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Some properties of superprocesses conditioned on non-extinction

LIU RongLi[1] & REN YanXia[2†]

[1]LMAM School of Mathematical Sciences, Peking University, Beijing 100871, China
[2]LMAM School of Mathematical Sciences, Peking University, Beijing 100871, China

Abstract We simply call a superprocess conditioned on being never extinct a conditioned superprocess. In this study, we investigate some properties of the conditioned superprocesses (subcritical or critical). Firstly, we give an equivalent description of the probability of the event that the total occupation time measure on a compact set is finite and some applications of this equivalent description. Our results are extensions of those of Krone [6] from particular branching mechanisms to general branching mechanisms. We also prove a claim of Krone [6] for the cases of $d = 3, 4$. Secondly, we study the local extinction property of the conditioned binary super-Brownian motion $\{X_t, P_\mu^\infty\}$. When $d = 1$, as t goes to infinity, $X_t/\sqrt{t}$ converges to $\eta\lambda$ in weak sense under P_μ^∞, where η is a nonnegative random variable and λ is the Lebesgue measure on $\mathbb{R}$. When $d \geqslant 2$, the conditioned binary super-Brownian motion is locally extinct under P_μ^∞.

Keywords: Conditioned superprocess, h-transform, Occupation time, Local extinct

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1 Introduction

Let E be a Polish space and $D([0, \infty), E)$ be the space of the right continuous functions with left limit from $[0, \infty)$ to E with the Skorohod topology. Suppose $\xi = \{\xi_t, t \geq 0; \Pi_x\}$ is a Feller Markov process defined on $(D([0, \infty), E), \mathcal{E}, (\mathcal{E}_t)_{t\geq 0})$ staring from $x \in E$ with infinitesimal generator A and semigroup $(\mathbb{P}_t)_{t\geqslant 0}$. Let $C_b(E)$ (resp. $C_c(E)$) be the space of bounded continuous functions (resp. continuous functions with compact support) on E. We denote by $C_b(E)_+$ (resp. $C_c(E)_+$) the subset of $C_b(E)$ (resp. $C_c(E)$), consisting of nonnegative members of $C_b(E)$ (resp. $C_c(E)$). We use $M_F(E)$ to denote the collection of all finite measures on E. Write $\langle \mu, \phi\rangle$ for $\int_E \phi\, d\mu$, where ϕ is a nonnegative measurable function and $\mu \in M_F(E)$. It is well known that $M_F(E)$ endowed with the weak topology is a Polish space. So we can define the Polish space $D([0, \infty), M_F(E))$. Let $X = \{X_t, t \geq 0; P_\mu\}$ be the time homogeneous Markov coordinate process on $(D([0, \infty), M_F(E)), \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ with a fixed initial value $\mu \in M_F(E)$ and with the semigroup given by its Laplace transform

$$P_\mu \exp\{-\langle X_t, \phi\rangle\} = \exp\{-\langle \mu, V_t(\phi)\rangle\}, \qquad \forall \phi \in C_b(E)_+,$$

where $V_t(\phi)$ denotes the unique non-negative bounded mild solution of the nonlinear evolution

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† Corresponding author Ren YanXia (email: yxren@math.pku.edu.cn)

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equation

$$\begin{cases} \dfrac{\partial V_t}{\partial t} = AV_t - \psi(V_t), \\ V_0 = \phi. \end{cases} \tag{1}$$

Here,

$$\psi(\lambda) = a\lambda + b\lambda^2 + \int_0^\infty [e^{-\lambda x} - 1 + \lambda x]\nu(dx), \tag{2}$$

$a,\ b \in \mathbb{R},\ b \geqslant 0$, and ν is a σ- finite measure on $(0,\ \infty)$ which satisfies $\int_0^\infty x^2 \wedge x d\nu(x) < \infty$. A $C_b(E)$-valued function $u(t)$ is called a *mild solution* of the evolution equation (1) if it satisfies the integral equation

$$u(t) = \mathbb{P}_t\phi + \int_0^t \mathbb{P}_{t-s}\psi(u(s))ds.$$

The measure-valued Markov process X is called $(A,\ \psi)-$ superprocess or $(\xi,\ \psi)-$ superprocess. In particular, if ξ is an $\alpha-$stable ($\alpha \in (0,2]$) processes on $\mathbb{R}^d$ and if $\psi(\lambda) = \lambda^{1+\beta}$, $\beta \in (0,1]$, we call the $(\xi,\ \psi)-$ superprocess (d,α,β)- superprocess.

Define $T = \inf\{t \geqslant 0; X_t(1) = 0\}$. Grey [1] showed that if $\psi'(\lambda) \geqslant 0$ for any $\lambda \geqslant 0$, and $\int^\infty \frac{d\lambda}{\psi(\lambda)} < \infty$, then

$$P_\mu(T < \infty) = 1 \quad \text{for any} \ \ \mu \in M_F(E).$$

That is to say the superprocess is extinct in finite time almost surely under that condition. In this article, we assume that

$$\psi'(\lambda) \geqslant 0, \quad \text{for any} \ \ \lambda \geqslant 0, \qquad \text{and} \ \int^\infty \frac{d\lambda}{\psi(\lambda)} < \infty.$$

Note that ψ' is a nondecreasing function on $[0,\ \infty)$ and $a = \psi'(0)$. So the assumption that $\psi'(\lambda) \geqslant 0$, for any $\lambda \geqslant 0$, is equivalent to that $a \geqslant 0$, which says that we only consider the subcritical ($a > 0$) or critical ($a = 0$) cases.

A few words about notations are in order. Without further mention, $V_t(\cdot,\ \phi)$ always denotes the unique mild solution of the equation (1), and $U_t(\cdot,\ \phi)$ denotes the unique mild solution of the following nonlinear partial differential equation

$$\begin{cases} \dfrac{\partial U_t}{\partial t} = AU_t - \psi(U_t) + \phi, \\ U_0 = 0. \end{cases} \tag{3}$$

Sometimes we may use $V_t(\phi)$ and $U_t(\phi)$ instead of $V_t(\cdot,\phi)$ and $U_t(\cdot,\phi)$ respectively. We write P_μ for both the probability and the expectation of X with initial value $\mu \in M_F(E)$.

In recent years, a series of papers investigated the Dawson-Watanabe superprocess conditioned on non-extinction (see, for example, [2], [3],[4], [5], [6], [7], [8] and [10], and the references therein). Some of these references are quoted more precisely later in this paper. We simply call the $(\xi,\ \psi)$-superprocesses conditioned on non-extinction the conditioned $(\xi,\ \psi)$-superprocesses. In particular, if $\psi(\lambda) = \lambda^{1+\beta}$, $\beta \in (0,1]$, and the underlying process is an α - stable ($\alpha \in (0,2]$) processes on $\mathbb{R}^d$, then we call the corresponding conditioned superprocesses the conditioned (d,α,β)- superprocesses. Moreover, when ξ is a Brownian motion and $\beta = 1$, the corresponding conditioned superprocess is called the conditioned binary super-Brownian motion.

In the coming section, we give the Laplace functionals of the conditioned $(\xi,\ \psi)$-superprocess and its occupation time measure. In Section 3, we give an equivalent description of the probability of the event that the conditioned total occupation time measure on a compact set is finite. Using this equivalent description, we give the judging conditions to some conditioned superpocesses, under which the corresponding conditioned total occupation time measures on compact sets are finite with probability one, and under which they are infinite with probability one. In the last section, we discuss the local extinction property of the conditioned binary super-Brownian motion. In this paper, we always assume the initial value $\mu \neq 0$.

2 Conditioned superprocess and its occupation time

In this section, we establish the existence of the conditioned superprocess and give the Laplace functionals of the conditioned superprocess and its occupation time at time t, $t \geqslant 0$. We see that the conditioned superprocess can be defined by means of Doob h-transformation of the original one.

Proposition 2.1 (Proposition 1 [9]) *Let $\{X_t,\ P_\mu\}$ be a $(\xi,\ \psi)$-superprocess.*

i) For any $\mu \in M_F(E)$ with compact support, as $t \to \infty$, $P_\mu(\cdot|T > t)$ converges to some probability measure P_μ^∞. More precisely, for any $t \geqslant 0$, $A \in \mathcal{F}_t$,

$$\lim_{s\to\infty} P_\mu(A|T > s) = P_\mu^\infty(A).$$

ii) The probability P_μ^∞ is an h-transform of P_μ. More precisely, put

$$Z_t = X_t(1)e^{at}.$$

Then $\{(Z_t, \mathcal{F}_t), P_\mu\}$ is a martingale defined on $D([0,\ \infty),\ M_F(E))$, and

$$dP_\mu^\infty\Big|_{\mathcal{F}_t} = \frac{Z_t}{\mu(1)}dP_\mu\Big|_{\mathcal{F}_t}.$$

The above results were also obtained by Krone [6] for particular branching mechanisms: $\psi(\lambda) = \gamma\lambda^{1+\beta}(0 < \beta \leq 1)$ with γ being some positive constant. We deduce from Proposition 2.1 that for any $\mathcal{F}_t$ measurable bounded function Φ,

$$P_\mu^\infty\left[\Phi\right] = \mu(1)^{-1}P_\mu\left[\Phi Z_t\right].$$

In particular, for any $f \in C_b(E)_+$

$$P_\mu^\infty[\exp(-\langle X_t,\ f\rangle)] = \mu(1)^{-1}P_\mu\left[\exp(-\langle X_t,\ f\rangle)X_t(1)e^{at}\right].$$

Note that

$$\exp(-\langle X_t, f\rangle)\langle X_t,\ 1\rangle = \frac{\partial}{\partial\lambda}\exp(-\langle X_t, f+\lambda\rangle)\big|_{\lambda=0},$$

and $P_\mu[X_t(1)] < \infty$. Applying the dominated convergence theorem, we can get that

$$\begin{aligned} & P_\mu\left[\exp(-\langle X_t, f\rangle)\langle X_t,\ 1\rangle\right] = P_\mu\left[\frac{\partial}{\partial\lambda}\exp(-\langle X_t, f+\lambda\rangle)\Big|_{\lambda=0}\right] \\ = \ & \frac{\partial}{\partial\lambda}P_\mu\left[\exp(-\langle X_t, f+\lambda\rangle)\right]\big|_{\lambda=0} = \frac{\partial}{\partial\lambda}\exp(-\langle\mu,\ V_t(f+\lambda)\rangle)\big|_{\lambda=0} \\ = \ & e^{-\langle\mu,\, V_t(f)\rangle}\langle\mu,\ G_t\rangle, \end{aligned}$$

where (G_t) is the unique non-negative bounded solution of the following partial differential equation

$$\begin{cases} \dfrac{\partial G_t}{\partial t} = AG_t - \psi'(V_t(f))G_t, \\ W_0 = 1. \end{cases} \tag{4}$$

Recall that Π_x is the probability of the Feller process ξ starting from x. We also use it to stand for the corresponding expectation. Applying the Feyman-Kac formula to equation (4), we get

$$G_t(x) = \Pi_x \left[\exp\left(-\int_0^t \psi'(V_{t-s}(\xi_s,\, f))ds \right) \right].$$

Define

$$\varphi(\lambda) = \psi'(\lambda) - a, \tag{5}$$

then $\varphi(\lambda) = 2b\lambda + \int_0^\infty (1 - e^{-\lambda r}) r d\nu(r)$, $\lambda \geqslant 0$. So the Laplace functional of the conditioned superprocess at time t is:

$$\begin{aligned} & P_\mu^\infty \left[\exp(-\langle X_t,\, f\rangle)\right] \\ = \quad & \mu(1)^{-1} e^{-\langle \mu,\, V_t(f)\rangle} \left\langle \mu,\, \Pi.\exp\left(-\int_0^t \varphi(V_{t-s}(\xi_s,\, f)ds \right) \right\rangle. \end{aligned} \tag{6}$$

Now let us consider the occupation time measure $Y_t(\cdot)$ of the superprocess $\{X_t\}$ defined by $Y_t(\cdot) = \int_0^t X_s(\cdot)ds$, for any $t \geqslant 0$. Under P_μ^∞, the process $\{Y_t, t \geq 0\}$ is called the conditioned occupation time process. A similar argument gives the Laplace functional of Y_t, for any $t \geqslant 0$.

$$\begin{aligned} & P_\mu^\infty \left[\exp(-\langle Y_t,\, \phi\rangle)\right] \\ = \quad & \mu(1)^{-1} e^{-\langle \mu,\, U_t(\phi)\rangle} \left\langle \mu,\, \Pi.\exp\left(-\int_0^t \varphi(U_{t-s}(\xi_s, \phi)ds \right) \right\rangle, \quad \phi \in C_b(E)_+, \end{aligned} \tag{7}$$

where $U_t(\phi)$ is the unique nonnegative mild solution of (3).

Note that $P_\mu^\infty \perp P_\mu$. This may explain why the properties of conditioned superprocesses are quite different from those of the corresponding unconditioned superprocesses.

3 Total occupation time measure of the conditioned superprocess

Define $\upsilon(\cdot) = \int_0^\infty \langle X_s,\, \cdot\rangle ds$. Under P_μ, υ is called the total occupation time measure of the superprocess X, and under P_μ^∞ υ is called the total occupation time measure of the conditioned superprocesses or simply called the conditioned total occupation time measure. We aim at finding out the P_μ^∞-probability of the event $\{\upsilon(C) < \infty\}$ for any compact subset C of E. The measure υ was studied by Serlet [10] for super-Brownian motion with $\psi(\lambda) = \lambda^2$ by using Brownian snake. It was showed in [10] that for any compact subset C of $\mathbb{R}^d$, if $d > 4$, then $\upsilon(C) < \infty$ P_μ^∞-a.s., while if $d \leq 4$, then $\upsilon(C) = \infty$ P_μ^∞-a.s.. Krone [6] investigated the finiteness of υ of the conditioned $(\xi,\ \lambda^{1+\beta})$-superprocess, the author proved that if the underlying process is Harris recurrent, then

$$\int_0^\infty \langle X_s,\, \phi\rangle ds = \infty, \quad P_\mu^\infty - \text{a.s.}$$

for any $\phi \in C_c(E)_+$. In this section, we consider the conditioned $(\xi,\ \psi)$-superprocesses with ψ given by (2). The following result is a generalization of those in [6] and [10].

Theorem 3.1 *For $\phi \in C_c(E)_+$, we have*

$$P_\mu^\infty \left(\int_0^\infty \langle X_s,\, \phi\rangle ds < \infty \right) = \Pi_{\bar\mu} \left(\int_0^\infty \varphi(U_\infty(\phi)(\xi_s)) ds < \infty \right),$$

where $U_\infty(\phi)(x) = \lim_{t\to\infty} U_t(\phi)(x), \quad \bar\mu(\cdot) = \mu(\cdot)/\mu(1)$.

We prove Theorem 3.1 by proving some lemmas.

Lemma 3.2 *For $\phi \in C_c(E)_+$, $U_t(\phi)$ denotes the mild solution of the equation* (3).

i) $\{U_t(\phi)\}_{t\ge 0}$ *is nonnegative and uniformly bounded;*

ii) If $x \in E$ satisfies $\phi(x) > 0$, then for any $t > 0$, $U_t(x, \phi) > 0$.

Proof. Assume that $\{X_t\}$ is the $(\xi,\, \psi)$-superprocess with initial value $\mu,\ \mu \in M_F(E)$. Then the occupation time process $\{\int_0^t X_s(\cdot)ds\}_{t\ge 0}$ has the Laplace functional

$$P_\mu \left[\exp\{-\int_0^t X_s(\phi)ds\} \right] = \exp\{-\langle \mu,\, U_t(\cdot,\, \phi)\rangle\}, \quad \text{for any } \phi \in C_c(E)_+.$$

It is obvious that $U_t(x,\, \phi) \geqslant 0$, for any $t > 0$.

We first prove the assertion ii). Assume that there exists $x \in E$, such that $\phi(x) > 0$, but there exists $t > 0$ such that $U_t(x,\, \phi) = 0$. Then

$$P_{\delta_x} \left[\exp\left\{ -\int_0^t X_u(\phi)du \right\} \right] = \exp\{-U_t(x,\, \phi)\} = 1,$$

which says that $\int_0^t X_u(\phi)du = 0$, P_{δ_x}-a.s. Note that $P_{\delta_x}[\langle X_s,\, \phi\rangle] = \mathrm{e}^{-as}\mathbb{P}_s\phi(x)$, for any $s > 0$. Then we have

$$0 = P_{\delta_x} \left[\int_0^t \langle X_s,\, \phi\rangle ds \right] = \int_0^t \mathbb{P}_s\phi(x)\mathrm{e}^{-as} ds,$$

which implies that $\mathbb{P}_s\phi(x) = 0$ for almost surely all $s \in (0, t]$. Recall that ξ is a right continuous process and $\Pi_x(\xi_0 = x) = 1$. So there exists $\varepsilon > 0$, such that $\mathbb{P}_s\phi(x) > 0$, for all $0 < s < \varepsilon$, due to $\mathbb{P}_0\phi(x) = \phi(x) > 0$, and then we get a contradiction. Therefore, $U_t(x, \phi) > 0$ for any $t > 0$.

Now we come to prove that $U_t(\phi)$ is uniformly bounded. Here we make use of the maximal principle and a argument similar to that used in the proof of Lemma 9 of [6]. We claim that there exist constants $c,\ d > 0$ such that

$$\psi(\lambda) \geqslant c\lambda - d, \quad \text{for any } \lambda > 0. \tag{8}$$

When $a > 0$, it is easy to check that the above domination holds for $c = a$ and d being an arbitrary positive number. When $a = 0$, we have

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \infty,$$

by our assumption that $\int^\infty \frac{1}{\psi(\lambda)} d\lambda < \infty$ and that $\psi(\lambda)$ is a nonnegative continuous function on $(0,\, \infty)$. So there exist c and d such that (8) holds. Consider the solution of the following evolution equation

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = Au_t - cu_t + \phi + d, \\ u(0) = 0. \end{cases}$$

The solution to this equation is $u(t) = \int_0^t \mathrm{e}^{-c(t-s)}\mathbb{P}_{t-s}(\phi + d)ds$, which satisfies

$$|u(t)| \leqslant (||\phi||_\infty + d)\int_0^t \mathrm{e}^{-cs}ds \leqslant c^{-1}(||\phi||_\infty + d).$$

Finally, the maximal principle and the comparison between $\psi(\lambda)$ and $c\lambda - d$ show that

$$\sup_{t\geqslant 0} ||U_t||_\infty \leqslant \sup_{t\geqslant 0} ||u_t||_\infty \leqslant c^{-1}(||\phi||_\infty + d) < \infty.$$

□

Recall that for any $\mu \in M_F(E)$,

$$P_\mu\left[\exp(-\langle Y_t,\, \phi\rangle)\right] = \exp(-\langle \mu,\, U_t(\phi)\rangle),$$

and Y_t is a nondecreasing function with respect to t. So $U_t(\phi)$ is nondecreasing in t. Since $\{U_t(\phi)\}$ are uniformly bounded by Lemma 3.2, $U_\infty(\phi) = \lim_{t\to\infty} U_t(\phi) < \infty$. Recall the definition of φ given by (5). $\varphi(\lambda)$ is an increasing function and $\varphi(0) = 0$. These come to the following lemma.

Lemma 3.3

$$\lim_{t\to\infty}\int_0^t \varphi(U_s(\phi)(\xi_{t-s}))ds = \lim_{t\to\infty}\int_0^t \varphi(U_{t-s}(\phi)(\xi_s))ds = \int_0^\infty \varphi(U_\infty(\phi)(\xi_s))ds,$$

where φ *is defined by* (5).

Denote by $\{U_t^\lambda\}$ the unique nonnegative bounded mild solution of the equation:

$$\begin{cases} \dfrac{\partial U_t}{\partial t} = AU_t - \psi(U_t) + \lambda\phi, \\ U_0 = 0, \end{cases} \tag{9}$$

where $\phi \in C_c(E)_+$. The argument above shows that $U_\infty^\lambda = \lim_{t\to\infty} U_t^\lambda$ exists. Since U_t^λ are nondecreasing in λ, U_∞^λ is also nondecreasing in λ. Hence $\lim_{\lambda\to 0^+} U_\infty^\lambda$ exists. Furthermore, we have the following result.

Lemma 3.4

$$\lim_{\lambda\to 0^+} U_\infty^\lambda(x) = 0, \qquad \forall x \in E.$$

Proof.

$$\begin{aligned} & P_\mu\left[\exp\{-\lambda\int_0^\infty \langle X_s,\, \phi\rangle ds\}\right] \\ = \;& \lim_{t\to\infty} P_\mu\left[\exp\left\{-\lambda\int_0^t \langle X_s,\, \phi\rangle ds\right\}\right] \\ = \;& \lim_{t\to\infty} \exp\left\{-\langle \mu,\, U_t^\lambda\rangle\right\} = \exp\{-\langle \mu,\, U_\infty^\lambda\rangle\}. \end{aligned} \tag{10}$$

Recall that the superprocess $\{X_t\}$ considered here satisfies $P_\mu(T < \infty) = 1$. So

$$P_\mu\left(\int_0^\infty \langle X_s,\, \phi\rangle ds < \infty\right) \geqslant P_\mu(\exists t > 0,\, X_t(1) = 0) = P_\mu(T < \infty) = 1.$$

That is

$$P_\mu\left(\int_0^\infty \langle X_s,\,\phi\rangle ds<\infty\right)=1.$$

Letting $\lambda\to 0+$ in (10), since μ is arbitrary, we have

$$\lim_{\lambda\to 0^+} U_\infty^\lambda(x)=0,\quad \text{for any } x\in E.$$

□

Lemma 3.5 *If there exists $\lambda>0$, such that $\int_0^\infty \varphi(U_\infty^\lambda(\xi_s))ds<\infty$, $\Pi_{\bar\mu}$-a.s., then for all $\lambda>0$,*

$$\int_0^\infty \varphi(U_\infty^\lambda(\xi_s))ds<\infty,\quad \Pi_{\bar\mu}-\ a.s.$$

Proof. Since $\varphi(u)$ is nondecreasing in u, $\varphi(U_\infty^\lambda)$ is also nondecreasing in λ. Define $U_\infty=U_\infty^1$. We only need to prove that if $\int_0^\infty \varphi(U_\infty(\xi_s))ds<\infty$, then $\int_0^\infty \varphi(U_\infty^\lambda(\xi_s))ds<\infty$, for any $\lambda>1$. By simple calculation, it can be obtained that $1-\mathrm{e}^{-cx}\leqslant c(1-\mathrm{e}^{-x})$, for any $x\geqslant 0$ and any $c\geqslant 1$. Therefore, for any $c\geqslant 1$ and any $\lambda>0$,

$$\begin{aligned}\psi(c\lambda) &= a(c\lambda)+b(c\lambda)^2+\int_0^\infty(\mathrm{e}^{-c\lambda x}-1+c\lambda x)\nu(dx)\\ &\geqslant c\left[a\lambda+b\lambda^2+\int_0^\infty(\mathrm{e}^{-\lambda x}-1+\lambda x)\nu(dx)\right]\\ &= c\psi(\lambda),\end{aligned}$$

and then for $\lambda>1$,

$$\frac{\partial(\lambda U_t)}{\partial t}-A(\lambda U_t)+\psi(\lambda U_t)-\lambda\phi\geqslant\lambda\left[\frac{\partial U_t}{\partial t}-AU_t+\psi(U_t)-\phi\right]=0.$$

(The maximum principle holds for mild solutions, see Appendix of [6].) Noticing that $U_0=0$, we have $U_t^\lambda\leqslant\lambda U_t$ by the maximal principle. Letting $t\to\infty$, we get $U_\infty^\lambda\leqslant\lambda U_\infty$. Note that for any $c>1$ and any $u\geqslant 0$, we have

$$\varphi(cu)=2bcu+\int_0^\infty(1-\mathrm{e}^{-cux})x\nu(dx)\leqslant c\left[2bu+\int_0^\infty(1-\mathrm{e}^{-ux})x\nu(dx)\right]=c\varphi(u).$$

Therefore,

$$\int_0^\infty \varphi(U_\infty^\lambda(\xi_s))ds\leqslant\lambda\int_0^\infty\varphi(U_\infty(\xi_s))ds<\infty,\quad\forall\lambda>1.$$

□

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Recall that $U_t^\lambda=U_t(\lambda\phi)$ and $U_\infty^\lambda=\lim_{t\to\infty}U_t^\lambda=\lim_{t\to\infty}U_t(\lambda\phi)$.

Using (7) and Lemma 3.3, we see that

$$\begin{aligned} & P_\mu^\infty\left(\int_0^\infty \langle X_s,\, \phi\rangle ds<\infty\right)\\ = & \lim_{\lambda\to 0^+} P_\mu^\infty\left[\mathrm{e}^{-\lambda\int_0^\infty\langle X_s,\,\phi\rangle ds}\right]\\ = & \lim_{\lambda\to 0^+}\lim_{t\to\infty} P_\mu^\infty\left[\mathrm{e}^{-\lambda\int_0^t\langle X_s,\,\phi\rangle ds}\right]\\ = & \lim_{\lambda\to 0^+}\lim_{t\to\infty} \mathrm{e}^{-\langle\mu,\,U_t(\lambda\phi)\rangle}\left\langle\bar{\mu},\,\Pi.\exp\left(-\int_0^t\varphi(U_{t-s}(\lambda\phi)(\xi_s))ds\right)\right\rangle\\ = & \lim_{\lambda\to 0^+}\mathrm{e}^{-\langle\mu,\,U_\infty^\lambda\rangle}\left\langle\bar{\mu},\,\Pi.\exp\left(-\int_0^\infty\varphi(U_\infty^\lambda(\xi_s))ds\right)\right\rangle. \end{aligned} \tag{11}$$

It follows from Lemma 3.4 that $\lim_{\lambda\to 0^+}\mathrm{e}^{-\langle\mu,\,U_\infty^\lambda\rangle}=1$. Since $\varphi(0)=0$, applying the dominated convergence theorem and Lemma 3.5, we can get

$$\lim_{\lambda\to 0^+}\int_0^\infty\varphi(U_\infty^\lambda(\xi_s))ds=\begin{cases}0, & \text{if}\displaystyle\int_0^\infty\varphi(U_\infty(\xi_s))ds<\infty;\\ \infty, & \text{otherwise.}\end{cases}$$

Applying the dominated convergence theorem again,

$$P_\mu^\infty\left(\int_0^\infty\langle X_s,\,\phi\rangle ds<\infty\right)=\Pi_{\bar{\mu}}\left(\int_0^\infty\varphi(U_\infty(\xi_s))ds<\infty\right).$$

□

In the remaining part of this section we give some applications of Theorem . For an open subset O of $\mathbb{R}^d$, we use $C_c^2(O)$ to denote the space of twice continuously differentiable functions on O, and $C_c^2(O)_+$ denote the subset of $C_c^2(O)$ of all nonnegative members in $C_c^2(O)$.

Theorem 3.6 *Let ξ be an α-stable process, $\alpha\in(0,2]$. Let (X_t,P_μ) be a $(d,\,\alpha,\,\beta)$-superprocess with $\beta\in(0,\,1]$ and $\mu\in M_F(\mathbb{R}^d)\backslash\{0\}$. Suppose that $\phi\in C_c^2(\mathbb{R}^d)_+$.*

i) If $d>\alpha+\frac{\alpha}{\beta}$, then $\int_0^\infty\langle X_s,\phi\rangle ds<\infty$, $P_\mu^\infty-a.s.$

ii) If ξ is a Brownian motion and $2<d\leqslant 2+2/\beta$, then

$$\int_0^\infty\langle X_s,\phi\rangle ds=\infty,\quad P_\mu^\infty-a.s.$$

To prove Theorem 3.6, we need two more lemmas.

Let $u(t,\,r),\ r\geqslant 0$ satisfies the radially symmetric version of equation (3)

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,\,r)=u''(r)+\dfrac{(d-1)}{r}u'(r)-u^{1+\beta}(r)+\phi(r),\\ u(0,\,r)=0, \qquad\qquad r\in\mathbb{R}_+^1,\,\phi\in C_c^2(\mathbb{R}_+^1)+.\end{cases} \tag{12}$$

($\mathbb{R}_+^1$ is the positive semi-axis of $\mathbb{R}^1$). Set $u(|x|)=\lim_{t\to\infty}u(t,|x|)$, $x\in\mathbb{R}^d$. By Theorem 3.3 in [13], $u(r),\ r\in\mathbb{R}_+^1$ is the solution of the partial differential equation

$$u''(r)+\frac{(d-1)}{r}u'(r)-u^{1+\beta}(r)+\phi(r)=0,\quad r\in\mathbb{R}_+^1,\,\phi\in C_c^2(\mathbb{R}_+^1). \tag{13}$$

Lemma 3.7 (Theorem 4.2[13]) *For $u(r)$ stated above, when $r \to \infty$, the following hold: ($f(r) \sim g(r)$ means $\lim_{r\to\infty} f(r)/g(r) = 1$)*

$$\begin{aligned} d = 1, \quad & u(r) \sim \Big[c + (\tfrac{\beta}{2})[2/(2+\beta)]^{\frac{1}{2}} \cdot r\Big]^{-2/\beta}; \\ d \geqslant 2, \quad & (d-2)\beta < 2: \quad u(r) \sim c_{\beta,\,d}\, r^{-2/\beta}; \\ & (d-2)\beta = 2: \quad u(r) \sim c_d\, [r^2 \log r]^{-1/\beta}; \\ & (d-2)\beta > 2: \quad u(r) \sim c_{\beta,\,d}\, r^{2-d}, \end{aligned}$$

where $c, c_{\beta,d}, c_d$ are some constants and may be different from place to place, $c_{\beta,d}$ dependents only on β and d, and c_d depends only on d.

Lemma 3.8 *If $\phi \in C_c^2(\mathbb{R}^d)_+$ is a radial function, ξ is a Brownian motion, and $(d-2)\beta \leqslant 2$, then $\int_0^\infty \langle X_s, \phi\rangle ds = \infty$, $\quad P_\mu^\infty - a.s.$*

Proof. If ξ is a Brownian motion and $d = 1,\ 2$, the conclusions have been proved in [6]. When $d \geqslant 3$, we have $\lim_{t\to\infty} |\xi_t| = \infty$, Π_x- a.s., due to the transience property of the Brownian motions. For any radial function f, with abuse of notation, we write $f(x) = f(|x|)$. Recall that ϕ is a radial function, so $U_t(\phi)$ is also radial and satisfies equation (12). Note that $U_\infty(\phi) = \lim_{t\to\infty} U_t(\phi)$. For simplicity, we denote $U_\infty(\phi)$ by U_∞. Then $U_\infty(x)$ is a radial function, i.e., $U_\infty(x) = U_\infty(|x|)$ for any $x \in \mathbb{R}^d$, and by similar arguments in the proof of Theorem 3.3 [13], we can prove that U_∞ is a solution of (13). By Theorem 3.1, we only need to prove that

$$\Pi_x\left(\int_0^\infty [U_\infty(\xi_s)]^\beta ds = \infty\right) = 1. \tag{14}$$

Note that, for any $t > 0$,

$$\begin{aligned} \Pi_x\left(\int_0^\infty [U_\infty(\xi_s)]^\beta ds = \infty\right) &= \Pi_0\left(\int_0^\infty [U_\infty(|\xi_s + x|)]^\beta ds = \infty\right) \\ &\geq \Pi_0\left(\int_t^\infty [U_\infty(|\xi_s + x|)]^\beta ds = \infty\right). \end{aligned}$$

Then we have the following domination:

$$\begin{aligned} & \Pi_x\left(\int_0^\infty [U_\infty(\xi_s)]^\beta ds = \infty\right) \\ \geqslant\ & \lim_{t\to\infty} \Pi_0\left(\int_t^\infty [U_\infty(|\xi_s + x|)]^\beta ds = \infty\right) \\ \geqslant\ & \lim_{t\to\infty} \Pi_0\left(\int_t^\infty [U_\infty(|\xi_s + x|)]^\beta ds = \infty, \sup_{s\geqslant t} \frac{|\xi_s|}{\sqrt{s\log\log s}} \leqslant 3\right) \\ =\ & \lim_{t\to\infty} \Pi_0\Big(\int_t^\infty [U_\infty(|\xi_s + x|)(|\xi_s + x|)^{2/\beta}]^\beta (|\xi_s + x|)^{-2} ds = \infty, \\ & \sup_{s\geqslant t} \frac{|\xi_s + x|}{\sqrt{s\log\log s}} \leqslant 3\Big) \\ \geqslant\ & \lim_{t\to\infty} \Pi_0\Big(\int_t^\infty [U_\infty(|\xi_s + x|)(|\xi_s + x|)^{2/\beta}]^\beta (s\log\log s)^{-1} ds = \infty, \\ & \sup_{s\geqslant t} \frac{|\xi_s + x|}{\sqrt{s\log\log s}} \leqslant 3\Big). \end{aligned}$$

In the cases of $0 < (d-2)\beta < 2$, according to Lemma 3.7, almost surely there exists $T > 0$, such that for any $t > T$, $U_\infty(|\xi_t + x|)(|\xi_t + x|)^{2/\beta} > c_{\beta,d}/2$. Also note that for arbitrary $t > 0$, $\int_t^\infty \frac{1}{s \log\log s} ds = \infty$. Then we get that

$$\Pi_x \left(\int_0^\infty [U_\infty(\xi_s)]^\beta ds = \infty \right) \geqslant \lim_{t\to\infty} \Pi_0 \left(\sup_{s \geqslant t} \frac{|\xi_s + x|}{\sqrt{s \log\log s}} \leqslant 3 \right).$$

By the iterate logarithm law, for any $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \Pi_0 \left(\sup_{s \geqslant t} \frac{|\xi_s + x|}{\sqrt{s \log\log s}} \leqslant 3) \right) = 1,$$

and then we get (14).

In the cases of $(d-2)\beta = 2$, the result can be proved by a similar argument and by noticing that $\int_t^\infty (r \log r \log\log r)^{-1} dr = \infty$ for any $t > 0$. We complete the proof. $\square$

Proof of Theorem 3.6. i) According to Theorem 3.1, we only need to prove that $\Pi_{\bar\mu} - a.s.$, $\int_0^\infty [U_\infty(\xi_s)]^\beta ds < \infty$, Since U_∞ is bounded, we only need to prove that $\int_1^\infty [U_\infty(\xi_s)]^\beta ds < \infty$, $\Pi_{\bar\mu}$-a.s. By Hölder's inequality,

$$\begin{aligned} \Pi_x \int_1^\infty [U_\infty(\xi_s)]^\beta ds &= \int_1^\infty \Pi_x [U_\infty(\xi_s)]^\beta ds \\ &\leqslant \int_1^\infty \Big[\Pi_x U_\infty(\xi_s) \Big]^\beta ds \\ &= \int_1^\infty [\mathbb{P}_s U_\infty(x)]^\beta ds. \end{aligned}$$

Since $U_\infty(x) \leq \int_0^\infty \mathbb{P}_u \phi(x) du$, we can continue the above domination:

$$\begin{aligned} \Pi_x \int_1^\infty [U_\infty(\xi_s)]^\beta ds &\leqslant \int_1^\infty \Big[\mathbb{P}_s \int_0^\infty \mathbb{P}_u \phi(x) du \Big]^\beta ds \\ &= \int_1^\infty \Big[\int_0^\infty \mathbb{P}_{s+u} \phi(x) du \Big]^\beta ds \\ &\leqslant c||\phi||_1^\beta \int_1^\infty \Big[\int_0^\infty (s+u)^{-\frac{d}{\alpha}} du \Big]^\beta ds \\ &= c||\phi||_1^\beta \int_1^\infty s^{(1-\frac{d}{\alpha})\beta} ds. \end{aligned}$$

In the last inequality of the above domination, we used the fact that the density $p(t,x)$ of an α-stable process satisfies $p(t,x) \leq C_{d,\alpha} t^{-d/\alpha}$ for some constant $C_{d,\alpha}$. When $d > \alpha + \frac{\alpha}{\beta}$, $\int_1^\infty s^{(1-\frac{d}{\alpha})\beta} ds < \infty$, and then $\int_0^\infty [U_\infty(\xi_s)]^\beta ds < \infty$, $\Pi_{\bar\mu}$-a.s. Therefore, $\int_0^\infty \langle X_s, \phi \rangle ds < \infty$, P_μ^∞-a.s.

ii) Now suppose that O is an open subset of $\mathbb{R}^d$. Choose $r > 0$ and $x \in O$ such that $B(x, r) \subset O$. Since the operator $\triangle$ and the Lebesgue measure are both translation invariant. There is no loss of generality we assume that $x = 0$. Then there exists a function $f \in C_c^2(\mathbb{R}^1)_+$, such that $f(|x|) \leqslant I_O(x)$. Lemma shows that $\int_0^\infty \langle X_s, f \rangle ds = \infty$, P_μ^∞-a.s., which implies that

$$\int_0^\infty X_s(O) = \infty \quad P_\mu^\infty - \text{a.s.}$$

For any function $\phi \in C_c^2(\mathbb{R}^d)_+$, there must be a constant $a > 0$, such that $\{x \in \mathbb{R}^d : \phi(x) > a\}$ is a nonempty open set. The conclusion we have got for open sets implies that

$$\int_0^\infty \langle X_s, \phi\rangle ds = \infty, \quad P_\mu^\infty - \text{a.s.} \qquad \square$$

Remark 3.9 *If $\psi'(0) > 0$ (subcritical case), using the argument used in the proof of i), we see that*

$$\begin{aligned} \Pi_x\left[\int_1^\infty [U_\infty(\xi_s)]^\beta ds\right] &\leqslant \int_1^\infty \left[\mathbb{P}_s \int_0^\infty \mathbb{P}_u\phi(x)e^{-\alpha u}du\right]^\beta ds \\ &= \int_1^\infty \left[\int_0^\infty \mathbb{P}_{u+s}\phi(x)e^{-\alpha u}du\right]^\beta ds \\ &\leqslant C||\phi||_1^\beta \int_1^\infty \left[\int_s^\infty u^{-\frac{d\beta}{\alpha}} e^{-\alpha(u-s)\beta}du\right] ds \\ &\leqslant C||\phi||_1^\infty \int_1^\infty s^{-\frac{d\beta}{\alpha}} ds, \end{aligned}$$

where C is a positive constant which may change values from line to line. Therefore, if $d > \dfrac{\alpha}{\beta}$, then

$$\int_0^\infty \langle X_s,\, \phi\rangle ds < \infty, \quad P_\mu^\infty - a.s.$$

4 Local extinction property of the conditioned binary super-Brownian motion

In this section, we will focus on the local extinct property of the conditioned binary super-Brownian motion in $\mathbb{R}^d$. Let's recall some definitions from [11] and [14]. A Borel measurable subset of $\mathbb{R}^d$ is *proper*, if it is bounded and has positive Lebesgue measure. A measure valued process $\{\mu_t, P\}$ is called *stochastically bounded*, if for any finite ball $B \subset \mathbb{R}^d$,

$$\lim_{M\to\infty} \limsup_{t\to\infty} P(\mu_t(B) \geqslant M) = 0.$$

$\{\mu_t, P\}$ is called *locally extinct*, if for any $\varepsilon > 0$, and any proper set B

$$\lim_{t\to\infty} P(\mu_t(B) > \varepsilon) = 0.$$

If $\{\mu_t, P\}$ is not stochastically bounded, then $\{\mu_t, P\}$ is called *unstable*.

Remark 4.1 *Traditionally, the local extinct properties of the superprocesses are discussed in almost sure sense. But here we only consider the local extinct properties of the conditioned superprocesses in the probability sense.*

Let B be any proper set, then by Theorem 3.6, we can get that when $d \geqslant 5$, for any $\mu \in M_F(\mathbb{R}^d)\backslash\{0\}$,

$$\int_0^\infty X_s(B)ds < \infty, \quad P_\mu^\infty - \text{a.s.}$$

So when $d \geqslant 5$, $\lim_{t\to\infty} X_t(B) = 0$, P_μ^∞-a.s. Now we consider the cases of $d \leq 4$. Without loss of generality, here we just consider the case of $\mu = \delta_x(\cdot)$. Through simple calculation, it is easy to obtain

$$P_{\delta_x}^\infty[X_t(B)] = (1 + 2t)\mathbb{P}_t I_B(x).$$

Therefore, when $d \geqslant 3$, $P^{\infty}_{\delta_x}[X_t(B)] \to 0$ as $t \to \infty$. Hence when $d = 3, 4$, $X_t(B) \to 0$ in $P^{\infty}_{\delta_x}$ probability as $t \to \infty$, which implies that when $d \geqslant 3$, the conditioned binary super-Brownian motion is locally extinct.

Theorem 4.2 *When $d = 1$, $\{X_t,\ P^{\infty}_{\mu}\}$ is unstable. Moreover, under P^{∞}_{μ}, $\frac{1}{\sqrt{t}} X_t \Rightarrow \eta\lambda$, as $t \to \infty$, where λ is Lebesgue measure and η is a nonnegative random variable with Laplace transform*

$$Ee^{-\theta\eta} = \Pi_0 \exp\left(-\int_0^1 G(1-s,\, \xi_s,\, (1-s)^{3/2}\theta)ds\right), \tag{15}$$

where $G(t,\, x,\, \theta)$ is the unique nonnegative mild solution of the following equation

$$\begin{cases} \dfrac{\partial G_t}{\partial t} = \dfrac{1}{2}\Delta G_t - G_t^2; \\ G_0 = \theta\delta_0. \end{cases} \tag{16}$$

To prove this theorem we need a result (see Lemma 4.3 below) from [14]. Define

$$M_{\exp} = \left\{ f \geqslant 0 :\ f \text{ is Borel measurable },\ \exists\ k,\ m > 0,\ \text{ such that }\ f(x) \leqslant k\mathrm{e}^{-|x|^2/2m} \right\}.$$

It is obvious that for any proper set B considered here, $I_B \in M_{\exp}$.

Lemma 4.3 (Theorem1.3[11]) *Assume $d = 1, h \in M_{\exp}$, $G(t,\ x,\ \theta)$ is the unique nonnegative mild solution of equation* (16)*, and $V(t,\ x,\ h)$ is the mild solution of equation* (16) *with initial value h, then*

$$\lim_{t\to\infty} tV(t-s,\, t^{1/2}y,\, t^{-1/2}h) = G(1,\, y,\, \bar{h}), \quad \forall s \in \mathbb{R},\ y \in \mathbb{R},$$

where $\bar{h} = \int h(x)dx$.

We also need the following well known result (see Exercise 3.8 in [12], for example).

Lemma 4.4 *If X is a nonnegative random variable, and $EX < \infty$, then for all $0 < r < 1$,*

$$P(X > rEX) \geqslant (1-r)^2 \frac{(EX)^2}{EX^2}.$$

Proof of Theorem 4.2. Note that $P^{\infty}_{\delta_x}\left[X_t^2(B)\right] = P_{\delta_x}\left[X_t(B)^2 X_t(1)\right]$. By Lemma 2.2 in [15], for any $\phi \in C_c^2(\mathbb{R}^d)_+$,

$$\begin{aligned} & P_{\delta_x}\left[\langle X_t, \phi\rangle^2 X_t(1)\right] \\ = \ & [\mathbb{P}_t\phi(x)]^2 + 4t[\mathbb{P}_t\phi(x)]^2 + 2\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s\phi(x))^2 ds + \\ & 4t\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s\phi(x))^2 ds + 4\int_0^t s\mathbb{P}_{t-s}(\mathbb{P}_s\phi(x))^2 ds. \end{aligned}$$

For any bounded ball $B \subset \mathbb{R}^d$, there exist a sequence of uniformly bounded $\{\phi_n\} \subset C_c^2(\mathbb{R}^d)_+$ such that ϕ_n converge to I_B a.e. with respect to Lebesgue measure. By the dominated convergence theorem,

$$\begin{aligned} P^{\infty}_{\delta_x}\left[X_t^2(B)\right] &= [\mathbb{P}_t I_B(x)]^2 + 2\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s I_B(x))^2 ds + 4t^2[\mathbb{P}_t I_B(x)]^2 \\ &\quad + 4t\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s I_B(x))^2 ds + 4\int_0^t s\mathbb{P}_{t-s}(\mathbb{P}_s I_B(x))^2 ds. \end{aligned}$$

Recall that

$$P^{\infty}_{\delta_x}[X_t(B)] = (1+2t)\mathbb{P}_t I_B(x).$$

So when $d=1$,

$$\limsup_{t\to\infty} \frac{[P^{\infty}_{\delta_x} X_t(B)]^2}{P^{\infty}_{\delta_x}[X_t^2(B)]} \geqslant \frac{\frac{2}{\pi}|B|^2}{\liminf_{t\to\infty} 8\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s I_B(x)^2)ds}.$$

Since

$$\int_0^t \mathbb{P}_{t-s}(\mathbb{P}_s I_B(x))^2 ds \leqslant \int_0^t \frac{1}{\sqrt{2\pi s}}|B|\mathbb{P}_t I_B(x)ds = \frac{\sqrt{t}|B|}{\sqrt{2\pi}}\mathbb{P}_t I_B(x) \leqslant \frac{|B|^2}{2\pi},$$

we obtain

$$\limsup_{t\to\infty} \frac{[P^{\infty}_{\delta_x} X_t(B)]^2}{P^{\infty}_{\delta_x}[X_t^2(B)]} \geqslant \frac{1}{2}.$$

By Lemma 4.4, for $0<r<1$,

$$\limsup_{t\to\infty} P_{\delta_x}(X_t(B) \geqslant rP^{\infty}_{\delta_x} X_t(B)) \geqslant \frac{1}{2}(1-r)^2.$$

Meanwhile, $\lim_{t\to\infty} P^{\infty}_{\delta_x}\big[X_t(B)\big] = \lim_{t\to\infty}(1+2t)\mathbb{P}_t I_B(x) = \infty$. So $\{X_t, P^{\infty}_{\mu}\}_{t\geqslant 0}$ is unstable.

Now let us prove the weak convergence. For any proper set B,

$$\begin{aligned} P^{\infty}_{\delta_x}\Big[\exp\{-\theta t^{-1/2}X_t(B)\}\Big] \;&=\; \Pi_x\left[\exp\left\{-\int_0^t V(t-s,\,\xi_s,\,\theta t^{-1/2}I_B)ds\right\}\right] \\ &\quad\cdot\exp\{-V(t,x,\theta t^{-1/2}I_B)\}, \end{aligned} \tag{17}$$

Where V is the unique mild solution of (1) with $A=\frac{1}{2}\Delta$, $\psi(\lambda)=\lambda^2$ and $\phi=\theta t^{-1/2}I_B$. Since

$$V(t,x,\theta t^{-1/2}I_B) = \mathbb{P}_t\theta t^{-1/2}I_B(x) - \int_0^t \mathbb{P}_{t-s}V^2(s,x,\theta t^{-1/2}I_B)ds \leqslant \theta t^{-1/2},$$

we just need to consider the limit of the first part of the right side of identity (17).

$$\begin{aligned} &\Pi_x \exp\left\{-\int_0^t V(t-s,\,\xi_s,\,\theta t^{-1/2}I_B)ds\right\} \\ =\;& \Pi_0 \exp\left\{-\int_0^t V(t-s,\,\xi_s+x,\,\theta t^{-1/2}I_B)ds\right\} \\ =\;& \Pi_0 \exp\left\{-\int_0^1 tV(t(1-s),\,t^{1/2}(\xi_s+xt^{-1/2}),\,\theta t^{-1/2}I_B)ds\right\}. \end{aligned}$$

If $s\neq 1$, then by the uniqueness of the nonnegative mild solution of equation (1),

$$\begin{aligned} &tV(t(1-s),\,t^{1/2}(\xi_s+xt^{-1/2}),\,\theta t^{-1/2}I_B) \\ =\;& V(1-s,\,\xi_s+xt^{-1/2},\,\theta t^{1/2}I_B(t^{1/2}\cdot)) \\ \leqslant\;& (1-s)^{-1/2}\theta|B|, \end{aligned}$$

and

$$tV(t(1-s),\,t^{1/2}(\xi_s+xt^{-1/2}),\,\theta t^{-1/2}I_B) = tV(t(1-s),\,t^{1/2}\xi_s,\,\theta t^{-1/2}I_B(\cdot+x)).$$

By Lemma 4.3, for any $s \neq 1$,

$$
\begin{aligned}
& \lim_{t\to\infty} tV(t(1-s),\, t^{1/2}(\xi_s + xt^{-1/2}),\, \theta t^{-1/2} I_B) \\
= & \lim_{t\to\infty} \frac{t(1-s)}{(1-s)} V\left(t(1-s), t^{1/2}(1-s)^{1/2}((1-s)^{-1/2}\xi_s), t^{-1/2} I_B(\cdot + x)\right) \\
= & (1-s)^{-1} G(1,\, (1-s)^{-1/2}\xi_s,\, (1-s)^{1/2}\theta|B|) \\
= & G(1-s,\, \xi_s,\, (1-s)^{3/2}\theta|B|).
\end{aligned}
$$

The solution of equation (16) is continuous depend on the initial value, so the right hand of the identity (15) is a Laplace function of some random variable denoted by η. By the dominated convergence theorem and the inner regularity property of the finite measure on $\mathcal{B}(\mathbb{R}^d)$, the result follows. $\square$

When $d = 2$, the result is quite different from the case of $d = 1$. Since any proper set can be covered by a finite ball, without lose of generality, we assume that set B considered in the remaining is a finite ball.

Theorem 4.5 *If $d = 2$, $\{X_t,\ P_\mu^\infty\}$ is locally extinct.*

To prove this we need to use the technology of super solution of an evolution equation and a result from [14], which we present here directly without proof.

Lemma 4.6 (Proposition 1[14]) *Suppose $V(t,\ x) \geqslant 0$ solves the partial differential equation in dimension $d = 2$,*

$$
\begin{cases} \dfrac{\partial V_t}{\partial t} = \dfrac{1}{2}\triangle V_t - V_t^2 \\ V(0,x) = I_B(x). \end{cases} \tag{18}
$$

Then there is $A > 0$, when $t > e^4$,

$$
V(t,\, x) \leqslant \phi(x, 8t), \quad \textit{where } \ \phi(x,t) = \frac{A}{t\log t}\exp\left(-\frac{|x|^2}{t}\right).
$$

Proof of Theorem 4.5**.** By identity (6), if the limits concerned below exist then

$$
\lim_{t\to\infty} P_{\delta_x}^\infty \mathrm{e}^{-X_t(B)} = \lim_{t\to\infty} \Pi_x \mathrm{e}^{-\int_0^t V(s,\,\xi_{t-s})ds} \lim_{t\to\infty} \mathrm{e}^{-V(t,x)},
$$

where V is defined as in Lemma . Note that $1/(1+t)$ is the solution of equation (18) with initial value 1. By the maximal principle, $V(t,\ x) \leqslant 1/(1+t)$, and then $\lim_{t\to\infty} V(t,\ x) = 0$. If we can prove that

$$
\lim_{t\to\infty} \Pi_x \int_0^t V(s,\ \xi_{t-s})ds = 0, \tag{19}
$$

then $\lim_{t\to\infty} P_{\delta_x}^\infty \mathrm{e}^{-X_t(B)} = 1$, from which we see that $X_t(B)$ approaches 0 as $t \to \infty$ in probability with respect to $P_{\delta_x}^\infty$.

Now we only need to prove (19). For $t \geqslant \mathrm{e}^4$, we have

$$
\begin{aligned}
\Pi_x \int_0^t V(s,\, \xi_{t-s})ds \quad &= \quad \Pi_x\left[\int_{\mathrm{e}^4}^t V(s,\, \xi_{t-s})ds\right] + \Pi_x\left[\int_0^{\mathrm{e}^4} V(s,\, \xi_{t-s})ds\right] \\
&\overset{def}{=} \quad I + II.
\end{aligned}
$$

From Lemma 4.6, we dominate I:

$$\begin{aligned}
I &\leqslant \Pi_x \int_{e^4}^t \frac{A}{8s\log 8s}\exp\left(-\frac{|\xi_{t-s}|^2}{8s}\right)ds \\
&= \int_{e^4}^t \frac{A}{8s\log 8s}\int_{\mathbb{R}^2}\frac{1}{2\pi(t-s)}\exp\left(-\frac{|y-x|^2}{2(t-s)}-\frac{|y|^2}{8s}\right)dyds \\
&= \int_{e^4}^t \frac{A}{8s\log 8s}\int_{\mathbb{R}^2}\frac{1}{2\pi(t-s)}\exp\left(-\frac{|y|^2}{2(t-s)}-\frac{|y+x|^2}{8s}\right)dyds \\
&\leq \exp\left(\frac{|x|^2}{8e^4}\right)\int_{e^4}^t \frac{A}{8s\log 8s}\int_{\mathbb{R}^2}\frac{1}{2\pi(t-s)}\exp\left(-\frac{|y|^2}{2(t-s)}-\frac{|y|^2}{8s}\right)dyds \\
&= \exp\left(\frac{|x|^2}{8e^4}\right)\int_{e^4}^t \frac{A}{8s\log 8s}\int_{\mathbb{R}^2}\frac{1}{2\pi(t-s)}\exp\left(-\frac{|y|^2}{8\frac{(t-s)s}{t+3s}}\right)dyds \\
&= \exp\left(\frac{|x|^2}{8e^4}\right)\int_{e^4}^t \frac{A}{2\log(8s)(t+3s)}ds \\
&\leq \exp\left(\frac{|x|^2}{8e^4}\right)\frac{1}{t}\int_{e^4}^t \frac{1}{\log(8s)}ds.
\end{aligned}$$

Since

$$\lim_{t\to\infty}\frac{1}{t}\int_{e^4}^t\frac{1}{\log(8s)}ds = \lim_{t\to\infty}\frac{1}{\log(8t)} = 0,$$

we have

$$\lim_{t\to\infty} I = 0. \tag{20}$$

To estimate II, note that $V(t,x)\leqslant \mathbb{P}_t I_B(x)$, so

$$II \leqslant \int_0^{e^4}\Pi_x\mathbb{P}_s I_B(\xi_{t-s})ds = \int_0^{e^4}\mathbb{P}_t I_B(x)ds \leqslant e^4\frac{1}{2\pi t}|B|\to 0, \text{ as } t\to\infty. \tag{21}$$

From (20) and (21), we get (19). We finish the proof. □

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