# Properties of Super-Poisson Processes and Super-Random Walks with Spatially Dependent Branching Rates

(Running head: Super-Poisson Processes and Super-RW)

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**Abstract** The global supports of super-Poisson processes and super-random walks with branching mechanism $\psi(z) = z^2$ and constant branching rate are known to be noncompact. It turns out that for any spatially dependent branching rate, this property remains true. However, the asymptotic extinction property for these two kinds of superprocesses depends on the decay rate of the branching rate function at infinity.

**Keywords** super-Poisson process, super-random walk, global support, asymptotic extinction

**MR(2000) Subject Classification** 60J80, 60J25

[1]Supported by NNSF of China (Grant No. 10471003), Foundation for Authors Awarded Excellent Ph. D. Dissertation

# 1 Introduction

Let $X = (X_t, P_\mu)$ be a super-process in a polish space $E$. Let $M_F(E)$ denote the space of finite measures on $E$, equipped with the topology of weak convergence. The global support of $X$, Gsupp$(X)$, is defined as the closure of $\bigcup_{t\geq 0} \mathrm{supp} X_t$.

**Definition 1** For $0 \neq \mu$ with compact support, We say that the global support of $X$ under $P_\mu$ is compact if

$$P_\mu\,(\mathrm{Gsupp}(X) \text{ is bounded}) = 1. \tag{1}$$

In [1], Perkins pointed out that for a super-Poisson process $X$ with positive constant branching rate $k$, the support of $X_t$ will propagate instantaneously to any points to which the underlying Poisson process can jump. More precisely,

$$\mathrm{supp}(X_t) = \{m_t, m_t + 1, \cdots\} \quad \text{a.s.} \quad \forall t > 0, \tag{2}$$

where $m_t = \inf \mathrm{supp}(X_t)$. This result follows from the following associated stochastic differential equation:

$$X_t(j) = X_0(j) + \int_0^t (X_s(j-1) - X_s(j))ds + \int_0^t \sqrt{kX_s(j)dB_s^j}, \quad j \in \mathbf{Z}^+. \tag{3}$$

Here $\{B^j_\cdot : j \in \mathbf{Z}^+\}$ is a collection of independent linear Brownian motions, and $X_s(-1) \equiv 0$.

For a super-random walk, we have similar stochastic differential equations, and

$$\mathrm{supp}(X_t) = \emptyset \quad \text{or} \quad \mathbf{Z}^d \quad \text{a.s.} \quad \forall t > 0. \tag{4}$$

In the first part of this paper we concern about the global support of $X$. We are interested in the following question: Is it possible to find a spatially dependent branching rate $k$ such that the global support of super-Poisson process is compact? The interest now is in seeing whether or not letting $k$ go to infinity fast enough would result in having compact global support. In section 2, It is proved that the answer for super-Poisson process and super-random walk is negative. Intuitively, this result is not too surprising as the associated sde's have positive drift from neighboring sites and even if the branching rates get large, over small time intervals the mass will immediately propagate to any far points. In this paper the author wants to seek a proof mainly depends on the log-Laplace functional of $X$.

In section 3, we will discuss the asymptotic extinction of super-Poisson processes and super-random walks. Let us give the definition of asymptotic extinction.

**Definition 2** For $0 \neq \mu \in M_F(E)$, we say that the measure-valued process $X$ under $P_\mu$ exhibits asymptotic extinction if

$$P_\mu\left(\lim_{t\uparrow\infty} X_t(E) = 0\right) = 1. \tag{5}$$

Our goal is to find necessary and sufficient conditions for each of these two processes to be extinct in weak sense. It turns out that the asymptotic extinction property depends on the decay rate of the branching rate $k$ at infinity (see Theorem 3, Theorem 5 and Theorem 7). Moreover, we will also present sufficient conditions on $k$ for super-Poisson processes and super-random walks to have positive extinction probability. See Theorem 4 and Theorem 6 respectively.

# 2 Super-Poisson processes and super-random walks with noncompact global support

**Theorem 1** Suppose $\{X_t, t \geq 0\}$ is a super-Poisson process with branching rate $k(i) > 0, i \in \mathbf{Z}^+$, If $\mathrm{supp}(\mu)$ is compact, then the global support of $X$ is not compact. More precisely,

$$P_\mu\left(\mathrm{Gsupp}(X) \text{ is bounded}\right) = 0. \tag{6}$$

**Proof** Without loss of generality, we may and do assume that $\mu = \delta_0$. Let $\tau_n$ be the first hitting time of $\{n\}$. Note that

$$P_{\delta_0}\left(\mathrm{Gsupp}(X) \text{ is bounded}\right) = \lim_{n\to\infty} P_{\delta_0}\left(X_{\tau_n} = 0\right)). \tag{7}$$

Step 1 We first prove that for every integer $n > 0$

$$P_{\delta_{n-1}}(X_{\tau_n} = 0) = 0. \tag{8}$$

By (1.6) and (1.7) in Dynkin [2],

$$P_{\delta_{n-1}}(\exp(-\lambda X_{\tau_n})) = \exp(-u_\lambda^n(n-1)), \tag{9}$$

where $u_\lambda^n(\cdot)$ is the unique solution of the following integral equation:

$$u_\lambda^n(i) + \Pi_i \int_0^{\tau_n} k(\xi_s)(u_\lambda^n)^2(\xi_s)ds = \lambda, \quad i \leq n. \tag{10}$$

In particular,

$$u_\lambda^n(n-1) + k(n-1)(u_\lambda^n(n-1))^2 \Pi_{n-1}(\tau_n) = \lambda.$$

Letting $\lambda \to \infty$, we get

$$\lim_{\lambda \to \infty} u_\lambda^n(n-1) = \infty.$$

Letting $\lambda \to \infty$ in (9), we get (8).

Step 2 By the special Markov property and (9),

$$\begin{aligned}
P_{\delta_0}(X_{\tau_n} = 0) = {} & P_{\delta_0}(P_{\tau_{n-1}}(X_{\tau_n} = 0)) \\
= {} & P_{\delta_0}(X_{\tau_{n-1}} = 0) \\
= {} & \cdots = 0.
\end{aligned}$$

Therefore, by (7), we proved (6).

**Corollary 1** Suppose $\{X_t, t \geq 0\}$ is a super-Poisson process with branching rate $k(i) > 0, i \in \mathbf{Z}^+$, If $\mathrm{supp}(\mu)$ is compact, then for every fixed $t > 0$, the support of $X_t$ is not compact.

**Proof** Without loss of generality, we may and do assume that $\mu = \delta_0$. Since for every $n$,

$$P_{\delta_0}(X_{t \wedge \tau_n} = 0) \leq P_{\delta_0}(X_{\tau_n} = 0) = 0,$$

we have

$$P_{\delta_0}\,(\mathrm{supp}(X_t) \text{ is bounded}) \leq \lim_{n \to \infty} P_{\delta_0}(X_{t \wedge \tau_n} = 0) = 0.$$

**Theorem 2** Suppose $\{X_t, t \geq 0\}$ is a super-random walk with branching rate $k(x) > 0, x \in \mathbf{Z}^d$, If $\mathrm{supp}(\mu)$ is compact, then

$$P_\mu\left(\mathrm{Gsupp}(X) \text{ is bounded}\right) = 0. \tag{11}$$

**Proof** We suppose $\mu = \delta_0$. Let $\tau_n$ be the first hitting time of $\{x \in \mathbf{Z}^d, \|x\| \geq n\}$, where for $x = (x_1, x_2, \cdots, x_d)$, we define $\|x\| = \sum_{k=1}^d |x_k|$. As in the proof of Theorem 1, we will prove that

$$P_{\delta_0}(X_{\tau_n} = 0) = 0. \tag{12}$$

Step 1 We prove that there exists $x_0 \in B_n$ such that

$$P_{\delta_{x_0}}(X_{\tau_n} = 0) = 0.$$

Since, for $x \in B_n$,

$$P_{\delta_x}(\exp(-\lambda X_{\tau_n})) = \exp(-u_\lambda(x)), \tag{13}$$

where $u_\lambda(\cdot)$ is the unique solution of the following integral equation:

$$u_\lambda(x) + \Pi_x \int_0^{\tau_n} k(\xi_s)(u_\lambda)^2(\xi_s)ds = \lambda, \quad x \in B_n. \tag{14}$$

Letting $\lambda \uparrow \infty$, we get

$$u_\infty(x) + \Pi_x \int_0^{\tau_n} k(\xi_s)(u_\infty)^2(\xi_s)ds = \infty, \quad x \in B_n. \tag{15}$$

where $u_\infty(x) = \lim_{\lambda \uparrow \infty} u_\lambda(x), x \in B_n$. Put

$$K_n = \max_{x \in B_n} k(x), \quad M = \max_{x \in B_n} u_\infty(x).$$

It follows from (15) that

$$M(1 + M\, K_n\, \Pi_0 \tau_n) \geq \infty,$$

which implies $M = \infty$. Hence there exists $x_0 \in B_n$ such that $u_\infty(x_0) = \infty$, which is equivalent to $P_{\delta_{x_0}}(X_{\tau_n} = 0) = 0$.

Step 2 Put $n_0 = \|x_0\|$. We prove that for $x \in B_n \setminus B_{n_0}$, $P_{\delta_x}(X_{\tau_n} = 0) = 0$. It is easy to see that for every $x \in S_{n_0} = \{x \in \mathbf{Z}^d; \|x\| = n_0\}$, $P_{\delta_x}(X_{\tau_n} = 0) = P_{\delta_{x_0}}(X_{\tau_n} = 0) = 0$. For every $x \in S_m$ with $n_0 < m < n$, we have

$$0 = P_{\delta_{x_0}}(X_{\tau_n} = 0) = P_{\delta_{x_0}}(P_{X_{\tau_m}}(X_{\tau_n} = 0)),$$

which implies that $P_{X_{\tau_m}}(X_{\tau_n}=0)=0$ a.s. $P_{\delta_{x_0}}$. Since $P_{\delta_{x_0}}(X_{\tau_m}=0)(\leq P_{\delta_{x_0}}(X_{\tau_n}=0))=0$, there exists $x\in S_m$ such that $P_{\delta_x}(X_{\tau_n}=0)=0$, which implies $P_{\delta_x}(X_{\tau_n}=0)=0$ for every $x\in S_m$.

Step 3  By the special Markov property,

$$P_{\delta_0}(X_{\tau_n}=0)=P_{\delta_x}(P_{X_{\tau_{n_0}}}(X_{\tau_n}=0))=P_{\delta_0}(X_{\tau_{n_0}}=0).$$

To prove (12), we only need to prove that $P_{\delta_0}(X_{\tau_{n_0}}=0)=0$. Noticing that $n_0<n$, by the inductive method, we only need to prove that $P_{\delta_0}(X_{\tau_1}=0)=0$, which is obviously true.

**Corollary 2**  Suppose $\{X_t,t\geq 0\}$ is a super-random walk with branching rate $k(x)>0, x\in \mathbf{Z}^d$, If supp$(\mu)$ is compact, then for every fixed $t>0$, the support of $X_t$ is not compact.

**Proof**  The argument is similar to that of Corollary 1.

# 3 Asymptotic extinctions of super-random walks and super-Poisson processes

**Theorem 3** Suppose $\{X_t, t \geq 0\}$ is a super-Poisson process with branching rate $k(i) > 0$, $i \in \mathbf{Z}^+$. Let $0 \neq \mu \in M(\mathbf{Z}^+)$.

(1) If $\sum_{i=1}^{\infty} k(i) < \infty$, then under $P_\mu$ the process $X$ exhibits asymptotic survival with positive probability, i.e.,

$$P_\mu(\lim_{t\to\infty} X_t(\mathbf{Z}^+) = 0) < 1. \tag{16}$$

(2) If $\sum_{i=1}^{\infty} k(i) = \infty$, then under $P_\mu$ the process $X$ exhibits asymptotic extinction.

**Proof** Note that $u_c(t,x) = -\log P_{\delta_x} \exp(-c\langle 1, X_t\rangle)$ is a solution of

$$u_c(t,x) + \Pi_x \int_0^t k(\xi_s) u_c^2(t-s,\xi_s) ds = c. \tag{17}$$

By the Markov property, for $s < t$, we have

$$P_{\delta_x}(\exp\langle -c, X_t\rangle / \mathcal{F}_s) = P_{X_s} \exp\langle -c, X_t\rangle = \exp\langle -u_c(t,\cdot), X_s\rangle \geq \exp\langle -c, X_s\rangle, \tag{18}$$

which means that $\exp\langle -c, X_t\rangle$ is a bounded submartingale. Therefore $\lim_{t\to\infty} \exp\langle -c, X_t\rangle$ exists $P_{\delta_x}$-a.s. and hence $\lim_{t\to\infty}\langle 1, X_t\rangle$ exists $P_{\delta_x}$-a.s. By (18), $u_c(t,x)$ is non-increasing in $t$. Put

$$u_c(x) := -\log P_{\delta_x} \exp(-c \lim_{t\to\infty} \langle 1, X_t\rangle).$$

$u_c(\cdot)$ is a radial function, i.e., $u_c(x) = u_c(\|x\|)$. We claim that $u_c(\|x\|)$ is increasing in $\|x\|$. Indeed, by the special Markov property,

$$\begin{aligned} \exp(-u_c(x)) = {} & P_{\delta_x} \exp(-c \lim\nolimits_{t\to\infty} \langle 1, X_t\rangle) \\ = {} & P_{\delta_x} \left( P_{X_{s\wedge\tau_1}} \exp(-c \lim\nolimits_{t\to\infty} \langle 1, X_t\rangle)\right) \\ = {} & P_{\delta_x} \exp\langle -u_c, X_{s\wedge\tau_1}\rangle. \end{aligned}$$

Letting $s \to \infty$, we get

$$\exp(-u_c(x)) = P_{\delta_x} \exp\langle -u_c, X_{\tau_1}\rangle.$$

By Jensen's inequality,

$$\exp(-u_c(x)) \geq \exp(-P_{\delta_x}\langle u_c, X_{\tau_1}\rangle) = \exp(-u_c(x+1)),$$

which implies $u_c(x) \leq u_c(x+1), \quad x \in \mathbf{Z}^+$.

(1) Suppose $\sum_{i=0}^{\infty} k(i) < \infty$. Letting $t \to \infty$ in (17), by the dominated convergence theorem, we see that $u_c$ satisfies

$$u_c(x) + \Pi_x \int_0^{\infty} k(\xi_s) u_c^2(\xi_s) ds = c. \tag{19}$$

Let $\tau_n$ be the $n$th jumping time of $\{\xi_s\}$ and put $\lambda = \Pi_x \tau_1$. Then

$$\lambda = \Pi_x \tau_1 = \Pi_x(\tau_n - \tau_{n-1}) > 0.$$

We rewrite (19) in the form:

$$u_c(x) + \lambda \sum_{j=x}^{\infty} k(j) u_c^2(j) = c, \quad x \in \mathbf{Z}^+. \tag{20}$$

It is easy to check that $u_c$ satisfies

$$u_c(x+1) - u_c(x) = \lambda k(x) u_c^2(x), \quad x \in \mathbf{Z}^+. \tag{21}$$

Hence,

$$u_c(x) \equiv 0 \quad \text{iff} \quad u_c(x) = 0, \quad \exists x \in \mathbf{Z}^+.$$

(19) implies that there exists $x \in \mathbf{Z}^+$ such that $u_c(x) > 0$, and therefore, $u_c(x) > 0, \forall x \in \mathbf{Z}^+$, which implies

$$P_{\delta_x}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) < 1, \quad x \in \mathbf{Z}^+, \tag{22}$$

which implies eqrefsuvuval.

(2) We only need to prove that if there exists $x_0 \in \mathbf{Z}^+$ such that $P_{\delta_{x_0}}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) < 1$, then $\sum_{i=0}^{\infty} k(i) < \infty$.

Since $u_c(t,x) \geq u_c(x), x \in \mathbf{Z}^+$, by (17),

$$u_c(t,x) + \Pi_x \int_0^t k(\xi_s) u_c^2(\xi_s) ds \leq c, \quad x \in \mathbf{Z}^+.$$

Letting $t \to \infty$, we get

$$\Pi_x \int_0^{\infty} k(\xi_s) u_c^2(\xi_s) ds \leq c, \quad x \in \mathbf{Z}^+.$$

Note that $P_{\delta_{x_0}}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) < 1$ is equivalent to $u(x_0) > 0$. Then $u_c(x_0) \uparrow u(x_0) > 0$ implies that there exists $c > 0$ such that $u_c(x_0) > 0$. Since for fixed $c > 0$, $u_c(x)$ is non-decreasing in $x$, one has

$$u_c^2(x_0)\Pi_{x_0}\int_{x_0}^{\infty} k(\xi_s)ds \le c.$$

Hence

$$\Pi_{x_0}\int_0^{\infty} k(\xi_s)ds < \infty,$$

which is equivalent to

$$\sum_{i=0}^{\infty} k(i) < \infty.$$

**Theorem 4** Suppose $\{X_t, t \ge 0\}$ is a super-Poisson process with branching rate $k(i) > 0$, $i \in \mathbf{Z}^+$. If $\sum_{i=0}^{\infty} k(i) < \infty$ and $\limsup_{x\to\infty} \dfrac{k(x)}{\sum_{j=x+1}^{\infty} k(j)} < \infty$, then $X$ has positive probability of asymptotic extinction, i.e.,

$$P_{\delta_i}(\lim_{t\to\infty} X_t(\mathbf{Z}^+) = 0) > 0, \quad i \in \mathbf{Z}^+.$$

To prove Theorem 4, we first give two lemmas.

**Lemma 1** Suppose $k(i) > 0$, $i \in \mathbf{Z}^+$, satisfies the condition of Theorem 4. Then there is a positive solution of the following problem:

$$\begin{cases} v(x+1) - v(x) \le k(x)v^2(x), \quad x \in \mathbf{Z}^+ \\ \lim_{x\to\infty} v(x) = \infty. \end{cases} \tag{23}$$

**Proof** Put

$$v(x) = \frac{C}{\sum_{j=x}^{\infty} k(j)}, \quad x \in \mathbf{Z}^+.$$

Then

$$\begin{aligned} v(x+1) - v(x) = &\ Ck(x)/(\textstyle\sum_{j=x+1}^{\infty} k(j))(\sum_{j=x}^{\infty} k(j)) \\ \le &\ C\frac{\sum_{j=x}^{\infty} k(j)}{\sum_{j=x+1}^{\infty} k(j)}\frac{k(x)}{(\sum_{j=x}^{\infty} k(j))^2} \\ = &\ \frac{1}{C}\left(1 + \frac{k(x)}{\sum_{j=x+1}^{\infty} k(j)}\right)k(x)v^2(x). \end{aligned}$$

The assumption implies that, for sufficiently large $C$, $v(x+1) - v(x) \le k(x)v^2(x)$. It is obvious that $v(x) \to \infty$ as $x \to \infty$.

**Lemma 2** (Maximum Principle) Suppose $k(x) > 0$, $x \in \mathbf{Z}^+$. If $v$ satisfies

$$\begin{cases} v(x+1) - v(x) \le k(x)v^2(x), \quad x \in \mathbf{Z}^+ \\ \limsup_{x\to\infty} v(x) = C_1 \end{cases}$$

and $u$ satisfies

$$\begin{cases} u(x+1) - u(x) = k(x)u^2(x), \quad x \in \mathbf{Z}^+ \\ \limsup_{x\to\infty} u(x) = C_2 \end{cases}$$

with constants $\infty \ge C_1 > C_2$, then $v(x) > u(x)$, $x \in \mathbf{Z}^+$.

**Proof** Suppose there exists $x_0 \in \mathbf{Z}^+$ such that $v(x_0) \le u(x_0)$. Then

$$v(x_0+1) \le v(x_0) + k(x_0)v^2(x_0);$$

$$u(x_0+1) = u(x_0) + k(x_0)u^2(x_0).$$

Therefore $v(x_0+1) \le u(x_0+1)$. By induction we can show that for every $x \ge x_0$,

$$v(x) \le u(x).$$

Letting $x \to \infty$, we get

$$\limsup_{x\to\infty} v(x) \le \limsup_{x\to\infty} u(x),$$

which means $C_1 \le C_2$, we got a contradiction.

**Proof of Theorem 4** Let

$$u_c(x) \uparrow u(x) = -\log P_{\delta_x}(\lim_{t\to\infty} \langle 1, X_t \rangle = 0) \tag{24}$$

If we can prove that

$$u(x) < \infty, \quad \forall x \in \mathbf{Z}^+,$$

then

$$P_{\delta_x}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) > 0, \quad x \in \mathbf{Z}^+. \tag{25}$$

Using (20) and (21) we can see that $u_c$ satisfies

$$\begin{cases} u_c(x+1) - u_c(x) = \lambda k(x) u_c^2(x), \quad x \in \mathbf{Z}^+, \\ \lim_{x\to\infty} u_c(x) = c. \end{cases}$$

By Lemma 2, $u_c(x) < v(x)$, where $v$ is the solution constructed in Lemma 1 with $k(x)$ replaced by $\lambda k(x)$. Letting $c \to \infty$, we get $u(x) \le v(x) < \infty, \quad x \in \mathbf{Z}^+$.

Now we begin to discuss the asymptotic extinction of super random walk.

**Theorem 5** Suppose $\{X_t, t \ge 0\}$ is a super-random walk with branching rate $k(x) = k(\|x\|) > 0$, $x \in \mathbf{Z}^d$ $(d \ge 3)$. Let $0 \ne \mu \in M(\mathbf{Z}^d)$. Assume that $k(i) \downarrow$ as $i \uparrow$.

(1) If $\sum_{i=1}^{\infty} ik(i) < \infty$, then under $P_\mu$ the process $X$ exhibits asymptotic survival with positive probability, i.e.,

$$P_\mu(\lim_{t\to\infty} X_t(\mathbf{Z}^d) = 0) < 1. \tag{26}$$

(2) If $\sum_{i=1}^{\infty} ik(i) = \infty$, then under $P_\mu$ the process $X$ exhibits asymptotic extinction.

**Proof** The proof essentially requires us to prove the following two facts. The first one is that if nonnegative radial function $u(x) = u(\|x\|)$ satisfies $\Delta u(x) = k(x)u^2(x)$ in $\mathbf{Z}^d$ $(d \ge 3)$, then $u(r)$ ( $r = \|x\|$ ) is increasing in $r$ and $u \equiv 0$ if and only if $u(0) = 0$. The second one is that

$$\Pi_x \int_0^\infty k(\xi_s)ds = \infty \quad \text{iff} \quad \sum_{i=1}^{\infty} ik(i) < \infty. \tag{27}$$

With these two facts, an argument similar to the one given in the proof of Theorem 3 will yield the conclusions of the theorem.

We continue then with the proofs of these two facts. First note that

$$u(1) - u(0) = k(0)u^2(0)$$

and

$$u(r+1) + u(r-1) - 2u(r) \ge \frac{1}{2}\Delta u(x) = \frac{1}{2}k(x)u^2(x) > 0.$$

Therefore,

$$u(r+1) - u(r) \geq u(r) - u(r-1) + \frac{1}{2}k(r)u^2(r), \quad r \geq 1.$$

So, $u(r)$ is increasing in $r$.

Since

$$\Pi_x \int_0^\infty k(\xi_s)ds = \int_{\mathbf{Z}^d} k(y)g(x,y)dy,$$

with $g(x,y)$ being the Green function of $\xi$ and

$$g(x,y) \sim \frac{K_d}{\|x-y\|^{d-2}} \quad (\text{ as } \|x-y\| \to \infty),$$

claim (27) holds.

**Theorem 6** Suppose $\{X_t, t \geq 0\}$ is a super-random walk with branching rate $k(x) = k(\|x\|) > 0$, $x \in \mathbf{Z}^d$ $(d \geq 3)$. Assume that $k(i) \downarrow$ as $i \uparrow$, and $k$ satisfies

$$\sum_{i=1}^\infty ik(i) < \infty, \quad \limsup_{r\to\infty} \frac{rk(r)}{\sum_{i=r+1}^\infty ik(i)} < \infty. \tag{28}$$

Then

$$P_{\delta_i}(\lim_{t\to\infty} X_t(\mathbf{Z}^d) = 0) > 0, \quad i \in \mathbf{Z}^d.$$

**Proof** The argument is similar to that of Theorem 4, by using Lemma 3 and Lemma 4 below instead of Lemma 1 and Lemma 2. We omit the detail.

**Lemma 3** Suppose the function $k$ satisfies the conditions of Theorem 6. There exists a constant $C > 0$, such that

$$u(x) = \frac{C}{(\sum_{i=[\|x\|]+1}^\infty ik(i))^2} \tag{29}$$

is a solution of

$$\begin{cases} \Delta u(x) \leq k(x)u^2, \quad x \in \mathbf{Z}^d, \\ \lim_{x\to\infty} u(x) = \infty, \end{cases}$$

where $\Delta$ is the discrete $d$-dimensional Laplacian:

$$\Delta u(x) := \frac{1}{2d} \sum_{\|y-x\|=1} (u(y) - u(x)), \quad x \in \mathbf{Z}^d.$$

**Proof** $u$ defined by (29) is radial, i.e., $u(x) = u(\|x\|)$, $x \in \mathbf{Z}^d$. For $x \in \mathbf{Z}^d$ with $[|x\|] = r > 0$,

$$\Delta u(x) \leq \frac{1}{2}\left(u(r+1) + u(r-1) - 2u(r)\right).$$

we have

$$\begin{aligned} u(r+1) - u(r) = &\quad 2C\frac{(r+1)k(r+1)(\sum_{i=r+1}^{\infty} ik(i) + \sum_{i=r+2}^{\infty} ik(i))}{(\sum_{i=r+1}^{\infty} ik(i))^2\,(\sum_{i=r+2}^{\infty} ik(i))^2} \\ \leq &\quad 4C\frac{(r+1)k(r+1)}{(\sum_{i=r+2}^{\infty} ik(i))^3}, \end{aligned}$$

and

$$\begin{aligned} u(r) - u(r-1) = &\quad 2C\frac{rk(r)(\sum_{i=r}^{\infty} ik(i) + \sum_{i=r+1}^{\infty} ik(i))}{(\sum_{i=r}^{\infty} ik(i))^2\,(\sum_{i=r+1}^{\infty} ik(i))^2} \\ \geq &\quad 4C\frac{rk(r)}{(\sum_{i=r}^{\infty} ik(i))^3}. \end{aligned}$$

Then

$$\begin{aligned} \Delta u(x) \leq &\quad 2C\left(\frac{(r+1)k(r+1)}{(\sum_{i=r+2}^{\infty} ik(i))^3} - \frac{rk(r)}{(\sum_{i=r}^{\infty} ik(i))^3}\right) \\ = &\quad 2C\frac{(r+1)k(r+1)(\sum_{i=r}^{\infty} ik(i))^3 - rk(r)(\sum_{i=r+2}^{\infty} ik(i))^3}{(\sum_{i=r+2}^{\infty} ik(i))^3(\sum_{i=r}^{\infty} ik(i))^3} \\ = &\quad 2C\frac{(r+1)k(r+1)\left[(\sum_{i=r}^{\infty} ik(i))^3 - (\sum_{i=r+2}^{\infty} ik(i))^3\right]}{(\sum_{i=r+2}^{\infty} ik(i))^3(\sum_{i=r}^{\infty} ik(i))^3} + \\ &\quad 2C\frac{[(r+1)k(r+1) - rk(r)]}{(\sum_{i=r}^{\infty} ik(i))^3} \\ \leq &\quad 2C\frac{(r+1)k(r+1)\left[rk(r) + (r+1)k(r+1)\right]}{(\sum_{i=r+1}^{\infty} ik(i))^4}\left[f^3(r) + f^2(r) + f(r)\right] + \\ &\quad 2C\frac{(r+1)\left[k(r+1) - k(r)\right] + k(r)}{(\sum_{i=r}^{\infty} ik(i))^3} \end{aligned}$$

where we set

$$f(r) = \frac{\sum_{i=r+1}^{\infty} ik(i)}{\sum_{i=r+2}^{\infty} ik(i)}.$$

Note that

$$k(r+1) \leq k(r), \quad r > 0,$$

and there exists $M > 1$ such that

$$\sum_{i=1}^{\infty} ik(i) \le M, \qquad \text{and} \qquad f(r) \le M \quad r > 0.$$

Hence,

$$\begin{aligned}
\Delta u(x) \le & \quad 2C \frac{k(r)}{(\sum_{i=r+1}^{\infty} ik(i))^4} \left[6M^3(r+1)^2 k(r) + M\right] \\
= & \quad k(x) u^2(x) \left[6M^3(r+1)^2 k(r) + M\right] /C.
\end{aligned}$$

The assumption (28) implies that $(r+1)^2 k(r)$ is bounded. Then we can choose $C$ large enough such that

$$\left[6M^3(r+1)^2 k(r) + M\right] /C \le 1.$$

And therefore,

$$\Delta u(x) \le k(x) u^2(x), \quad x \in \mathbf{Z}^d.$$

The following Maximum principle for nonlinear differential equation corresponding to super-random walk is easy to prove.

**Lemma 4** (Maximum Principle) Suppose $k(x) > 0$, $x \in \mathbf{Z}^d$. If $v$ satisfies

$$\begin{cases} \Delta v(x) \le k(x) v^2(x), \quad x \in \mathbf{Z}^d, \\ \limsup_{x \to \infty} v(x) = C_1 \end{cases}$$

and $u$ satisfies

$$\begin{cases} \Delta u(x) = k(x) u^2(x), \quad x \in \mathbf{Z}^d, \\ \limsup_{x \to \infty} u(x) = C_2 \end{cases}$$

with constants $\infty \ge C_1 > C_2$, then $v(x) > u(x)$, $x \in \mathbf{Z}^d$.

**Theorem 7** Suppose $\{X_t, t \ge 0\}$ is a super-random walk with branching rate $k(x)$, $x \in \mathbf{Z}^d$ ($d \le 2$). If there exists $x_0 \in \mathbf{Z}^d$ such that $k(x_0) > 0$, then for every $0 \ne \mu \in M_F(\mathbf{Z}^d)$, the process $X$ under $P_\mu$ exhibits asymptotic extinction.

**Proof** As in the proof of Theorem 3, $u_c(t,x) := -\log P_{\delta_x} \exp(-c\langle 1, X_t\rangle)$ satisfies

$$u_c(t,x) + \Pi_x \int_0^t k(\xi_s) u_c^2(t-s,\xi_s)ds = c, \quad x \in \mathbf{Z}^d,$$

and $u_c(t,x) \downarrow u_c(x) := -\log P_{\delta_x} \exp(-c \lim_{t\to\infty}\langle 1, X_t\rangle)$. Then we have

$$u_c(t,x) + \Pi_x \int_0^t k(\xi_s) u_c^2(\xi_s)ds \leq c, \quad x \in \mathbf{Z}^d.$$

Letting $t \to \infty$, we get

$$\Pi_x \int_0^\infty k(\xi_s) u_c^2(\xi_s)ds \leq c, \quad x \in \mathbf{Z}^d. \tag{30}$$

Because the random walk in $\mathbf{Z}^d$ with $d \leq 2$ is recurrent, (30) implies that for every $c > 0$, $u_c(x_0) = 0$. Therefore, $P_{\delta_{x_0}}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) = \exp(-\lim_{c\to\infty} u_c(x_0)) = 1$. Then by the special Markov property, for every $0 \neq \mu \in M_F(\mathbf{Z}^d)$,

$$P_\mu(\lim_{t\to\infty}\langle 1, X_t\rangle = 0) = P_\mu \left[P_{X_{T_{x_0}}}(\lim_{t\to\infty}\langle 1, X_t\rangle = 0)\right] = 1,$$

where $T_{x_0}$ is the first hitting time of $x_0$.

# References


[1] Perkins E., Dawson-Watanable superprocesses and measure-valued diffusions. Lecture Notes in Mathematics, Vol. 1781. Springer-Verlag, Heidelberg, 2002, 135-192.

[2] Dynkin E. B., A probabilistic approach to one class of nonlinear differential equations, *Probab. Th. Rel. Fields*, 89(1991), 89-115.