(\cdot, \lambda)$ is bounded on D .

Let $(Y, \Pi_{r,x})$ denote a diffusion with generator L , birth time r , and starting point x . For any $\mu \in M_F(D)$, the family of all finite Borel measures on D , we will use $(X, P_{r,\mu})$ to denote a $(Y, \psi(\lambda) - \beta\lambda)$ -superprocess with starting time r such that $P_{r,\mu}(X_r = \mu) = 1$. We will simply denote $(X, P_{0,\mu})$ as (X, P_μ) . Let $X^{t,D}$ be the exit measure from $[0, t] \times D$, and let $\partial^{t,D}$ be the union of $(0, t) \times \partial D$ and $\{t\} \times D$.

Define $\phi^t: [0, t] \times \bar{D} \rightarrow [0, \infty)$ for each fixed $t \geq 0$, such that $\phi^t(u, x) = \phi(x)$ for $(u, x) \in [0, t] \times D$ and $\phi^t(u, x) = 0$ for $(u, x) \in [0, t] \times \partial D$. In particular, we extend ϕ to \bar{D} by setting it to be 0 on the boundary. Then

$$\{M_t(\phi) := \exp\{-\lambda_1 t\} \langle \phi^t, X^{t,D} \rangle, t \geq 0\} \quad (1.2)$$

is a P_μ -martingale with respect to $\mathcal{F}_t := \sigma(X^{s,D}, s \leq t)$ (see Lemma 2.1, below) and $P_\mu(M_t(\phi)) = \langle \phi, \mu \rangle$, $t \geq 0$. It is easy to check that $\{M_t(\phi), t \geq 0\}$ is a multiplicative functional of $X^{t,D}$.

To state our main result, we first define a new kernel $n^\phi(x, dr)$ from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that, for any nonnegative measurable function f on \mathbb{R}_+ ,

$$\int_0^\infty f(r) n^\phi(x, dr) = \int_0^\infty f(r\phi(x)) n(x, dr), \quad x \in D.$$

The following theorem is the main result of the paper.

Theorem 1.1. *Suppose that (X_t) is a $(Y, \psi(\lambda) - \beta\lambda)$ -superdiffusion starting from time 0 and with initial value μ . Set*

$$l(y) := \int_1^\infty r \log r n^\phi(y, dr).$$

1. If $\int_D \tilde{\phi}(y) l(y) dy < \infty$ then $M_\infty(\phi)$ is nondegenerate under P_μ for any $\mu \in M_F(D)$.
2. If $\int_D \tilde{\phi}(y) l(y) dy = \infty$ then $M_\infty(\phi)$ is degenerate for any $\mu \in M_F(D)$.

The proof of this theorem is accomplished by combining the ideas from [14] with the ‘spine decomposition’ of [5]. The new feature here is that we consider a different branching mechanism. The new branching mechanism considered here is essential. With this branching mechanism, we can establish a strong (that is, almost-sure) version of the spine decomposition, as opposed to the weak (that is, in distribution) version in [5]. The reason is that the branching mechanism we consider here results in *discrete* immigration points, as opposed to the quadratic branching case where immigration is continuous in time.

In the next section we first give a spine decomposition of the superdiffusion X under a martingale change of measure with the help of Poisson point processes. Then, in Section 3 we use this decomposition to give a proof of Theorem 1.1.

2. Decomposition of superdiffusions under the martingale change of measure

Let $\mathcal{F}_t = \sigma(X^s, s \leq t)$. We define a probability measure \tilde{P}_μ by the martingale change of measure:

$$\left. \frac{d\tilde{P}_\mu}{dP_\mu} \right|_{\mathcal{F}_t} = \frac{1}{\langle \phi, \mu \rangle} M_t(\phi).$$

The purpose of this section is to give a spine decomposition of X under \tilde{P}_μ .

The most important step in proving Theorem 1.1 is a decomposition of X under \tilde{P}_μ . We could decompose X under \tilde{P}_μ as the sum of two independent measure-valued processes. The first process is a copy of X under P_μ . The second process is, roughly speaking, obtained by taking an ‘immortal particle’ that moves according to the law of Y under $\Pi_{\phi\mu}^\phi$ and spins off pieces of mass that continue to evolve according to the dynamics of X .

To give a rigorous description of this decomposition of X under \tilde{P}_μ , let us first recall some results on Poisson point processes. Let (S, \mathcal{S}) be a measurable space. We will use \mathcal{M} to denote the family of σ -finite counting measures on (S, \mathcal{S}) and $\mathcal{B}(\mathcal{M})$ to denote the smallest σ -field on \mathcal{M} with respect to which all $\nu \in \mathcal{M} \mapsto \nu(B) \in \mathbb{Z}^+ \cup \{\infty\}$, $B \in \mathcal{S}$, are measurable. For any σ -finite measure \hat{N} on \mathcal{S} , we call an $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ -valued random variable ξ a Poisson random measure with intensity \hat{N} if

- (a) for each $B \in \mathcal{S}$ with $\hat{N}(B) < \infty$, $\xi(B)$ has a Poisson distribution with parameter $\hat{N}(B)$;
- (b) for $B_1, \dots, B_n \in \mathcal{S}$ disjoint, the variables $\xi(B_1), \dots, \xi(B_n)$ are independent.

Suppose that \hat{N} is a σ -finite measure on $(0, \infty) \times S$, if $e = (e(t), t \geq 0)$ is a process taking values in $S \cup \{\Upsilon\}$, where Υ is an isolated additional point and $e(0) = \Upsilon$, such that the random counting measure $\xi = \sum_{t \geq 0} \delta_{(t, e(t))}$ is a Poisson random measure on $(0, \infty) \times S$ with intensity \hat{N} , then e is called a Poisson point process with compensator \hat{N} . If, for every $t > 0$, $\hat{N}((0, t] \times S) < \infty$ then e can also be expressed as $e = ((\sigma_i, e_i), i = 1, \dots, N_t), t \geq 0$, where $e_i = e(\sigma_i)$ and N_t is a Poisson process with instant intensity $\hat{N}(dt \times S)$. The following proposition follows easily from [15, Proposition 19.5].

Proposition 2.1. *Suppose that $e = (e(t), t \geq 0)$ is a Poisson point process with compensator \hat{N} . Let f be a nonnegative Borel function on $(S \cup \{\Upsilon\}) \times [0, \infty)$ with $f(\Upsilon, t) = 0$ for*

all $t > 0$. If $\int_{(0,t]} \int_S |1 - e^{-f(x,s)}| \widehat{N}(ds, dx) < \infty$ for all $t > 0$ then

$$\mathbb{E} \left(\exp \left\{ - \sum_{0 \leq s \leq t} f(e(s), s) \right\} \right) = \exp \left\{ - \int_0^t \int_S (1 - e^{-f(x,s)}) \widehat{N}(ds, dx) \right\}.$$

Moreover, if $\int_0^\infty \int_S f(s, x) \widehat{N}(ds, dx) < \infty$ then

$$\mathbb{E} \left(\int_0^\infty \int_S f(x, s) N(ds, dx) \right) = \int_0^\infty \int_S f(x, s) \widehat{N}(ds, dx). \quad (2.1)$$

To give a formula for the one-dimensional distribution of the exit measure process under $\tilde{\mathbb{P}}_\mu$, we recall some results from [4] first.

According to [4], for any nonnegative bounded continuous function $f: \partial^{t,D} \rightarrow \mathbb{R}$, we have

$$\mathbb{P}_{r,\mu}(\exp\langle -f, X^{t,D} \rangle) = \exp\langle -U^t(f)(r, \cdot), \mu \rangle, \quad (2.2)$$

where $U^t(f)$ denotes the unique nonnegative solution to

$$\begin{aligned} -\frac{\partial U(s, x)}{\partial s} &= LU + \beta U(s, x) - \psi(U(s, x)), & x \in D, s \in (0, t), \\ U &= f \quad \text{on } \partial^{t,D}. \end{aligned} \quad (2.3)$$

More precisely, $U^t(f)$ satisfies the following integral equation:

$$\begin{aligned} U^t(f)(r, x) + \Pi_{r,x} \int_r^{t \wedge \tau_r} [\psi(U^t(f))(s, Y_s) - \beta(Y_s)U^t(f)(s, Y_s)] ds \\ = \Pi_{r,x} f(t \wedge \tau_r, Y_{t \wedge \tau_r}), \quad r \leq t, x \in D, \end{aligned} \quad (2.4)$$

where $\tau_r = \inf\{t \geq r: X_t \notin D\}$. Since Y is a time-homogeneous process, we find that $X^{t,D}$ under $\mathbb{P}_{r,\mu}$ has the same distribution as $X^{t-r,D}$ under \mathbb{P}_μ . The first moment of $\langle f, X^{t,D} \rangle$ is given by

$$\mathbb{P}_{r,x} \langle f, X^{t,D} \rangle = \Pi_{r,x} \left(f(t \wedge \tau_r, Y_{t \wedge \tau_r}) \exp \left\{ \int_r^{t \wedge \tau_r} \beta(Y_s) ds \right\} \right). \quad (2.5)$$

Lemma 2.1. $\{M_t(\phi), t \geq 0\}$ is a \mathbb{P}_μ -martingale with respect to \mathcal{F}_t .

Proof. It follows from the first moment formula (2.5) that

$$\begin{aligned} \mathbb{P}_{r,x} \langle \phi^t, X^{t,D} \rangle &= \Pi_{r,x} \left(\phi(Y_t) \exp \left\{ \int_r^t \beta(Y_s) ds \right\}, t < \tau_r \right) \\ &= \mathbb{P}_{t-r} \phi(x) \quad \text{for } r \leq t, x \in D. \end{aligned}$$

It is obvious that $\mathbb{P}_{r,x} \langle \phi^t, X^{t,D} \rangle = 0$ for $x \in \partial D$. By the special Markov property of X and the invariance of ϕ under $\exp\{-\lambda_1 t\} P_t$,

$$\begin{aligned} \mathbb{P}_\mu(M_t(\phi) \mid \mathcal{F}_s) &= \exp\{-\lambda_1 s\} \mathbb{P}_{X^s,D}(\exp\{-\lambda_1(t-s)\} \langle \phi^t, X^{t,D} \rangle) \\ &= \exp\{-\lambda_1 s\} \langle \exp\{-\lambda_1(t-s)\} P_{t-s} \phi, X^{s,D}|_D \rangle \\ &= \exp\{-\lambda_1 s\} \langle \phi^s, X^{s,D} \rangle \\ &= M_s(\phi) \quad \text{for } s \leq t, \end{aligned}$$

where $X^{s,D}|_D$ is the restriction of the measure $X^{s,D}$ on $\{s\} \times D$.

Now we give a formula for the one-dimensional distribution of X under \tilde{P}_μ .

Theorem 2.1. *Suppose that μ is a finite measure on D and that $g \in C_b^+(\partial^{t,D})$. Then*

$$\begin{aligned} \tilde{P}_\mu(\exp\langle -g, X^{t,D} \rangle) &= P_\mu(\exp\langle -g, X^{t,D} \rangle) \\ &\quad \times \Pi_{\phi\mu}^\phi \left(\exp \left\{ - \int_0^{t \wedge \tau} \psi'(Y_s, U^t(g)(s, Y_s)) ds \right\} \right), \end{aligned} \quad (2.6)$$

where $U^t(g)$ is the unique solution of (2.3) or, equivalently, (2.4) with f replaced by g .

Proof. This theorem can be proved using the same argument as that given in [5] to obtain Theorem 5 therein, with some obvious modifications. We omit the details.

From (2.6) we can see that the superprocess $(X^{t,D}, \tilde{P}_\mu)$ can be decomposed into two independent parts in the sense of distributions. The first part is a copy of the original superprocess and the second part is an immigration process. To explain the second part more precisely, we need to introduce another measure-valued process (\hat{X}_t) . Now we construct the measure-valued process (\hat{X}_t) as follows.

- (a) Suppose that $\tilde{Y} = (\tilde{Y}_t, t \geq 0)$ is defined on some probability space $(\Omega, P_{\mu,\phi})$ and that $\tilde{Y} = (\tilde{Y}_t, t \geq 0)$ has the same law as $(Y, \Pi_{\phi\mu}^\phi)$. Here \tilde{Y} serves as the spine or the immortal particle, which visits every part of D for large times since it is an ergodic diffusion.
- (b) Suppose that $m = \{m_t, t \geq 0\}$ is a point process taking values in $(0, \infty) \cup \{\Upsilon\}$ such that, conditional on $\sigma(\tilde{Y}_t, t \geq 0)$, m is a Poisson point process with intensity $rn(\tilde{Y}_t, dr)$. Now $(0, \infty)$ is the ‘space of mass’ and $m_t = \Upsilon$ simply means that there is no immigration at t . We suppose that $\{m_t, t \geq 0\}$ is also defined on $(\Omega, P_{\mu,\phi})$. Set $\mathcal{D}_m = \{t: m_t(\omega) \neq \Upsilon\}$. Note that \mathcal{D}_m is almost surely (a.s.) countable. The process m describes the immigration mechanism: along the path of \tilde{Y} , at the moment $t \in \mathcal{D}_m$, a particle with mass m_t is immigrated into the system at the position \tilde{Y}_t .
- (c) Once the particles are in the system, they begin to move and branch according to a $(Y, \psi(\lambda) - \beta\lambda)$ -superprocess independently.

We use $(X_t^\sigma, t \geq \sigma)$ to denote the measure-valued process generated by the mass immigrated at time σ and position \tilde{Y}_σ . Conditional on $\{\tilde{Y}_t, m_t, t \geq 0\}$, $\{X^\sigma, \sigma \in \mathcal{D}_m\}$ are independent $(Y, \psi - \beta\lambda)$ -superprocesses. The birth time of X^σ is σ and the initial value of X^σ is $m_\sigma \delta_{\tilde{Y}_\sigma}$. Set

$$\hat{X}^{t,D} = \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} X^{\sigma,(t,D)},$$

where, for each $\sigma \in \mathcal{D}_m$, $X^{\sigma,(t,D)}$ is the exit measure of the superprocess X^σ from $[0, t) \times D$. The Laplace functional of $\hat{X}^{t,D}$ is described in the following proposition.

Proposition 2.2. *The Laplace functional of $\hat{X}^{t,D}$ under $P_{\mu,\phi}$ is*

$$\Pi_{\phi\mu}^\phi \left(\exp \left\{ - \int_0^t \psi'(Y_s, U^t(g)(Y_s, s)) ds \right\} \right).$$

Proof. For any $g \in C_b^+(\partial^{t,D})$, using (2.2), we have

$$\begin{aligned} P_{\mu,\phi}(\exp\{-\langle g, \widehat{X}^{t,D} \rangle\}) &= P_{\mu,\phi}\left(P_{\mu,\phi}\left(\exp\left\{-\sum_{\sigma \in (0,t] \cap \mathcal{D}_m} \langle g, X^{\sigma,(t,D)} \rangle\right\} \middle| \widetilde{Y}, \sigma, m\right)\right) \\ &= P_{\mu,\phi}\left(\prod_{\sigma \in (0,t] \cap \mathcal{D}_m} \exp\{-m_\sigma U^t(g)(\widetilde{Y}_\sigma, \sigma)\}\right) \\ &= P_{\mu,\phi}\left(P_{\mu,\phi}\left(\exp\left\{-\sum_{\sigma \in (0,t] \cap \mathcal{D}_m} m_\sigma U^t(g)(\widetilde{Y}_\sigma, \sigma)\right\} \middle| \widetilde{Y}\right)\right). \end{aligned}$$

Using Proposition 2.1, we obtain

$$\begin{aligned} P_{\mu,\phi}(\exp\{-\langle g, \widehat{X}^{t,D} \rangle\}) &= \Pi_{\phi\mu}^\phi \exp\left\{-\int_0^t \int_0^\infty (1 - \exp\{-r U^t(g)(Y_s, s)\}) r n(Y_s, dr) ds\right\} \\ &= \Pi_{\phi\mu}^\phi \left(\exp\left\{-\int_0^t \psi'(Y_s, U^t(g)(s, Y_s)) ds\right\}\right). \end{aligned}$$

Without loss of generality, we suppose that $(X_t, t \geq 0; P_{\mu,\phi})$ is a superdiffusion defined on $(\Omega, P_{\mu,\phi})$, equivalent to $(X_t, t \geq 0; P_\mu)$ and independent of \widehat{X} . Proposition 2.2 says that we have the following decomposition of $X^{t,D}$ under \widetilde{P}_μ : for any $t > 0$,

$$(X^{t,D}, \widetilde{P}_\mu) = (X^{t,D} + \widehat{X}^{t,D}, P_{\mu,\phi}) \quad \text{in distribution,} \quad (2.7)$$

where $X^{t,D}$ is the exit measure of X from $[0, t) \times D$. Since $(X_t, t \geq 0; \widetilde{P}_\mu)$ is generated from the time-homogeneous Markov process $(X_t, t \geq 0; P_\mu)$ via a nonnegative martingale multiplicative functional, $(X_t, t \geq 0; \widetilde{P}_\mu)$ is also a time-homogeneous Markov process (see [17, Section 62]). From the construction of $(\widehat{X}^{t,D}, t \geq 0; P_{\mu,\phi})$ we see that $(\widehat{X}^{t,D}, t \geq 0; P_{\mu,\phi})$ is a time-homogeneous Markov process. For a rigorous proof of $(\widehat{X}^{t,D}, t \geq 0; P_{\mu,\phi})$ being a time-homogeneous Markov process, we refer the reader to [6]. Although [6] dealt with the representation of the superprocess conditioned to stay alive forever, we can check that the arguments there work in our case. Therefore, (2.7) implies that

$$(X^{t,D}, t \geq 0; \widetilde{P}_\mu) = (X^{t,D} + \widehat{X}^{t,D}, t \geq 0; P_{\mu,\phi}) \quad \text{in distribution.}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need some preparations. The following elementary result is taken from [3].

Lemma 3.1. ([3, Exercise 1.3.8].) *Let $Y \geq 0$ with $E(Y) < \infty$, and let $0 \leq a < E(Y)$. Then*

$$P(Y > a) \geq \frac{(E(Y) - a)^2}{E(Y^2)}.$$

Proposition 3.1. *Set $h(x) = P_{\delta_x}(M_\infty(\phi))/\phi(x)$.*

1. *h is nonnegative and invariant for the process (Y^D, Π_x^ϕ) .*
2. *Either M_∞ is nondegenerate under P_μ for all $\mu \in M_F(D)$ or M_∞ is degenerate under P_μ for all $\mu \in M_F(D)$.*

Proof. 1. Since $\phi^t(\cdot, u) = \phi(\cdot)$ for each $u \in [0, t]$ and ϕ is identically 0 on ∂D , we have, by the special Markov property of X ,

$$\begin{aligned} h(x) &= \frac{1}{\phi(x)} P_{\delta_x} \left(\lim_{s \rightarrow \infty} \langle \exp\{-\lambda_1(t+s)\} \phi^{t+s}, X^{t+s,D} \rangle \right) \\ &= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} P_{\delta_x} \left(P_{X^t,D} \left(\lim_{s \rightarrow \infty} \langle \exp\{-\lambda_1 s\} \phi^s, X^{s,D} \rangle \right) \right) \\ &= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} P_{\delta_x} (P_{X^t,D} (M_\infty)) \\ &= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} P_{\delta_x} (\langle (h\phi)^t, X^{t,D} \rangle) \\ &= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} \Pi_x \left(\exp \left\{ \int_0^t \beta(Y_s) ds \right\} (h\phi)(Y_t), t < \tau \right) \\ &= \frac{1}{\phi(x)} \Pi_x \left(\exp \left\{ \int_0^{t \wedge \tau} (\beta - \lambda_1)(Y_s) ds \right\} (h\phi)(Y_t^D) \right), \quad x \in D. \end{aligned}$$

By the definition of Π_x^ϕ we obtain $h(x) = \Pi_x^\phi(h(Y_t^D))$. So, h is an invariant function of the process (Y^D, Π_x^ϕ) . The nonnegativity of h is obvious.

2. Since h is nonnegative and invariant, if there exists an $x_0 \in D$ such that $h(x_0) = 0$, then $h \equiv 0$ on D . Since $P_\mu(M_\infty(\phi)) = \langle h\phi, \mu \rangle$, we then have $P_\mu(M_\infty(\phi)) = 0$ for any $\mu \in M_F(D)$. If $h > 0$ on D then $P_\mu(M_\infty(\phi)) > 0$ for any $\mu \in M_F(D)$.

Using Proposition 3.1, we see that, to prove Theorem 1.1, we only need to consider the case $\mu(dx) = \tilde{\phi}(x) dx$. So, in the remaining part of this paper we will always suppose that $\mu(dx) = \tilde{\phi}(x) dx$.

Lemma 3.2. *Let $(m_t, t \geq 0)$ be the Poisson point process constructed in Section 2. Define*

$$\sigma_0 = 0, \quad \sigma_i = \inf\{s \in \mathcal{D}_m : s > \sigma_{i-1}, m_s \phi(\tilde{Y}_s) > 1\}, \quad \eta_i = m_{\sigma_i}, \quad i = 1, 2, \dots$$

If $\int_D \tilde{\phi}(y) l(y) dy < \infty$ then

$$\sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) < \infty \quad P_{\mu, \phi} \text{-a.s.}$$

If $\int_D \tilde{\phi}(y) l(y) dy = \infty$ then

$$\limsup_{i \rightarrow \infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\tilde{Y}_{\sigma_i}) = \infty \quad P_{\mu, \phi} \text{-a.s.}$$

Proof. Since ϕ is bounded from above, σ_i is strictly increasing with respect to i . We first prove that if $\int_D \tilde{\phi}(y) l(y) dy < \infty$ then

$$\sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) < \infty \quad P_{\mu, \phi} \text{-a.s.}$$

For any $\varepsilon > 0$, we write the sum above as

$$\begin{aligned}
 & \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \\
 &= \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{\phi(\tilde{Y}_s) m_s \leq e^{\varepsilon s}\}} + \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{m_s \phi(\tilde{Y}_s) > e^{\varepsilon s}\}} \\
 &= \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{\phi(\tilde{Y}_s) m_s \leq e^{\varepsilon s}\}} + \sum_{i=1}^{\infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\tilde{Y}_{\sigma_i}) \mathbf{1}_{\{\eta_i \phi(\tilde{Y}_{\sigma_i}) > \exp\{\varepsilon \sigma_i\}\}} \\
 &=: I + II.
 \end{aligned} \tag{3.1}$$

By (2.1) we have

$$\begin{aligned}
 \sum_{i=1}^{\infty} \mathbb{P}_{\mu, \phi}(\eta_i \phi(\tilde{Y}_{\sigma_i}) > \exp\{\varepsilon \sigma_i\}) &= \sum_{i=1}^{\infty} \mathbb{P}_{\mu, \phi}(\mathbb{P}_{\mu, \phi}(\eta_i \phi(\tilde{Y}_{\sigma_i}) > \exp\{\varepsilon \sigma_i\} \mid \sigma(\tilde{Y}))) \\
 &= \mathbb{P}_{\mu, \phi} \left(\mathbb{P}_{\mu, \phi} \left(\sum_{i=1}^{\infty} \mathbf{1}_{\{\eta_i > \exp\{\varepsilon \sigma_i\} \phi(\tilde{Y}_{\sigma_i})^{-1}\}} \mid \sigma(\tilde{Y}) \right) \right) \\
 &= \Pi_{\phi\mu}^{\phi} \left(\int_0^{\infty} \left(\int_{\phi(Y_s)^{-1} e^{\varepsilon s}}^{\infty} r n(Y_s, dr) \right) ds \right).
 \end{aligned}$$

Recall that, under $\Pi_{\phi\mu}^{\phi}$, Y starts at the invariant measure $\phi(x)\mu(dx) = \phi(x)\tilde{\phi}(x)dx$. So we have

$$\begin{aligned}
 \sum_{i=1}^{\infty} \mathbb{P}_{\mu, \phi}(\eta_i \phi(\tilde{Y}_{\sigma_i}) > \exp\{\varepsilon \sigma_i\}) &= \int_0^{\infty} ds \int_D dy \phi(y) \tilde{\phi}(y) \int_{\phi(y)^{-1} e^{\varepsilon s}}^{\infty} r n(y, dr) \\
 &= \int_D \phi(y) \tilde{\phi}(y) dy \int_{\phi(y)^{-1}}^{\infty} r n(y, dr) \int_0^{\ln(r\phi(y))/\varepsilon} ds \\
 &= \varepsilon^{-1} \int_D \tilde{\phi}(y) l(y) dy.
 \end{aligned}$$

By the assumption that $\int_D \tilde{\phi}(y) l(y) dy < \infty$ and the Borel–Cantelli lemma, we obtain

$$\mathbb{P}_{\mu, \phi}(\eta_i \phi(\tilde{Y}_{\sigma_i}) > \exp\{\varepsilon \sigma_i\} \text{ infinitely often}) = 0 \quad \text{for all } \varepsilon > 0,$$

which implies that

$$II < \infty \quad \mathbb{P}_{\mu, \phi}\text{-a.s.} \tag{3.2}$$

Meanwhile, for $\varepsilon < \lambda_1$,

$$\begin{aligned}
 \mathbb{P}_{\mu, \phi} I &= \mathbb{P}_{\mu, \phi} \left(\sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \mathbf{1}_{\{m_s \leq e^{\varepsilon s} \phi(\tilde{Y}_s)^{-1}\}} \right) \\
 &= \Pi_{\phi\mu}^{\phi} \int_0^{\infty} dt \exp\{-\lambda_1 t\} \int_0^{\phi(Y_t)^{-1} e^{\varepsilon t}} \phi(Y_t) r^2 n(Y_t, dr) \\
 &\leq \|\phi\|_{\infty} \Pi_{\phi\mu}^{\phi} \int_0^{\infty} dt \exp\{-\lambda_1 t\} \int_0^1 r^2 n(Y_t, dr) \\
 &\quad + \Pi_{\phi\mu}^{\phi} \int_0^{\infty} dt \exp\{-(\lambda_1 - \varepsilon)t\} \int_1^{\infty} r n(Y_t, dr),
 \end{aligned}$$

where for the second term of the last inequality we used the fact that $r \leq \phi(Y_t)^{-1} e^{\varepsilon t}$ implies that $r\phi(Y_t) \leq e^{\varepsilon t}$. By the assumption that $\sup_{x \in D} \int_0^\infty (r \wedge r^2) n(x, dr) < \infty$ we have $P_{\mu, \phi} I < \infty$, which implies that

$$I < \infty \quad P_{\mu, \phi} \text{-a.s.} \quad (3.3)$$

Combining (3.1), (3.2), and (3.3), we see that $\sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) < \infty$, $P_{\mu, \phi}$ -a.s.

Next we prove that if $\int_D \tilde{\phi}(y) l(y) dy = \infty$ then

$$\limsup_{i \rightarrow \infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\tilde{Y}_{\sigma_i}) = \infty \quad P_{\mu, \phi} \text{-a.s.}$$

It suffices to prove that, for any $K > 0$,

$$\limsup_{i \rightarrow \infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\tilde{Y}_{\sigma_i}) > K \quad P_{\mu, \phi} \text{-a.s.} \quad (3.4)$$

Set $K_0 := 1 \vee (\max_{x \in D} \phi(x))$. Then, for $K \geq K_0$,

$$K \inf_{x \in D} \phi(x)^{-1} \geq 1.$$

Note that, for any $T \in (0, \infty)$, conditional on $\sigma(\tilde{Y})$,

$$\sharp\{i : \sigma_i \in (0, T]; \eta_i > K \phi(\tilde{Y}_{\sigma_i})^{-1} \exp\{\lambda_1 \sigma_i\}\}$$

is a Poisson random variable with parameter $\int_0^T dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}}^\infty r n(\tilde{Y}_t, dr)$ a.s. Since $(\tilde{Y}, P_{\mu, \phi})$ has the same distribution as $(Y, \Pi_{\mu, \phi}^\phi)$, we have

$$\begin{aligned} & P_{\mu, \phi} \int_0^T dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}}^\infty r n(\tilde{Y}_t, dr) \\ &= \int_0^T dt \int_D dy \phi(y) \tilde{\phi}(y) \int_{K \phi(y)^{-1} \exp\{\lambda_1 t\}}^\infty r n(y, dr) \\ &< \infty; \end{aligned}$$

thus,

$$\int_0^T dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}}^\infty r n(\tilde{Y}_t, dr) < \infty \quad P_{\mu, \phi} \text{-a.s.}$$

Consequently, we have

$$\sharp\{i : \sigma_i \in (0, T]; \eta_i > K \phi(\tilde{Y}_{\sigma_i})^{-1} \exp\{\lambda_1 \sigma_i\}\} < \infty \quad P_{\mu, \phi} \text{-a.s.}$$

So, to prove (3.4), we need to prove that

$$\int_0^\infty dt \int_{K \phi(\tilde{Y}_t)^{-1} \exp\{\lambda_1 t\}}^\infty r n(\tilde{Y}_t, dr) = \infty \quad P_{\mu, \phi} \text{-a.s.,}$$

which is equivalent to

$$\int_0^\infty dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^\infty r n(Y_t, dr) = \infty \quad \Pi_{\mu, \phi}^\phi \text{-a.s.} \quad (3.5)$$

For this purpose, we first prove that

$$\Pi_{\phi\mu}^{\phi} \left(\int_0^{\infty} dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr) \right) = \infty. \quad (3.6)$$

Applying Fubini's theorem, we obtain

$$\begin{aligned} & \Pi_{\phi\mu}^{\phi} \left(\int_0^{\infty} dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr) \right) \\ &= \int_D \phi(y) \tilde{\phi}(y) dy \int_0^{\infty} dt \int_{K\phi(y)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(y, dr) \\ &= \int_D \phi(y) \tilde{\phi}(y) dy \int_{K\phi(y)^{-1}}^{\infty} rn(y, dr) \int_0^{(1/\lambda_1) \ln(r\phi(y)/K)} dt \\ &= \frac{1}{\lambda_1} \int_D \phi(y) \tilde{\phi}(y) dy \int_{K\phi(y)^{-1}}^{\infty} (\ln[r\phi(y)] - \ln K) rn(y, dr) \\ &\geq \frac{1}{\lambda_1} \int_D \phi(y) \tilde{\phi}(y) dy \left(\int_{K\phi(y)^{-1}}^{\infty} r \ln[r\phi(y)] n(y, dr) - A \right) \\ &= \frac{1}{\lambda_1} \int_D \tilde{\phi}(y) dy \int_K^{\infty} r \ln rn^{\phi}(y, dr) - \frac{A}{\lambda_1} \int_D \tilde{\phi}(y) \phi(y) dy \end{aligned}$$

for some positive constant A , where in the inequality we used the facts that $K\phi(y)^{-1} > 1$ for any $y \in D$ and $\sup_{y \in D} \int_1^{\infty} rn(y, dr) < \infty$. Since

$$\int_D \tilde{\phi}(y) dy \int_1^{\infty} r \ln rn^{\phi}(y, dr) = \infty$$

and

$$\int_D \tilde{\phi}(y) dy \int_1^K r \ln rn^{\phi}(y, dr) \leq K \log K \int_D \tilde{\phi}(y) n(y, [\|\phi\|_{\infty}^{-1}, \infty)) dy < \infty,$$

we obtain

$$\int_D \tilde{\phi}(y) dy \int_K^{\infty} r \ln rn^{\phi}(y, dr) = \infty,$$

and, therefore, (3.6) holds.

By (1.1), there exists a constant $c > 0$ such that, for any $t > c$ and any $f \in L_+^{\infty}(D)$,

$$\frac{1}{2} \int_D \phi(y) \tilde{\phi}(y) f(y) dy \leq \int_D p^{\phi}(t, x, y) f(y) dy \leq 2 \int_D \phi(y) \tilde{\phi}(y) f(y) dy, \quad x \in D. \quad (3.7)$$

For $T > c$, we define

$$\xi_T = \int_0^T dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr), \quad A_T = \int_c^T dt \int_D \tilde{\phi}(y) dy \int_{K \exp\{\lambda_1 t\}}^{\infty} rn^{\phi}(y, dr).$$

Our goal is to prove (3.5), which is equivalent to

$$\xi_{\infty} := \int_0^{\infty} dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr) = \infty \quad \Pi_{\phi\mu}^{\phi}\text{-a.s.}$$

Since $\{\xi_\infty = \infty\}$ is an invariant event, by the ergodic property of Y under $\Pi_{\phi\mu}^\phi$, it is enough to prove that

$$\Pi_{\phi\mu}^\phi(\xi_\infty = \infty) > 0. \quad (3.8)$$

Note that

$$\Pi_{\phi\mu}^\phi \xi_T = \int_0^T dt \int_D \tilde{\phi}(y) dy \int_{K \exp\{\lambda_1 t\}}^\infty rn^\phi(y, dr) \geq A_T \quad (3.9)$$

and

$$\begin{aligned} \lim_{T \rightarrow \infty} \Pi_{\phi\mu}^\phi \xi_T &\geq A_\infty \\ &= \int_c^\infty dt \int_D \tilde{\phi}(y) dy \int_{K \exp\{\lambda_1 t\}}^\infty rn^\phi(y, dr) \\ &= \int_D \tilde{\phi}(y) dy \int_{K \exp\{\lambda_1 c\}}^\infty \left(\frac{1}{\lambda_1} (\log r - \log K) - c \right) rn^\phi(y, dr) \\ &\geq C \int_D \tilde{\phi}(y) l(y) dy \\ &= \infty, \end{aligned} \quad (3.10)$$

where C is a positive constant. By Lemma 3.1,

$$\Pi_{\phi\mu}^\phi \left(\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T \right) \geq \frac{(\Pi_{\phi\mu}^\phi \xi_T)^2}{4 \Pi_{\phi\mu}^\phi (\xi_T^2)}. \quad (3.11)$$

If we can prove that there exists a constant $\widehat{C} > 0$ such that, for all $T > c$,

$$\frac{(\Pi_{\phi\mu}^\phi \xi_T)^2}{4 \Pi_{\phi\mu}^\phi (\xi_T^2)} \geq \widehat{C}, \quad (3.12)$$

then by (3.11) we would obtain

$$\Pi_{\phi\mu}^\phi (\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \widehat{C},$$

and, therefore,

$$\Pi_{\phi\mu}^\phi (\xi_\infty \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \Pi_{\phi\mu}^\phi (\xi_T \geq \frac{1}{2} \Pi_{\phi\mu}^\phi \xi_T) \geq \widehat{C} > 0.$$

Since $\lim_{T \rightarrow \infty} \Pi_{\phi\mu}^\phi \xi_T = \infty$ (see (3.10)), the above inequality implies (3.8). Now we only need to prove (3.12). For this purpose, we first estimate $\Pi_{\phi\mu}^\phi (\xi_T^2)$:

$$\begin{aligned} \Pi_{\phi\mu}^\phi \xi_T^2 &= \Pi_{\phi\mu}^\phi \int_0^T dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^\infty rn(Y_t, dr) \int_0^T ds \int_{K \phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^\infty un(Y_s, du) \\ &= 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^\infty rn(Y_t, dr) \int_t^T ds \int_{K \phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^\infty un(Y_s, du) \\ &= 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^\infty rn(Y_t, dr) \int_t^{(t+c) \wedge T} ds \int_{K \phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^\infty un(Y_s, du) \\ &\quad + 2 \Pi_{\phi\mu}^\phi \int_0^T dt \int_{K \phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^\infty rn(Y_t, dr) \int_{(t+c) \wedge T}^T ds \int_{K \phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^\infty un(Y_s, du) \\ &= III + IV, \end{aligned}$$

where

$$III = 2\Pi_{\phi\mu}^{\phi} \int_0^T dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr) \int_t^{(t+c)\wedge T} ds \int_{K\phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^{\infty} un(Y_s, du)$$

and

$$\begin{aligned} IV &= 2\Pi_{\phi\mu}^{\phi} \int_0^T dt \int_{K\phi(Y_t)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(Y_t, dr) \int_{(t+c)\wedge T}^T ds \int_{K\phi(Y_s)^{-1} \exp\{\lambda_1 s\}}^{\infty} un(Y_s, du) \\ &= 2 \int_0^T dt \int_D \phi(y) \tilde{\phi}(y) dy \int_{K\phi(y)^{-1} \exp\{\lambda_1 t\}}^{\infty} rn(y, dr) \\ &\quad \times \int_{(t+c)\wedge T}^T ds \int_D p^{\phi}(s-t, y, z) dz \int_{K\phi(z)^{-1} \exp\{\lambda_1 s\}}^{\infty} un(z, du). \end{aligned}$$

By our assumption on the kernel n we have $\| \int_1^{\infty} rn(\cdot, dr) \|_{\infty} < \infty$. Since $K \inf_{x \in B} \phi(x)^{-1} \geq 1$, we have

$$III \leq C_1 \Pi_{\phi\mu}^{\phi} \xi_T$$

for some positive constant C_1 which does not depend on T . Using (3.7) and the definition of n^{ϕ} , we obtain

$$\begin{aligned} &\int_{(t+c)\wedge T}^T ds \int_D p^{\phi}(s-t, y, z) dz \int_{K\phi(z)^{-1} \exp\{\lambda_1 s\}}^{\infty} un(z, du) \\ &\leq 2 \int_{(t+c)\wedge T}^T ds \int_D \phi(z) \tilde{\phi}(z) dz \int_{K\phi(z)^{-1} \exp\{\lambda_1 s\}}^{\infty} un(z, du) \\ &\leq 2 \int_c^T ds \int_D \tilde{\phi}(z) dz \int_0^{\infty} (\phi(z)u) \mathbf{1}_{\{\phi(z)u > k \exp\{\lambda_1 s\}\}} n(z, du) \\ &= 2 \int_c^T ds \int_D \tilde{\phi}(z) dz \int_{k \exp\{\lambda_1 s\}}^{\infty} rn^{\phi}(z, dr) \\ &= 2A_T. \end{aligned}$$

Then, using (3.9), we have

$$IV \leq 4A_T \Pi_{\phi\mu}^{\phi} \xi_T \leq 4(\Pi_{\phi\mu}^{\phi} \xi_T)^2.$$

Combining the above estimates for III and IV , we find that there exists a $C_2 > 0$ independent of T such that, for $T > c$,

$$\Pi_{\phi\mu}^{\phi} (\xi_T^2) \leq 4(\Pi_{\phi\mu}^{\phi} \xi_T)^2 + C_1 \Pi_{\phi\mu}^{\phi} \xi_T \leq C_2 (\Pi_{\phi\mu}^{\phi} \xi_T)^2.$$

Then we have (3.12) with $\widehat{C} = 1/C_2$, and the proof of the theorem is now complete.

Definition 3.1. Suppose that (Ω, \mathcal{F}, P) is a probability space, that $\{\mathcal{F}_t, t \geq 0\}$ is a filtration on (Ω, \mathcal{F}) , and that \mathcal{G} is a sub- σ -field of \mathcal{F} . A real-valued process U_t on (Ω, \mathcal{F}, P) is called a $P(\cdot | \mathcal{G})$ -martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ if

- (i) it is adapted to $\{\mathcal{F}_t \vee \mathcal{G}, t \geq 0\}$;
- (ii) for any $t \geq 0$, $E(|U_t| | \mathcal{G}) < \infty$; and

(iii) for any $t > s$,

$$E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) = U_s \quad \text{a.s.}$$

We say that U_t on (Ω, \mathcal{F}, P) is a $P(\cdot \mid \mathcal{G})$ -submartingale or a $P(\cdot \mid \mathcal{G})$ -supermartingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ if, in addition to (i) and (ii), for any $t > s$,

$$E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \geq U_s \quad \text{a.s.}$$

or, respectively,

$$E(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \leq U_s \quad \text{a.s.}$$

The following result is a folklore. Since we could not find a reference for this result, we provide a proof for completeness.

Lemma 3.3. *Suppose that (Ω, \mathcal{F}, P) is a probability space, that $\{\mathcal{F}_t, t \geq 0\}$ is a filtration on (Ω, \mathcal{F}) , and that \mathcal{G} is a σ -field of \mathcal{F} . If U_t is a $P(\cdot \mid \mathcal{G})$ -submartingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ satisfying*

$$\sup_{t \geq 0} E(|U_t| \mid \mathcal{G}) < \infty \quad \text{a.s.}, \quad (3.13)$$

then there exists a finite random variable U_∞ such that U_t converges a.s. to U_∞ .

Proof. By Definition 3.1, U_t is a submartingale with respect to $\{\mathcal{F}_t \vee \mathcal{G}, t \geq 0\}$. Let $\Omega_n = \{\sup_{t \geq 0} E(|U_t| \mid \mathcal{G}) \leq n\}$. Assumption (3.13) implies that $P(\Omega_n) \uparrow 1$. Note that, for each fixed n , $\mathbf{1}_{\Omega_n} U_t$ is a submartingale with respect to $\{\mathcal{F}_t \vee \mathcal{G}, t \geq 0\}$ with

$$\begin{aligned} \sup_{t \geq 0} E(\mathbf{1}_{\Omega_n} |U_t|) &= \sup_{t \geq 0} E(E(\mathbf{1}_{\Omega_n} |U_t| \mid \mathcal{G})) \\ &= \sup_{t \geq 0} E(\mathbf{1}_{\Omega_n} E(|U_t| \mid \mathcal{G})) \\ &\leq E\left(\sup_{t \geq 0} E(|U_t| \mid \mathcal{G}); \Omega_n\right) \\ &< \infty. \end{aligned}$$

The martingale convergence theorem says that there exists a finite random variable U_∞ defined on Ω_n such that U_t converges to U_∞ on Ω_n as $t \rightarrow \infty$. Therefore, there exists a finite U_∞ on the whole space Ω such that U_t converges to U_∞ a.s.

The next result is basically [3, Theorem 4.3.3].

Lemma 3.4. *Suppose that (Ω, \mathcal{F}) is a measurable space and that $(\mathcal{F}_t)_{t \geq 0}$ is a filtration on (Ω, \mathcal{F}) with $\mathcal{F}_t \uparrow \mathcal{F}$. If P and Q are two probability measures on (Ω, \mathcal{F}) such that, for some nonnegative P -martingale Z_t with respect to $(\mathcal{F}_t)_{t \geq 0}$,*

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = Z_t.$$

Then the limit $Z_\infty := \limsup_{t \rightarrow \infty} Z_t$ exists and is finite a.s. under P . Furthermore, for any $F \in \mathcal{F}$,

$$Q(F) = \int_F Z_\infty dP + Q(F \cap \{Z_\infty = \infty\}),$$

and, consequently,

$$\begin{aligned} P(Z_\infty = 0) = 1 &\iff Q(Z_\infty = \infty) = 1, \\ \int_\Omega Z_\infty dP = \int_\Omega Z_0 dP &\iff Q(Z_\infty < \infty) = 1. \end{aligned}$$

Proof of Theorem 1.1. We first prove that if $\int_D \tilde{\phi}(y)l(y) dy < \infty$ then M_∞ is nondegenerate under P_μ . Since $M_t^{-1}(\phi)$ is a positive supermartingale under \tilde{P}_μ , $M_t(\phi)$ converges to some nonnegative random variable $M_\infty(\phi) \in (0, \infty]$ under \tilde{P}_μ . By Lemma 3.4, we only need to prove that

$$\tilde{P}_\mu(M_\infty(\phi) < \infty) = 1. \quad (3.14)$$

By (2.7), $(X^{t,D}, \tilde{P}_\mu)$ has the same law as $(X^{t,D} + \hat{X}^{t,D}, P_{\mu,\phi})$, where $X^{t,D}$ is the first exit measure of the superprocess X from $(0, t) \times D$ and $\hat{X}^{t,D} = \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} X^{\sigma,(t,D)}$. Define

$$W_t(\phi) := \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} \langle \phi^t, X^{\sigma,(t,D)} \rangle \exp\{-\lambda_1 t\}.$$

Then,

$$(M_t(\phi), t \geq 0; \tilde{P}_\mu) = (M_t(\phi) + W_t(\phi), t \geq 0; P_{\mu,\phi}) \quad \text{in distribution}, \quad (3.15)$$

where $\{M_t(\phi), t \geq 0\}$ is copy of the martingale defined in (1.2) and is independent of $W_t(\phi)$. Let \mathcal{G} be the σ -field generated by $\{\tilde{Y}_t, m_t, t \geq 0\}$. Then, conditional on \mathcal{G} , $(X_t^\sigma, t \geq \sigma; P_{\mu,\phi})$ has the same distribution as $(X_{t-\sigma}, t \geq \sigma; P_{m_\sigma \delta_{\tilde{Y}_\sigma}})$ and the $(X_t^\sigma, t \geq \sigma; P_{\mu,\phi})$ are independent for $\sigma \in \mathcal{D}_m$. Then we have

$$W_t(\phi) \stackrel{D}{=} \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} \exp\{-\lambda_1 \sigma\} M_{t-\sigma}^\sigma(\phi), \quad (3.16)$$

where, for each $\sigma \in \mathcal{D}_m$, $M_t^\sigma(\phi)$ is a copy of the martingale defined by (1.2) with $\mu = m_\sigma \delta_{\tilde{Y}_\sigma}$ and, conditional on \mathcal{G} , the $\{M_t^\sigma(\phi), \sigma \in \mathcal{D}_m\}$ are independent. Here ‘ $\stackrel{D}{=}$ ’ denotes equality in distribution. To prove (3.14), by (3.15), it suffices to show that

$$P_{\mu,\phi} \left(\lim_{t \rightarrow \infty} [M_t(\phi) + W_t(\phi)] < \infty \right) = 1.$$

Since $(M_t(\phi), t \geq 0)$ is a nonnegative martingale under the probability $P_{\mu,\phi}$, it converges $P_{\mu,\phi}$ -a.s. to a finite random variable $M_\infty(\phi)$ as $t \rightarrow \infty$. So we only need to prove that

$$P_{\mu,\phi} \left(\lim_{t \rightarrow \infty} W_t(\phi) < \infty \right) = 1. \quad (3.17)$$

Define $\mathcal{H}_t := \mathcal{G} \vee \sigma(X^{\sigma,(s-B)}; \sigma \in [0, t] \cap \mathcal{D}_m, s \in [\sigma, t])$. Then $(W_t(\phi))$ is a $P_{\mu,\phi}(\cdot | \mathcal{G})$ -nonnegative submartingale with respect to (\mathcal{H}_t) . By (3.16) and Lemma 3.2,

$$\begin{aligned} \sup_{t \geq 0} P_{\mu,\phi}(W_t(\phi) | \mathcal{G}) &= \sup_{t \geq 0} \sum_{s \in [0, t] \cap \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \\ &\leq \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\tilde{Y}_s) \\ &< \infty \quad P_{\mu,\phi} \text{ -a.s.} \end{aligned}$$

Then, by Lemma 3.3, $W_t(\phi)$ converges $P_{\mu,\phi}$ -a.s. to $W_\infty(\phi)$ as $t \rightarrow \infty$ and $P_{\mu,\phi}(W_\infty(\phi) < \infty) = 1$; therefore, (3.17) holds.

Now we turn to the proof of the second part of the theorem. Assume that $\int_D \tilde{\phi}(y)l(y)dy = \infty$. We are going to prove that $M_\infty(\phi) := \lim_{t \rightarrow \infty} M_t(\phi)$ is degenerate with respect to P_μ .

By [7, Proposition 2], $1/M_t(\phi)$ is a supermartingale under \tilde{P}_μ , and, thus, $1/(M_t(\phi) + W_t(\phi))$ is a nonnegative supermartingale under $P_{\mu,\phi}$. Recall that $M_t(\phi)$ is a nonnegative martingale under $P_{\mu,\phi}$. Then the limits $\lim_{t \rightarrow \infty} M_t(\phi)$ and $1/\lim_{t \rightarrow \infty} (M_t(\phi) + W_t(\phi))$ exist and are finite $P_{\mu,\phi}$ -a.s. Therefore, $\lim_{t \rightarrow \infty} W_t(\phi)$ exists in $[0, \infty]$ $P_{\mu,\phi}$ -a.s. Recall the definition of $(\eta_i, \sigma_i, i = 1, 2, \dots)$ in Lemma 3.2, and note that $\lim_{i \rightarrow \infty} \sigma_i = \infty$. By Lemma 3.2,

$$\limsup_{t \rightarrow \infty} W_t(\phi) \geq \limsup_{i \rightarrow \infty} W_{\sigma_i}(\phi) \geq \limsup_{i \rightarrow \infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\tilde{Y}_{\sigma_i}) = \infty \quad P_{\mu,\phi}\text{-a.s.}$$

So we have

$$\lim_{t \rightarrow \infty} W_t(\phi) = \infty \quad P_{\mu,\phi}\text{-a.s.}$$

By (3.15),

$$\tilde{P}_\mu(M_\infty(\phi) = \infty) = 1.$$

It follows from Lemma 3.4 that $P_\mu(M_\infty = 0) = 1$.

Remark 3.1. The argument of this paper actually works for general superprocesses. Our main result remains valid for any general $(Y, \psi(\lambda) - \beta\lambda)$ -superprocess with Y being a reasonable Markov process such that Assumptions 1.1 and 1.2 are satisfied. For examples of discontinuous Markov processes satisfying Assumption 1.2, we refer the reader to [11] and the references therein.

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