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A LARGE DEVIATION FOR OCCUPATION TIME OF SUPER α-STABLE PROCESS

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We derive a large deviation principle for occupation time of super α-stable process in $\mathbb{R}^d$ with $d > 2\alpha$. The decay of tail probabilities is shown to be exponential and the rate function is characterized. Our result can be considered as a counterpart of Lee's work on large deviations for occupation times of super-Brownian motion in $\mathbb{R}^d$ for dimension $d > 4$ (see Ref. 10).

Keywords: Large deviation; super α-stable process; occupation time; semi-linear PDE.

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1. Introduction and Main Result

Assume that $\xi = \{\xi_t, t \geq 0\}$ is a symmetric α-stable process in $\mathbb{R}^d$, whose characteristic function is $e^{-t|z|^\alpha}$. Let S_t^α denote the corresponding semigroup and $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ the infinitesimal generator. The domain of Δ_α is denoted by $\mathcal{D}(\Delta_\alpha)$.

Let $M(\mathbb{R}^d)$ denote the space of positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$, the Borel σ-algebra of $\mathbb{R}^d$. $M(\mathbb{R}^d)$ carries the vague topology. $C(\mathbb{R}^d)$ denotes the Banach space of continuous bounded function on $\mathbb{R}^d$ equipped with the usual sup norm $\|\cdot\|$. We also define:

$$M_r(\mathbb{R}^d) = \{\mu \in M(\mathbb{R}^d) : (1 + |x|^r)^{-1} d\mu(x) \text{ is a finite measure}\}\,, \quad r > 0\,.$$

$$C_r(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \|f(x) \cdot |x|^r\| < \infty\}\,, \quad C_r(\mathbb{R}^d)_+ = \{f \in C_r(\mathbb{R}^d), f \geq 0\}\,.$$

Let $D(\mathbb{R}^1_+, M_r(\mathbb{R}^d))$ denote the Polish space of cádlág paths form $\mathbb{R}^1_+$ to $M_r(\mathbb{R}^d)$ with the Skorokhod J_1-topology. And let $0 < \alpha \leq 2$ and $r < d+\alpha$ in case $\alpha < 2$. According to Ref. 13, there exists an $M_r(\mathbb{R}^d)$-valued Markov process X_t, with sample paths in $D(\mathbb{R}^1_+, M_r(\mathbb{R}^d))$ almost surely, such that

$$E_\mu[\exp(-\langle \psi, X_t\rangle)] = \exp[-\langle u(t), \mu\rangle]\,, \qquad \mu \in M_r(\mathbb{R}^d), \psi \in C_r(\mathbb{R}^d)_+\,,$$

where $u(t)$ is the mild solution of the evolution equation:

$$\begin{cases} \dot{u}(t) = \Delta_\alpha u(t) - u^2(t)\,, \\ u(0) = \psi\,. \end{cases}$$

We call $X = (X_t; t \geq 0)$ the super α-stable process. Specially, when $\alpha = 2$, the process is known as super Brownian-motion.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be Hölder-continuous with compact support. Define the V-weighted occupation time $D_{T,V}$ by

$$D_{T,V} = \frac{1}{T}\int_0^T \int V(x) X_s(dx)ds\,.$$

Assume that $V \geq 0$ and V is not identically equal to zero. Iscoe and Lee[14] studied the large deviations for occupation times of super-Brownian motion in dimensions 3 and 4, and their results show slower than exponential decay of tail probabilities. Lee[10] studied the large deviation results of super-Brownian motion in dimension $d > 4$. Lee proved that for dimension $d > 4$ the tail (as $T \to \infty$) probability that $\frac{1}{T}\int_0^T X_s ds$ deviates from Lebesgue measure (i.e. $D_{T,V}$ deviates from $\int_{\mathbb{R}^d} V(x)dx$) decays exponentially, and showed that the rate function is

$$I(\gamma) = \int_{\mathbb{R}^d} \frac{(\Delta\gamma)(x)^2}{4\gamma(x)} dx\,, \qquad \gamma > 0, \gamma - 1 \in C^2(\mathbb{R}^d) \text{ and has compact support}\,.$$

Where Δ is the Laplace operator on $\mathbb{R}^d$. Now assume that $\alpha \in (0,2)$ and that $X = (X_t, t \geq 0)$ is a super α-stable process on $\mathbb{R}^d$. There have been some results about Super α-stable process when dimension d satisfies $d < 2\alpha$. Zhang[18] generalized results of Iscoe and Lee[14] to super α-stable process with dimension $d = 2$ $(< 2\alpha)$. We note that Zhang's result in Ref. 18 is just for dimension $d = 2$, but his result remains true when $d < 2\alpha$ and his proofs also work for the general case $d < 2\alpha$.

The question that we are going to address in this paper is the following: can one estimate the asymptotic probability that $\frac{1}{T}\int_0^T X_s ds$ deviates from Lebesgue measure when dimension $d > 2\alpha$? As far as we know, this question has not been addressed in the literature. Lee's proof in Ref. 10 was based on analytic techniques of PDEs related to the operator Δ. But these techniques are unclear for nonlocal operator Δ_α, the generator of symmetric α-stable process. It seems that, to answer the question above, one has to develop some results on solutions of differential equations related to Δ_α. In this paper, we are going to tackle the question above by using Dirichlet form of symmetric α-stable process and results on Schrödinger operator $\Delta_\alpha + \mu$ with μ in class $K^{\infty}_{d,\alpha}$ (see Definition 2.1) developed by Takeda and Uemura (see Ref. 15).

For the occupation time process $\frac{1}{T}\int_0^T X_s ds$ of super α-stable process X, Fleischman and Gärtner[7] proved the strong law that as $T \to \infty$, $\frac{1}{T}\int_0^T X_s ds$ converges (in the vague topology) with probability one to the Lebesgue measure λ when $d > \alpha$. This is the starting point of large deviation theory.

The following theorem is our main result:

Theorem 1.1. *Let $d > 2\alpha$, define*

$$A = \{V : V \textit{ is a Hölder continuous function in } \mathbb{R}^d \textit{ with compact support}\},$$

$$\mathcal{G} = \Bigg\{\gamma : \gamma \textit{ is a Borel measurable function in } \mathbb{R}^d \text{ satisfying:}$$

$$\inf_x \gamma(x) > 0\,, \qquad \gamma - 1 \in \mathcal{D}(\Delta_\alpha), \qquad \int_{\mathbb{R}^d} |\Delta_\alpha(\gamma-1)(x)|^2 dx < \infty \Bigg\}.$$

Assume that the functions $V_1, V_2, \ldots, V_n$ are non-negative and belong to the class A. Define

$$\mathbf{D}_{T,V} = \left(\frac{1}{T}\int_0^T \int V_1(x)X_s(dx)ds, \frac{1}{T}\int_0^T \int V_2(x)X_s(dx)ds, \ldots, \frac{1}{T}\int_0^T \int V_n(x)X_s(dx)ds\right).$$

Then there exists a neighborhood O of

$$\left(\int_{\mathbb{R}^d} V_1(x)dx, \int_{\mathbb{R}^d} V_2(x)dx, \ldots, \int_{\mathbb{R}^d} V_n(x)dx\right)$$

such that if $U \subset O$ is open and $C \subset O$ is closed, then

$$\liminf_{T\to\infty} \frac{1}{T}\log P(\mathbf{D}_{T,V} \in U) \geq -\inf_{\lambda\in U} I(\gamma)\,,$$

$$\limsup_{T\to\infty} \frac{1}{T}\log P(\mathbf{D}_{T,V} \in C) \leq -\inf_{\lambda\in C} I(\gamma)\,,$$

where $\lambda = \int V(x)\gamma(x)dx$, $I(\gamma) = \int_{\mathbb{R}^d} \frac{(\Delta_\alpha(\gamma-1)(x))^2}{4\gamma(x)}dx, \gamma \in \mathcal{G}$.

Remark 1.1. When the function V satisfies $\int_{\mathbb{R}^d} V(x)dx = 0$, the large deviation for $D_{T,V}$ of super-Brownian motion with dimension $d = 3$ was first studied by Lee and Remillard[11] in 1995, and left a conjecture. In 1998, Deuschel and Rosen[5] developed complete large deviations for $D_{T,V}$ of super-α stable process $(1 < \alpha \leq 2)$ with dimension $d < 2\alpha < 2 + d$, and answered Lee and Remillard's conjecture.

To prove the main theorem, we need to develop some results on the following two nonlinear differential equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\alpha u + |u|^p + V(x)\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\,, \\ u(0) = 0\,, \end{cases} \tag{1.1}$$

and

$$\Delta_\alpha u + |u|^p + V(x) = 0\,, \qquad x \in \mathbb{R}^d\,. \tag{1.2}$$

We will prove that if $d > \frac{\alpha p}{p-1}$ $(p > 1)$, $V \in A$ and $|V| \le M(1+|x|)^{(\alpha-d)p}$, $x \in \mathbb{R}^d$ for some constant $M > 0$, then Eq. (1.1) has a unique proper solution (see definition given in Sec. 3) $u(t,x,V)$; furthermore, the limit function

$$u(x,V) \equiv \lim_{t\to\infty} u(t,x,V)$$

exists pointwise and is a proper solution of Eq. (1.2). This result is of independent interest and will be addressed in Sec. 3.

2. Preparation: α-stable Schrödinger Operator

In this section, we will give some results about symmetric α-stable process and related Schrödinger operator $\Delta_\alpha + \mu$. The symmetric α-stable process $\xi = \{\xi_t, t \ge 0\}$ in $\mathbb{R}^d$ has a transition density $p(t,x,y) = p(t,x-y)$ with respect to the Lebesgue measure. When $\alpha < d$, the process ξ is transient and its potential density $G(x,y) = G(x-y)$ is given by

$$G(x,y) = \int_0^\infty p(t,x,y)dt = \mathcal{A}_1(d,\alpha)|x-y|^{\alpha-d}\,,$$

where $\mathcal{A}_1(d,\alpha) = 2^{-\alpha}\pi^{-\frac{\alpha}{2}}\Gamma(\frac{d-\alpha}{2})\Gamma(\frac{\alpha}{2})^{-1}$. For any function V, we define

$$GV(x) = \int G(x-y)V(y)dy\,.$$

By a signed measure we mean in this paper the difference of two non-negative measures at most one of which can have infinite total mass. For any signed measure on $\mathbb{R}^d$, we use μ^+ and μ^- to denote its positive and negative parts, and $|\mu| = \mu^+ + \mu^-$ its total variation. For any signed measure μ, we define

$$G\mu(x) = \int G(x-y)\mu(dy)\,.$$

The Dirichlet form $(\mathcal{E},\mathcal{F})$ of ξ is given by

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}_2(d,-\alpha)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}dxdy\,,$$

$$\mathcal{F} = \left\{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy < \infty\right\}\,,$$

where

$$\mathcal{A}_2(d,-\alpha) = \frac{|\alpha|\Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}\,.$$

The following result is well known (see Corollary 1.3.1 in Ref. 9).

Proposition 2.1. *$\mathcal{D}(\Delta_\alpha) \subset \mathcal{F}$, and*

$$\mathcal{E}(u,v) = (-\Delta_\alpha u, v), u \in \mathcal{D}(\Delta_\alpha), v \in \mathcal{F}\,.$$

Definition 2.1. (1) A signed Radon measure μ on $\mathbb{R}^d$ is said to be in the Kato class $K_{d,\alpha}$, if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}\frac{|\mu|(dy)}{|x-y|^{d-\alpha}} = 0\,,$$

where $|\mu|$ is the total variation measure of μ.

(2) A signed Radon measure μ on $\mathbb{R}^d$ is said to be in $K^\infty_{d,\alpha}$, if for every $\epsilon > 0$, there exist a compact set K and a constant $\delta > 0$, such that

$$\sup_{x\in\mathbb{R}^d}\int_{K^c} G(x,y)|\mu|(dy) < \epsilon\,,$$

and for any measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{x\in\mathbb{R}^d}\int_{B} G(x,y)|\mu|(dy) < \epsilon\,.$$

(3) A function f on $\mathbb{R}^d$ is said to be in the class $K_{d,\alpha}$ or $K^\infty_{d,\alpha}$, if $\mu := f(x)dx$ is in the corresponding spaces (see Ref. 2).

The following proposition was shown by Chen[2] and Chen and Song[4]:

Proposition 2.2. *Assume that μ is a measure in $K_{d,\alpha}$.*

(i) *μ is in the class $K^\infty_{d,\alpha}$ if and only if*

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|>r} G(x,y)|\mu|(dy) = 0\,. \tag{2.1}$$

(ii) *If $\mu \in K^\infty_{d,\alpha}$, then*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)|\mu|(dy) < \infty\,. \tag{2.2}$$

Proposition 2.3. *Suppose $d > \alpha$. For $\rho > \alpha$, $\{V \in K_{d,\alpha} : V(x) = O(|x|^{-\rho})$ as $|x| \to \infty\} \subseteq K^\infty_{d,\alpha}$.*

Proof. When ξ is a Brownian motion with dimension $d > 2$, this was proved in Ref. 19. An argument similar to that of Proposition 2 in Ref. 19 shows that the result holds for symmetric α-stable process with $d > \alpha$. □

Definition 2.2. Suppose V and f are measurable functions.

(1) We say that u is a solution of $\frac{\partial u}{\partial t} = \Delta_\alpha u + Vu + f$, if u a continuously differentiable curve: $R_1^+ \to \mathcal{D}(\Delta_\alpha)$ satisfying $\frac{\partial u(t,x)}{\partial t} = \Delta_\alpha u(x) + V(x)u(x) + f(x)$, $x \in \mathbb{R}^d$.

(2) We say that u is a solution of $\Delta_\alpha u + Vu + f = 0$ in $\mathbb{R}^d$, if $u \in \mathcal{D}(\Delta_\alpha)$ and satisfies $\Delta_\alpha u(x) + V(x)u(x) + f(x) = 0$, $x \in \mathbb{R}^d$.

Proposition 2.4. *Let $d > \alpha$. Suppose that $f \in K_{d,\alpha}$ and $|f| = O(|x|^{-\rho})$ with $\rho > \alpha$. Then the function Gf is a bounded continuous solution of $\Delta_\alpha u = -f$. Conversely, if u is a bounded continuous solution of $\Delta_\alpha u = -f$, then $u = Gf + c$ for some constant c.*

Proof. By Proposition 2.3, $f \in K_{d,\alpha}^\infty$. Then by Proposition 2.2, Gf is bounded in $\mathbb{R}^d$. Since

$$S_t Gf = Gf - \int_0^t S_s f ds$$

and $S_s f \to f$ uniformly as $s \to 0$ it follows that $Gf \in \mathcal{D}(\Delta_\alpha)$ and $\Delta_\alpha(Gf) = -f$. Assume that u is a bounded continuous solution of $\Delta_\alpha u = -f$. Then $u - Gf$ is bounded, continuous and satisfies $\Delta_\alpha(u - Gf) = 0$. Since ξ is transient, we get that $u - Gf = c$ for some constant c. □

Note that the Harnack inequality holds for the operator $\Delta_\alpha + V$ (see, for instances, Chen and Song[3]). If the Green's function, denoted by $G^V(x, y)$, exists for the operator $\Delta_\alpha + V$, from the general theory of Markov processes and their potential theory (see, for instance, Ref. 8), we know that there exists a positive solution of $(\Delta_\alpha + V)u = 0$. This motivates the following classification of operators.

Definition 2.3. (1) We say that V is subcritical if the operator $\Delta_\alpha + V$ admits a Green function.

(2) We say that V is critical if $\Delta_\alpha + V$ is not subcritical but admits a positive solution of the equation $(\Delta_\alpha + V)u = 0$.

(3) We say that V is supercritical if $\Delta_\alpha + V$ does not admit a positive solution of the equation $(\Delta_\alpha + V)u = 0$.

The following proposition is well known for Brownian motion (see, for instance, Ref. 12). But for symmetric α-stable processes, we were unable to point out an actual reference and we give the proof in detail. The idea of the proof comes from Theorem C.8.1 in Ref. 16 and Theorem 2.4 in Ref. 12.

Proposition 2.5. *If there exists an essentially nonzero function $q \geq 0$ such that $(\Delta_\alpha + V + q)u = 0$ has a positive solution, then V is subcritical.*

Proof. Assume that u is a positive solution of $(\Delta_\alpha + V + q)u = 0$. We first prove that for $\varphi \in \mathcal{D}(\Delta_\alpha)$

$$((\Delta_\alpha + V + q)\varphi, \varphi) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi(x)u(y) - \varphi(y)u(x))^2}{u(x)u(y)|x-y|^{d+\alpha}} dxdy\,, \tag{2.3}$$

that is to say, $\Delta_\alpha + V + q \le 0$. Since $V + q = -u^{-1}\Delta_\alpha u$ we have

$$\begin{aligned}((\Delta_\alpha + V + q)\varphi, \varphi) &= -\mathcal{E}(\varphi, \varphi) + (V + q, \varphi^2)\\ &= -\mathcal{E}(\varphi, \varphi) - (u^{-1}\Delta_\alpha u, \varphi^2)\\ &= -\mathcal{E}(\varphi, \varphi) - (\Delta_\alpha u, u^{-1}\varphi^2)\,.\end{aligned}$$

By Proposition 2.1, $-(\Delta_\alpha u, u^{-1}\varphi^2) = \mathcal{E}(u, u^{-1}\varphi^2)$, so

$$((\Delta_\alpha + V + q)\varphi, \varphi) = -\mathcal{E}(\varphi, \varphi) + \mathcal{E}(u, u^{-1}\varphi^2)\,.$$

By a direct calculation, we get

$$-\mathcal{E}(\varphi, \varphi) + \mathcal{E}(u, u^{-1}\varphi^2) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi(x)u(y) - \varphi(y)u(x))^2}{u(x)u(y)|x-y|^{d+\alpha}}dxdy\,.$$

Then we proved (2.3).

We can choose a bounded Borel set A of positive measure such that $\bar{A} \subset \mathbb{R}^d$ and

$$a = \inf\{q(x); x \in A\} > 0\,.$$

Put $w(x) = aI_A(x)$, where I_A is the indicator function of A. Let $\{D_j, j \ge 1\}$ be a sequence of bounded smooth domains such that

$$\bar{D}_j \subset D_{j+1} \quad \text{and} \quad D_j \uparrow \mathbb{R}^d\,,$$

where $\bar{D}_j$ is the closure of D_j. Denote by $G_{D_j}^{V+q-w}(x, y)$ the Green's function for the operator $\Delta_\alpha + V + q - w$ in D_j with the zero Dirichlet boundary condition. For j with $\bar{A} \subset D_j$, define a function h_j on D_j by

$$h_j(x) = G_{D_j}^{V+q-w}w(x)\,,$$

where $G_{D_j}^{V+q-w}$ is the integral operator with kernel $G_{D_j}^{V+q-w}(x, y)$. We have that h_j belongs to $\mathcal{D}(\Delta_\alpha)$ and $(\Delta_\alpha + V + q - w)h_j = -w$. By (2.3),

$$((\Delta_\alpha + V + q)h_j, h_j) \le 0\,.$$

Then we have that

$$\int wh_j^2dx \le \int wh_jdx \le \left(\int wh_j^2dx\right)^{1/2}\left(\int wdx\right)^{1/2}.$$

Thus

$$\int wh_j^2dx \le \int wdx\,,$$

which implies

$$a^2\int_A dx\left(\int_A G_{D_j}^{V+q-w}(x, y)dy\right)^2 \le a\int_A dx\,.$$

This shows that $G_{D_j}^{V+q-w}(x,y)$ converges, which means that $\Delta_\alpha + V + q - w$ is subcritical. Denoting by $G_{D_j}^V(x,y)$ the Green's function for the operator $\Delta_\alpha + V$ in D_j with the zero Dirichlet boundary condition, we have

$$(\Delta_\alpha + V + q - w)(G_{D_j}^{V+q-w} - G_{D_j}^V) = (q-w)G_{D_j}^V\,.$$

Since $q - w \geq 0$ and $G_{D_j}^V \geq 0$, we thus obtain that $G_{D_j}^V \leq G_{D_j}^{V+q-w}$. This together with the convergence of $G_{D_j}^{V+q-w}$ shows that $\Delta_\alpha + V$ is subcritical. □

Definition 2.4. Let $\mu \in K_{d,\alpha}^\infty$ and $A^\mu(t)$ be the continuous additive functional of ξ associated with μ. We say that u is μ-harmonic in $\mathbb{R}^d$ if

$$u(x) = E_x[u(\xi_{\tau_U})e^{A^\mu(\tau_U)}]\,, \qquad x \in U$$

for every bounded open set U.

The following proposition is taken from Takeda and Uemura[15]:

Proposition 2.6. *Let $d > \alpha$, $\mu \in K_{d,\alpha}^\infty$ and $A^\mu(t)$ be the continuous additive functional of ξ associated with μ. Then the following conditions are equivalent*:

(i) $u_0(x) = E_x[e^{A^\mu(\infty)}] \not\equiv \infty$, $x \in \mathbb{R}^d$;
(i') $\sup_{x\in\mathbb{R}^d} u_0(x) < \infty$;
(ii) *the operator $\Delta_\alpha + \mu$ admits a Green function $G^\mu(x,y)$ satisfying $G^\mu(x,y) < \infty$ for x, $y \in \mathbb{R}^d$, $x \neq y$;*
(iii) *there exists μ-harmonic function u so that $\inf_{x\in\mathbb{R}^d} u(x) > 0$.*

Moreover, if one of the above conditions hold, then $u_0(x) = E_x[e^{A^\mu(\infty)}]$ is a μ-harmonic function satisfying $0 < \inf_{x\in\mathbb{R}^d} u_0(x) \leq \sup_{x\in\mathbb{R}^d} u_0(x) < \infty$.

Throughout this paper, the notation C always denotes a constant which may change values from line to line.

3. Nonlinear Differential Equations

To prove the main theorem, we need to develop some results on nonlinear differential equations. Let $d > \frac{\alpha p}{p-1}$ and $p > 1$. Consider the following two equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_\alpha u + |u|^p + V(x)\,, & (t,x) \in (0,\infty) \times \mathbb{R}^d\,, \\ u(0) = 0\,, \end{cases} \tag{3.1}$$

and

$$\Delta_\alpha u + |u|^p + V(x) = 0\,, \qquad x \in \mathbb{R}^d\,. \tag{3.2}$$

As in Ref. 10, a function $u(x)$ is called *proper* if $|u(x)| \leq C(1+|x|)^{\alpha-d}$ for some $C > 0$ and $u(t,x)$ is called proper if $\sup_{t>0}|u(t,x)|$ is proper. Lee's analytic techniques do not work for nonlocal operator Δ_α. To overcome these difficulties, we first consider their mild solutions, i.e. solutions of their corresponding integral equations:

$$u(t,x) = \int_0^t S_{t-s}^{\alpha}(|u(s,\cdot)|^p + V)(x)ds\,, \tag{3.3}$$

and

$$u(x,V) = G(|u|^p + V)(x)\,, \qquad x \in \mathbb{R}^d\,, \tag{3.4}$$

and then prove that the mild solutions are solutions in the sense of Definition 2.2 under some conditions. Results proved in this section are not only sufficient to prove our main results, but also have independent interest.

Lemma 3.1. *If $d > \frac{\alpha p}{p-1}$, $p > 1$, then there exists a constant $M > 0$ such that $\varphi(x) = M(1+|x|)^{(\alpha-d)p}$, $x \in \mathbb{R}^d$ satisfies*

$$G\varphi < \frac{1}{2}\varphi^{\frac{1}{p}}\,.$$

Proof. First by $G(x) \le C|x|^{\alpha-d}$, we have

$$\frac{G\varphi(x)}{\varphi^{\frac{1}{p}}(x)} = \frac{\int_{\mathbb{R}^d} G(x-y)\varphi(y)\mathrm{d}\,y}{\varphi^{\frac{1}{p}}(x)} \le CM^{(1-1/p)} \int_{\mathbb{R}^d} \left(\frac{1+|x|}{|x-y|}\right)^{d-\alpha} \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}}\,.$$

For simplicity, we define

$$J(x) = \int_{\mathbb{R}^d} \left(\frac{1+|x|}{|x-y|}\right)^{d-\alpha} \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}}\,.$$

We assert that $J(x)$ is bounded in $\mathbb{R}^d$. Then we can choose M sufficiently small such that $G\varphi < \frac{1}{2}\varphi^{\frac{1}{p}}$.

Now we are left to prove the assertion. Fix $M > 0$. We estimate $J(x)$ separately on the set $\{|x| \le M\}$ and the set $\{|x| > M\}$. On $\{|x| \le M\}$, we have

$$\begin{aligned} J(x) &= \left(\int_{|y-x|\ge 1} + \int_{|y-x|<1}\right) \left(\frac{1+|x|}{|x-y|}\right)^{d-\alpha} \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}} \\ &\le C\left(\int \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}} + \int_{|x-y|<1} \frac{\mathrm{d}\,y}{|x-y|^{d-\alpha}}\right) \le C\,. \end{aligned}$$

On $\{|x| > M\}$, we have

$$\begin{aligned} J(x) &= \left(\int_{|y-x|\ge\frac{1+|x|}{2}} + \int_{|y-x|<\frac{1+|x|}{2}}\right) \left(\frac{1+|x|}{|x-y|}\right)^{d-\alpha} \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}} \\ &\le C\left(\int \frac{\mathrm{d}\,y}{(1+|y|)^{(d-\alpha)p}} + (1+|x|)^{d-\alpha}(1+|x|)^{-(d-\alpha)p} \int_0^{\frac{1+|x|}{2}} r^{\alpha-1}dr\right) \\ &\le C\,. \end{aligned}$$

Then we have $J(x)$ is bounded in $\mathbb{R}^d$, and the lemma is proved. ◻

Theorem 3.1. *Let $d > \frac{\alpha p}{p-1}$ $(p > 1)$, and φ as in Lemma* 3.1. *If $V \in A$ and $|V| \leq \varphi$, then Eq.* (3.1) *has a unique proper solution $u(t, x, V)$; furthermore, the limit function*

$$u(x, V) \equiv \lim_{t \to \infty} u(t, x, V)$$

exists pointwise and is a proper solution of Eq. (3.2).

To prove the theorem, we need some lemmas.

Lemma 3.2. *Let $d > \frac{\alpha p}{p-1}$ $(p > 1)$, and φ as in Lemma* 3.1. *Suppose $V \in A$ and $|V| \leq \varphi$. The integral Eq.* (3.3) *has a unique proper solution $u(t, x, V)$.*

Proof. We begin with the existence of a solution. Define the usual Picard iteration scheme $\{u_n(t, x, V)\}$:

$$\begin{cases} u_0 = 0\,, \\ u_1 = \displaystyle\int_0^t S_{t-s}^{\alpha} V(x) ds\,, \\ \cdots \\ u_{n+1} = \displaystyle\int_0^t S_{t-s}^{\alpha} (|u_n|^p + V)(x) ds\,, \\ \cdots \end{cases}$$

then we have

$$-G\varphi(x) \leq u_n(t, x) \leq 2G\varphi(x)\,.$$

In fact, by the construction of u_n we easily have

$$u_n(t, x) \geq \int_0^t S_{t-s}^{\alpha} V(x) ds \geq -\int_0^t S_{t-s}^{\alpha} \varphi(x) ds \geq -G\varphi(x)\,.$$

On the other hand, it is easy to see $u_0 \leq 2G\varphi(x)$, $u_1 \leq G\varphi(x) \leq 2G\varphi(x)$. Then it follows by induction and by noticing that $G\varphi < \frac{1}{2}\varphi^{\frac{1}{p}}$, we have for each n,

$$u_n(t, x) \leq \int_0^t S_{t-s}^{\alpha} (|2G\varphi|^p + \varphi)(x) ds \leq 2G\varphi(x)\,.$$

Thus we get

$$|u_n(t, x)| \leq 2G\varphi(x) \leq \varphi^{1/p}(x)\,,$$

and then $|u_n(t, x)|$ is bounded in $[0, \infty) \times \mathbb{R}^d$. To prove the limit function $u(t, x, V) = \lim_{n \to \infty} u_n(t, x, V)$ exists, we only need to prove, for any fixed $t \geq 0$ and any $x \in \mathbb{R}^d$, the series

$$\sum_{n=1}^{\infty} [u_{n+1}(t, x, V) - u_n(t, x, V)] \tag{3.5}$$

is convergent. We claim that

$$\|u_{n+1}(t) - u_n(t)\| \le \frac{\|V\|}{C} \cdot \frac{(Ct)^{n+1}}{(n+1)!}\,. \tag{3.6}$$

We prove this claim by induction. In fact, for $n = 0$, we have $\|u_1(t) - u_0(t)\| = \| \int_0^t S^\alpha_{t-s} V(x)ds\| \le \|V\|t$. Now suppose that the claim (3.6) holds for $n = k$. Using the fact that $|x_1^p - x_2^p| \le p\max(|x_1|, |x_2|)|x_1 - x_2|$, we have

$$\begin{aligned}
\|u_{k+1}(t) - u_k(t)\| &= \left\| \int_0^t S^\alpha_{t-s}(|u_k|^p - |u_{k-1}|^p)(x)ds \right\| \\
&\le C \left\| \int_0^t S^\alpha_{t-s} |u_k - u_{k-1}|(x)ds \right\| \\
&\le C \int_0^t \|u_k - u_{k-1}\| ds \\
&\le C \cdot \frac{\|V\|}{C} \int_0^t \frac{(Cs)^k}{k!} ds \\
&= \frac{\|V\|}{C} \cdot \frac{(Ct)^{k+1}}{(k+1)!}\,.
\end{aligned} \tag{3.7}$$

That is (3.6) also holds in the case when $n = k+1$. Therefore the claim above is valid. It follows from the claim above that, for any $t \ge 0$, the series $\sum_{n=1}^\infty [u_{n+1}(t, x, V) - u_n(t, x, V)]$ is convergent in the sup norm $\|\cdot\|$. Now by the boundedness of φ and the bounded convergence theorem, we obtain that u is a solution of (3.3). Since $\sup_{t\ge 0}|u_n(t, x, V)| \le 2G\varphi(x)$, by Lemma 3.1, $u(t, x, V)$ is a proper solution of (3.3).

Then we are left to prove the uniqueness of the solution of Eq. (3.3). Assume that u, w are two solutions of (3.3). Then we have

$$u(t, x) - w(t, x) = \int_0^t S^\alpha_{t-s}(|u(s)|^p - |w(s)|^p)(x)ds\,.$$

By an argument similar to that of (3.7), we have

$$\|u(t) - w(t)\| \le C \int_0^t \|u(s) - w(s)\| ds\,.$$

By Gronwall's inequality, we have $\|u(t) - w(t)\| = 0$. So $u(t, x) = w(t, x)$ for any $t \ge 0$ and any $x \in \mathbb{R}^d$. □

Lemma 3.3. *Let $d > \frac{\alpha p}{p-1}$ $(p > 1)$, and φ as in Lemma 3.1. Suppose $V \in A$ and $|V| \le \varphi$. Consider the following integral equation*

$$w(t, x) = S^\alpha_t V(x) + \int_0^t S^\alpha_{t-s}[p|u(s)|^{p-1}\mathrm{sgn}(u(s))w(s)](x)ds\,, \tag{3.8}$$

where u is the unique solution of (3.3)*. There exists a unique solution to the above equation, and*

$$\frac{d^+}{dt}u(t) = w(t)\,.$$

Proof. The existence and uniqueness of solution to (3.8) can be proved by a similar argument as that in the proof of Lemma 3.2. We omit the details. Set

$$w_h(t,x) = \frac{u(t+h,x) - u(t,x)}{h}\,, \qquad h > 0\,.$$

By (3.3),

$$\begin{aligned}
u(t+h,x) &= \int_0^{t+h} S^\alpha_{t+h-s}(|u(s)|^p + V)(x)ds \\
&= \int_0^h S^\alpha_{t+h-s}(|u(s)|^p + V)(x)ds + \int_h^{t+h} S^\alpha_{t+h-s}(|u(s)|^p + V)(x)ds \\
&= \int_0^h S^\alpha_{t+s}(|u(h-s)|^p + V)(x)ds + \int_0^t S^\alpha_{t-s}(|u(h+s)|^p + V)(x)ds\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
w_h(t,x) = {}& \frac{1}{h}\int_0^h S^\alpha_{t+s}(|u(h-s)|^p + V)(x)ds \\
&+ \frac{1}{h}\int_0^t S^\alpha_{t-s}(|u(s+h)|^p - |u(s)|^p)(x)ds\,.
\end{aligned}$$

Note that $w_h(0,x) = \frac{1}{h}\int_0^h S^\alpha_s(|u(h-s)|^p + V)(x)ds$ and S^α_t is a contraction semigroup. We have

$$\|w_h(t)\| \le \|w_h(0)\| + C\int_0^t \|w_h(s)\|ds\,.$$

By an application of Gronwall's inequality, we have

$$\|w_h(t)\| \le \|w_h(0)\|e^{Ct}\,.$$

Since u and V are both bounded, it is easy to see that $\|w_h(0)\|$ is bounded in h, so $\|w_h(t)\|$ is also bounded in h.

Similarly we have, for $x \in \mathbb{R}^d$,

$$\begin{aligned}
&w_h(t,x) - w(t,x) \\
&\quad = \frac{1}{h}\int_0^h S^\alpha_t(S^\alpha_s[|u(h-s)|^p + V] - V)(x)ds \\
&\qquad + \int_0^t S^\alpha_{t-s}\left(\frac{|u(s+h)|^p - |u(s)|^p}{h} - p|u(s)|^{p-1}\mathrm{sgn}(u(s))w(s)\right)(x)ds\,.
\end{aligned}$$

Note that

$$w_h(0,x) - w(0,x) = \frac{1}{h}\int_0^h S_s^\alpha(|u(h-s)|^p + V)(x)ds - V(x)\,,$$

and that

$$\left\| \frac{|u(s+h)|^p - |u(s)|^p}{h} - p|u(s)|^{p-1}\mathrm{sgn}(u(s))w_h(s) \right\| \to 0\,, \quad \text{as } h \to 0\,.$$

Then we have $\forall\, \epsilon > 0$, there exists a $\delta > 0$, such that for every h satisfying $|h| < \delta$,

$$\|w_h(t) - w(t)\| \le \|w_h(0) - w(0)\| + CT\epsilon + C\int_0^t \|w_h(s) - w(s)\|ds\,, \quad t \le T\,.$$

Using Gronwall's inequality, we get

$$\|w_h(t) - w(t)\| \le (\|w_h(0) - w(0)\| + CT\epsilon)e^{Ct}\,, \qquad t \le T\,.$$

It is easy to see that $\|w_h(0) - w(0)\| \to 0$, as $h \to 0$. Therefore, for any $t \ge 0$, $\frac{d^+}{dt}u(t) = w(t)$. □

The next lemma says that to consider solution of Eq. (3.1) is equivalent to consider solution of the integral Eq. (3.3).

Lemma 3.4. *Let $d > \frac{\alpha p}{p-1}$ $(p > 1)$, and φ as in Lemma* 3.1. *Suppose $V \in A$ and $|V| \le \varphi$. u is a solution of the integral Eq.* (3.3) *if and only if it is a solution of Eq.* (3.1).

Proof. By Lemma 3.3 and a standard lemma (see p. 239, Ref. 17), $u(t,x,V)$ is actually continuously differentiable, and it is straightforward to check that $u(t,x,V)$ satisfies

$$\begin{cases} \dot{u}(t) = \Delta_\alpha u + |u|^p + V(x)\,, \\ u(0) = 0\,. \end{cases}$$

Conversely, the solution of (3.1) also satisfies the integral Eq. (3.3). In fact, if u satisfies (3.1), let $w(t) = \int_0^t S_{t-s}^\alpha(|u(s,\cdot)|^p + V)(x)ds$, so w satisfies $\frac{\partial w}{\partial t} = \Delta_\alpha w(t) + |u|^p + V(x)$ and we have $\frac{\partial[u(t)-w(t)]}{\partial t} = \Delta_\alpha[u(t) - w(t)]$ and $u(0) = w(0) = 0$, so $w(t) = u(t)$. □

Proof of Theorem 3.1. It follows from Lemmas 3.2 and 3.3 that Eq. (3.1) has a unique proper solution $u(t,x,V)$. From the integral equation (3.3), it is easy to see that $|u(t,x,V)| \le u(t,x,\varphi)$. Let $w(t,\varphi) = \frac{\partial u(t,x,\varphi)}{\partial t}$, $w(t,V) = \frac{\partial u(t,x,V)}{\partial t}$. By Lemma 3.3, $w(t,\varphi)$ and $w(t,V)$ satisfy

$$w(t,\varphi) = S_t^\alpha \varphi(x) + \int_0^t S_{t-s}^\alpha[p|u(s,\varphi)|^{p-1}\mathrm{sgn}(u(s,\varphi)) \cdot w(s,\varphi)](x)ds$$

and

$$w(t,V) = S_t^\alpha V(x) + \int_0^t S_{t-s}^\alpha[p|u(s,V)|^{p-1}\mathrm{sgn}(u(s,V)) \cdot w(s,V)](x)ds\,,$$

respectively. Let $\{w_n(t,\varphi), n \ge 0\}$ and $\{w_n(t,V), n \ge 0\}$ be the Picard iteration sequences corresponding to $w(t,\varphi)$ and $w(t,V)$, respectively. By induction, it is easy to check that $|w_n(t,V)| \le w_n(t,\varphi)$, and then $w(t,V) \le w(t,\varphi)$. That is to say

$$\frac{\partial[u(t,x,\varphi) - u(t,x,V)]}{\partial t} \ge 0\,, \qquad t \ge 0\,, \qquad x \in \mathbb{R}^d\,.$$

Then $u(t,x,\varphi) - u(t,x,V)$ is increasing in t, and then

$$u(t_2,x,V) - u(t_1,x,V) \le u(t_2,x,\varphi) - u(t_1,x,\varphi)\,, \quad 0 \le t_1 \le t_2\,, \quad x \in \mathbb{R}^d\,. \tag{3.9}$$

Similarly, we can also prove that $u(t,x,\varphi) + u(t,x,V)$ is increasing in t and

$$-[u(t_2,x,V) - u(t_1,x,V)] \le u(t_2,x,\varphi) - u(t_1,x,\varphi)\,, \quad 0 \le t_1 \le t_2\,, \quad x \in \mathbb{R}^d\,. \tag{3.10}$$

Combining the above (3.9) and (3.10) to arrive at

$$|u(t_2,x,V) - u(t_1,x,V)| \le u(t_2,x,\varphi) - u(t_1,x,\varphi)\,, \quad 0 \le t_1 \le t_2\,, \quad x \in \mathbb{R}^d\,. \tag{3.11}$$

Since φ is non-negative, $u(t,x,\varphi)$ is increasing in t and then the limit $u(x,\varphi) \equiv \lim_{t\to\infty} u(t,x,\varphi)$ exists. By the above inequalities, $u(x,V) = \lim_{t\to\infty} u(t,x,V)$ also exists. By Lemma 3.1 and the dominated convergence theorem, we have

$$u(x,V) = \int_0^\infty S_s^\alpha(|u|^p + V)(x)ds\,, \qquad x \in \mathbb{R}^d\,,$$

which can be written as

$$u(x,V) = G(|u|^p + V)(x)\,, \qquad x \in \mathbb{R}^d\,.$$

Since $u(t,x,V)$ is proper, the limit $u(x,V)$ is also proper. Then Proposition 2.4 implies that $u(x,V)$ satisfies

$$\Delta_\alpha u + |u|^p + V(x) = 0\,, \qquad x \in \mathbb{R}^d\,.$$

Now the theorem is proved. □

Lemma 3.5. *Let $d > \frac{\alpha p}{p-1}$ $(p > 1)$, and φ as in Lemma* 3.1. *If $V \in A$ and $|V| \le \varphi$, then the differential equation*

$$\begin{cases} \Delta_\alpha(f-1) + p(\mathrm{sgn}(u(x,V))|u(x,V)|^{p-1}f = 0\,, & x \in \mathbb{R}^d\,, \\ f > 0\,, & f(x) \to 1, \text{ as } x \to \infty\,, \end{cases} \tag{3.12}$$

has a unique solution, written as $f(x,u(\cdot,V))$, where $u(x,V)$ is the proper solution of (3.2) *constructed in Theorem* 3.1.

Proof. Let $w(x) = p(\mathrm{sgn}(u(x,V)))|u(x,V)|^{p-1}$, $\mu(dx) = w(x)dx$. Since $|u(x,V)| \le C\varphi^{\frac{1}{p}}(x)$, we have $|w(x)| \le C\varphi^{\frac{(p-1)}{p}}(x) = O(|x|^{-\alpha+\alpha p-d(p-1)})$ as $x \to \infty$. So the assumption $d > \frac{\alpha p}{p-1}$ implies that $-\alpha + \alpha p - d(p-1) < -\alpha$. Then we have $\mu \in K^\infty_{d,\alpha}$ by Proposition 2.3.

Let $U(x) \equiv u(x,\varphi) - u(x,V)$. It is easy to see that $U > 0$ and

$$\left(\Delta_\alpha + \frac{u(x,\varphi)^p - u(x,V)^p}{u(x,\varphi) - u(x,V)} + \frac{\varphi(x) - V(x)}{u(x,\varphi) - u(x,V)}\right) U = 0\,.$$

We denote the above equation simply by $(\Delta_\alpha + w + q)U = 0$, where

$$q = \frac{u(x,\varphi)^p - u(x,V)^p}{u(x,\varphi) - u(x,V)} - p(\operatorname{sgn}(u(x,V)))|u(x,V)|^{p-1} + \frac{\varphi(x) - V(x)}{u(x,\varphi) - u(x,V)}\,.$$

By the convexity, we have

$$\frac{u(x,\varphi)^p - u(x,V)^p}{u(x,\varphi) - u(x,V)} \geq p(\operatorname{sgn}(u(x,V)))|u(x,V)|^{p-1}\,,$$

then $q \geq 0$. Thus the Schrödinger equation $(\Delta_\alpha + w + q)u = 0$ has a bounded solution U. This implies that $\Delta_\alpha + w$ is subcritical by Proposition 2.5. By Proposition 2.6, $u_0(x) = E_x[\exp(\int_0^\infty w(\xi_s))ds]$ is a bounded w-harmonic function.

Now we show that $u_0(x) \to 1$, as $|x| \to \infty$. Similar to the proof of (32) in Zhao,[19] we can also prove

$$u_0(x) - 1 = \int_{\mathbb{R}^d} G(x,y)w(y)u_0(y)dy\,.$$

And by (2.1), $\{\int_{\mathbb{R}^d} G(x,y)w(y)dy, x \in \mathbb{R}^d\}$ is uniformly integrable. Since u_0 is bounded, we easily get

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^d} G(x,y)w(y)u_0(y)dy = 0\,.$$

Therefore, $\lim_{|x|\to\infty} u_0(x) = 1$. By Proposition 2.4, u_0 satisfies $\Delta_\alpha(u_0 - 1) = -wu_0$ in $\mathbb{R}^d$, which means that u_0 is a solution of (3.12).

We are left to prove the uniqueness. Assume that u is a solution of $(\Delta_\alpha + w)u = 0$ and satisfy $\lim_{|x|\to\infty} u(x) = 0$. It suffices to prove that $u \equiv 0$. By Proposition 2.4 and that $\lim_{|x|\to\infty}(u + G(wu)) = 0$, we have $u + G(wu) \equiv 0$, which implies that $u \equiv 0$. $\square$

4. Proof of the Main Result

For $d > \frac{\alpha p}{p-1}$ and $1 < p \leq 2$, define

$$I_p(\gamma) = (p^{-1/(p-1)} - p^{-p/(p-1)}) \int |\Delta_\alpha(\gamma - 1)(x)|^{p/(p-1)} \gamma(x)^{-1/(p-1)} dx\,, \quad \gamma \in \mathcal{G}\,. \tag{4.1}$$

This definition makes sense because $\forall\ \gamma \in \mathcal{G}$, we have

$$\inf_x \gamma(x) > 0\,, \qquad \int_{\mathbb{R}^d} |\Delta_\alpha(\gamma - 1)(x)|^{\frac{p}{p-1}} dx < \infty\,,$$

by which we can get $I_p(\gamma) < \infty$.

Lemma 4.1. *Suppose $d > \frac{\alpha p}{p-1}$, $p > 1$, and φ as in Lemma* 3.1. *If $V \in A$ and $|V| \leq \varphi$, then*

$$\lim_{t\to\infty} \frac{1}{t} \int u(t,x,V)dx = \int [|u(x,V)|^p + V(x)]dx (\doteq J(V)),$$

where u is the unique solution of (3.1).

Proof.

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}^d} u(t,x,V)dx &= \frac{1}{t}\int_0^t \int S^\alpha_{t-s}(|u(s,\cdot,V)|^p + V)(x)dxds \\ &= \frac{1}{t}\int_0^t \iint P_\alpha(t-s,x-y)(|u(s,y,V)|^p + V(y))dxdyds \\ &= \frac{1}{t}\int_0^t \int (|u(s,y,V)|^p + V(y))dyds\,.\end{aligned}$$

Since $|u(s,V)|^p \leq C\varphi$ and $\int_{\mathbb{R}^d}\varphi(x)dx < \infty$, by the dominated convergence theorem,

$$\lim_{t\to\infty}\int_{\mathbb{R}^d}(|u(t,y,V)|^p + V(y))dy = \int_{\mathbb{R}^d}(|u(y,V)|^p + V(y))dy\,.$$

Hence,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_{\mathbb{R}^d} u(t,x,V)dx &= \lim_{t\to\infty}\int_{\mathbb{R}^d}(|u(t,y,V)|^p + V(y))dy \\ &= \int_{\mathbb{R}^d}(|u(y,V)|^p + V(y))dy\,.\end{aligned}$$

□

Lemma 4.2.

$$\sup_{0\leq V<\varphi}\left[\int \gamma(x)V(x)dx - J(V)\right] \leq I_p(\gamma)\,, \qquad \forall\, \gamma \in \mathcal{G}\,, \tag{4.2}$$

$$\sup_{\gamma\in\mathcal{G}}\left[\int \gamma(x)V(x)dx - I_p(\gamma)\right] = J(V)\,, \qquad \forall\, 0 \leq V < \varphi\,. \tag{4.3}$$

Proof. By Lemma 4.1,

$$J(V) = \int [u(x,V)^p + V(x)]dx\,.$$

Then

$$\begin{aligned}&\int \gamma(x)V(x)dx - J(V) \\ &\quad= \int [\gamma(x)V(x) - u(x,V)^p - V(x)]dx \\ &\quad= \int \{(\gamma(x)-1)[V(x) + u(x,V)^p] - \gamma(x)u(x,V)^p\}dx\end{aligned}$$

$$= \int \{(\gamma(x)-1)(-\Delta_\alpha u) - \gamma(x)u(x,V)^p\}dx$$

$$= \mathcal{E}(\gamma-1,u) - \int \gamma(x)u(x,V)^p dx\,.$$

Note that in the third equality above we used the fact that $-\Delta_\alpha u = V(x)+u(x,V)^p$, which holds by (3.2). By Proposition 2.1, $\mathcal{E}(\gamma-1,u) = \int(-\Delta_\alpha(\gamma-1)(x))u(x,V)dx$, and then

$$\int \gamma(x)V(x)dx - J(V) = \int \{[-\Delta_\alpha(\gamma-1)(x)]u(x,V) - \gamma(x)u(x,V)^p\}dx\,.$$

An argument similar to the proof of Lemma 1.8 in Ref. 10 shows that (4.2) holds and the supremum in (4.3) is attained when $\gamma(x)$ satisfies

$$-\Delta_\alpha(\gamma-1)(x) = p(\operatorname{sgn} u(x,V))|u(x,V)|^{p-1}\gamma(x) \text{ in } \mathbb{R}^d\,.$$

By Lemma 3.5, there exists a bounded positive solution of the above equation, written as $f(x,u(\cdot,V))$, such that $\inf_{x\in\mathbb{R}^d} f(x,u(\cdot,V)) > 0$. If we can prove that $f\in\mathcal{G}$, then (4.3) holds and the lemma is proved. To prove $f\in\mathcal{G}$ it suffices to check that $\int_{\mathbb{R}^d} |\Delta_\alpha(f-1)|^{\frac{p}{p-1}}(x)dx < \infty$. Obviously, $|\Delta_\alpha(f-1)|^{\frac{p}{p-1}} \le C|u|^p$. Recall that u is a proper solution of (3.2). Then

$$\int |\Delta_\alpha(f-1)|^{\frac{p}{p-1}}dx \le C\int |u|^p dx \le C\int |\varphi| dx < \infty\,. \qquad \square$$

With the help of Lemma 3.5 and using the argument of Lemma 1.7 in Ref. 10, we have

Lemma 4.3. *Let $V_1, V_2, \dots, V_n$ be as in Theorem 1.1 and define*

$$\bar{a}_i = \min_{x\in\mathbb{R}^d} \frac{\varphi(x)}{V_i(x)}\,, \qquad 1\le i\le n$$

$$\Lambda(a_1,a_2,\dots,a_n) = J\left(\sum_{i=1}^n a_i V_i\right)\,, \qquad -\bar{a}_i \le a_i \le \bar{a}_i\,.$$

Then the functional Λ is strictly convex, continuously differentiable and

$$(\Delta\Lambda)(\mathbf{0}) = \left(\int_{\mathbb{R}^d} V_1(x)dx, \int_{\mathbb{R}^d} V_2(x)dx, \dots, \int_{\mathbb{R}^d} V_n(x)dx\right)\,,$$

where $\mathbf{0}$ is the origin in $\mathbb{R}^d$.

In the following we state two lemmas for super α-stable process without proof, which can be considered as counterparts of the super-Brownian motion case in Ref. 10.

Lemma 4.4. *Suppose $p=2$, $d>2\alpha$, $V\in A$ and $0\le V<\varphi$, where φ is as in Lemma 3.1. Then there exist analytic functions $F(t,z_1,z_2,\dots,z_n)$, $|z_1|,\dots,|z_n|<1$, $t\ge 0$, such that*

$$F(t,a_1,a_2,\dots,a_n) = \int_{\mathbb{R}^d} u\left(t,x,\sum a_i V_i\right)dx \text{ for } 0\le t, -1<a_1,\dots,a_n<1\,.$$

Lemma 4.5. *Let $p = 2$ and $d > 2\alpha$. Suppose $u(t, x, V)$ is the unique proper solution of Eq.* (3.1). *Then for every $V \in A$ and $|V| < \varphi$,*

$$E\left\{\exp\left[\int_0^t \int V(x)X_s(dx)ds\right]\right\} = \exp\left[\int u(t, x, V)dx\right], \qquad t \geq 0\,.$$

The above two results are the analogues of Lemmas 1.9 and 1.10 in Ref. 10.

Proof of Theorem 1.1. The argument is similar to that of the proof of Theorem 1.1 in Ref. 10. Here we only give an outline for the proof. Set

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)\,, \qquad \bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)\,.$$

From Lemmas 4.1, 4.3 and 4.5, we have

$$\lim_{T\to\infty} \frac{1}{T} \log E\{\exp(T\mathbf{a} \cdot \mathbf{D}_{T,V})\} = \Lambda(\mathbf{a})$$

for $-\bar{a}_i \leq a_i \leq \bar{a}_i$. Define $O \equiv \{(\nabla\Lambda)(\mathbf{a}) : -\bar{a}_i \leq a_i \leq \bar{a}_i\}$, which is an open neighborhood of $(\nabla\Lambda)(\mathbf{0})$. A general large deviation result (see Ref. 6, for instance) ensures two estimates:

$$\liminf_{T\to\infty} \frac{1}{T} \log P(\mathbf{D}_{T,V} \in U) \geq -\inf_{\boldsymbol{\sigma}\in U} \sup_{-\bar{a}_i \leq a_i \leq \bar{a}_i, 1\leq i\leq n} [\boldsymbol{\sigma} \cdot \mathbf{a} - \Lambda(\mathbf{a})]\,,$$

$$\limsup_{T\to\infty} \frac{1}{T} \log P(\mathbf{D}_{T,V} \in C) \leq -\inf_{\boldsymbol{\sigma}\in C} \sup_{-\bar{a}_i \leq a_i \leq \bar{a}_i, 1\leq i\leq n} [\boldsymbol{\sigma} \cdot \mathbf{a} - \Lambda(\mathbf{a})]\,.$$

The discussion in the proof of Theorem 1.1 in Ref. 10 implies that

$$\inf_{\gamma=\sigma} I_2(\gamma) = \sup_{-\bar{a}_i \leq a_i \leq \bar{a}_i, 1\leq i\leq n} [\sigma \cdot \mathbf{a} - \Lambda(\mathbf{a})]\,, \quad \text{for } \boldsymbol{\sigma} \in O\,,$$

where γ is as in Theorem 1.1. The theorem is proved. □

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References

1. K. Bogdan and T. Byczkowski, Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains, *Stud. Math.* **133** (1999) 53–92.
2. Z.-Q. Chen, Gaugeability and conditional gaugeability, *Trans. Amer. Math. Soc.* **354** (2002) 4639–4679.
3. Z.-Q. Chen and R. Song, Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains, *Illinois J. Math.* **44** (2000) 138–160.
4. Z.-Q. Chen and R. Song, General gauge and conditional gauge theorems, *Ann. Probab.* **30** (2002) 1313–1339.
5. J.-D. Deuschel and J. Rosen, Occupation time large deviations for critical branching Brownian motion, super-Brownian motion and related processes, *Ann. Probab.* **26** (1998) 602–643.

6. R. S. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer, 1985).
7. K. Fleischmann and J. Gärtner, Occupation time process at a critical point, *Math. Nachr.* **125** (1986) 275–290.
8. H. Kunita and T. Watanabe, Markov processes and related problems of analysis, *Illinois J. Math.* **9** (1965) 485–526.
9. M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes* (Walter De Gruyter, 1994).
10. T.-Y. Lee, Some limit theorems for super-Brownian motion and semilinear differential equations, *Ann. Probab.* **21** (1993) 979–995.
11. T.-Y. Lee and B. Remillard, Large deviations for the three-dimensional super-Brownian motion, *Ann. Probab.* **23** (1995) 1755–1771.
12. M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^d$, *Duke Math. J.* **53** (1986) 869–943.
13. I. Iscoe, A weighted occupation time for a class of measure-valued branching process, *Probab. Th. Rel. Fields* **71** (1986) 85–116.
14. I. Iscoe and T.-Y. Lee, Large deviations for occupation times of measure-valued branching Brownian motions, *Stoch. Stoch. Rep.* **45** (1993) 177–209.
15. M. Takeda and T. Uemura, Subcriticality and gaugeability for symmetric α-stable processes, *Forum Math.* **16** (2004) 505–517.
16. B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982) 447–526.
17. K. Yosida, *Functional Analysis* (Springer, 1968).
18. L. Zhang, A large deviation for occupation times of super-stable process, *Acta Math. Appl. Sin.* **15** (1999) 240–248.
19. Z. Zhao, Subcriticality and gaugeability of the Schrödinger operator, *Trans. Amer. Math. Soc.* **334** (1992) 75–96.