Absolutely continuous states of exit measures for super-Brownian motions with branching restricted to a hyperplane*

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Abstract The exit measures of super-Brownian motions with branching mechanism $\psi(z)=z^{\alpha}$, $1<\alpha\leqslant 2$, from a bounded smooth domain D in $\mathbb{R}^{d+1}$ are known to be absolutely continuous with respect to the surface area on ∂D if $d<\frac{2}{\alpha-1}$, whereas in the case $d>1+\frac{2}{\alpha-1}$, they are singular. However, if the branching is restricted to a singular hyperplane, it is proved that they have absolutely continuous states for all $d\geqslant 1$.

Keywords: exit measure, super-Brownian motion, absolutely continuous state, singular state.

Assume that

$$\xi(r,\mathrm{d}y)=\xi_d(r,y_d)\mathrm{d}y_d\xi_1(r,\mathrm{d}y_1),\ y=[y_d,y_1]\in\mathbb{R}^d\times\mathbb{R}, \tag{0.1}$$

where ξ_1 is a one-dimensional kernel, whereas ξ_d is a bounded measurable function on $\mathbb{R}^+\times\mathbb{R}^d$. Consider the $d+1$-dimensional super-Brownian motion $X=\{X_t;\ t\geqslant 0\}$ $(d\geqslant 1)$ with the above factored branching rate kernel $\xi(r,\mathrm{d}y)$. Let $A_\xi(\mathrm{d}t):=\mathrm{d}t\int\xi(t,\mathrm{d}y)\delta_y(W_t)$ be a continuous additive function associated with the branching rate kernel $\xi(r,\ \mathrm{d}y)$ given by (0.1). It is well known that if $\xi(r,\mathrm{d}y)=\rho\mathrm{d}y$, where $\rho\geqslant 0$ is the constant branching rate, X_t is singular about the Lebesgue measure on $\mathbb{R}^{d+1}$. But if $\xi(r,\mathrm{d}y)$ is given by (0.1) such that the additive function $A_\xi(\mathrm{d}t)$ is a.e.-regular, Dawson and Fleischmann[1] showed that X_t is absolutely continuous for all $d\geqslant 1$. For example if $\xi_1(\mathrm{d}y_1)=\delta_c$, $c\in\mathbb{R}$ (the branching effect is restricted to a single hyperplane) the additive function $A_\xi(\mathrm{d}t)$ is a.e.-regular. Dawson and Fleischmann[1] also gave two examples of randomly selected branching rate function ξ (The branching is allowed only at a countable collection of hyperplanes) such that the associated additive function $A_\xi(\mathrm{d}t)$ is a.e.-regular.

In the case $\xi(r,\mathrm{d}y)=\rho\mathrm{d}y$, where $\rho\geqslant 0$ is a constant, an enhanced model of super-Brownian motion was introduced by Dynkin[2]. For every open set D in $\mathbb{R}^{d+1}$, as a special case of Dynkin[2], there is a corresponding random exit measure X_D related to the boundary value problem

$$\frac{1}{2}\Delta u=\xi u^2\quad\text{in } D,\qquad u=f\quad\text{on }\partial D, \tag{0.2}$$

where f is a positive bounded measurable function on ∂D. The states of the random exit measures

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X_D were studied by Abraham and Le Gall[3] and Sheu[4]. It was shown that if $d \leqslant 1$, the states of X_D are absolutely continuous with respect to the surface area on ∂D, whereas in the case $d > 1$, they are singular. But if $\xi(r, \mathrm{d}y)$ is given by (0.1) such that the associated additive function $A_\xi(\mathrm{d}t)$ is a.e.-regular, the question as to what properties the random exit measures X_D have left open. In the present paper we focus our attention on the extremely simple case of branching allowed only at a single, non-moving and non-random hyperplane. More precisely, we consider the exit measures X_D related to the (formal) equation

$$\begin{cases} \frac{1}{2}\Delta u(x) = \delta_c(x_1)\psi(x, u(x)), x = [x_d, x_1] \in D, \\ \qquad u = f \quad \text{on } \partial D. \end{cases} \tag{0.3}$$

Here $c \in \mathbb{R}$, f is a positive bounded function on ∂D and ψ is given by the formula

$$\psi(x, z) = a(x)z + b(x)z^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz)n_x(\mathrm{d}u), \tag{0.4}$$

where n is a kernel from $\mathbb{R}^{d+1}$ to $(0, \infty)$ and $a(x), b(x)$ and $A(x) = \int_0^\infty u \wedge u^2 n_x(\mathrm{d}u)$ are positive bounded Borel functions. We prove that the random exit measure X_D related to the above boundary value problem (0.3) is absolutely continuous for all $d \geqslant 1$. But before we study the absolutely continuous property of X_D, we must first prove that the random measure X_D related to problem (0.3) exists.

1 Preliminaries and main results

1.1 Preliminaries

Fix dimension $d + 1 (d \geqslant 1)$. For a Borel set E in $\mathbb{R}^{d+1}$, we denote by $\mathscr{B}(E)$ the Borel σ-algebra of E. We write $f \in \mathscr{B}(E)$, if f is a $\mathscr{B}(E)$- measureable function. Writing $f \in p\mathscr{B}(E)(b\mathscr{B}(E))$ means that, in addition, f is positive (bounded). We put $bp\mathscr{B}(E) = (b\mathscr{B}(E) \cap p\mathscr{B}(E))$. If $E = \mathbb{R}^{d-1}$, we simply write $\mathscr{B}$ instead of $\mathscr{B}(\mathrm{R}^{d+1})$. We will use the symbol $\xrightarrow{bp}$ to denote bounded pointwise convergence (recall that functions converge boundedly pointwise if they are uniformfly bounded and converge pointwise).

Write $M(E)$ for the set of all finite measures on E, endowed with the topology of weak convergence. The expression $\langle f, \mu \rangle$ stands for the integral of f with respect to μ. Let $W = (W_t, \Pi_x)$ be a Brownian motion in $\mathbb{R}^{d+1}$. In this paper we will concentrate on the following simple additive function

$$A_c(\mathrm{d}t) = \mathrm{d}t \int_{\mathbb{R}^{d+1}} \delta_c(y_1)\delta_y(W_t)\mathrm{d}y_d \times \mathrm{d}y_1 = \delta_c(W_t^1)\mathrm{d}t,$$

where $c \in \mathbb{R}$ and $W_t = [W_t^d, W_t^1]$. For a point $c \in \mathbb{R}$, $A_c(\mathrm{d}t)$ can be interpreted as the collision local time of W at the hyperplane $\{(x_d c); x_d \in \mathbb{R}^d\}$ or as the Brownian local time $L^c(\mathrm{d}t)$ of W_t^1 at point c. Without loss of generality we assume $c = 0$ and denote

$$A(\mathrm{d}t) = A_0(\mathrm{d}t) = \delta_0(W_t^1)\mathrm{d}t.$$

Throughout this paper τ_D denotes the first exit time of W from an open set D, i.e. $\tau_D = \inf\{t; W_t \notin D\}$. For $x \in D$, the exit distribution $H_D(x, \cdot)$ of D for Brownian motion starting at x is defined by

$$H_D(x, A) = \Pi_x(\tau_D < \infty, W_{\tau_D} \in A), \ A \in \mathscr{B}.$$

It is abvious that $H_D(x, \cdot)$ is concentrated on the boundary ∂D of D. For $f \in bp\mathscr{B}$, define

$$H_Df(x) = \int f(y) H_D(x, \mathrm{d}y) = \Pi_x f(W_{\tau_D}).$$

If D is a bounded smooth domain, then $H_D(x, \mathrm{d}y) = k(x,y)S(\mathrm{d}y)$, where $k(x,y)$ is the Poisson kernel, and $S(\mathrm{d}y)$ is the surface area on ∂D. For $\nu \in M(\partial D)$ define

$$H_D\nu(x) = \int k(x,y)\nu(\mathrm{d}y).$$

Abviously if $\nu = f(y)S(\mathrm{d}y)$, $H_D\nu = H_Df$.

1.2 Main results

First of all we remark that we will always interpret eq. (0.3) in its mild form, i.e. as an integral equation:

$$u(x) + \Pi_x\int_0^{\tau_D} \psi(W_t, u(W_t))A(\mathrm{d}t) = H_Df(x), \tag{1.1}$$

where D is an open set in $\mathbb{R}^{d+1}$, $f \in bp\mathscr{B}$. If D is a bounded smooth domain we will also study the fundamental solutions of the following integral equation

$$u(x) + \Pi_x\int_0^{\tau_D} \psi(W_t, u(W_t))A(\mathrm{d}t) = H_D\nu, \tag{1.2}$$

where $\nu = \sum_{i=1}^{k} \lambda_i \delta_{z_i}$, $z_i \in \partial D$, $\lambda_i \in \mathbb{R}^+$, $i = 1, 2, \cdots, k$.

Theorem 1.1. *Let ψ be given by (0.4). The following results hold.*

(1) *There corresponds a Markov process $X = (X_t, P_\mu)$ in $M(\mathbb{R}^{d+1})$ such that for every $f \in bp\mathscr{B}$ and $\mu \in M(\mathbb{R}^{d+1})$, we have*

$$P_\mu \exp\{-\langle f, X_t\rangle\} = \exp\{-\langle u_t, \mu\rangle\}, \tag{1.3}$$

where u_t is the unique bounded solution to the equation

$$u_t(x) + \Pi_x\int_0^t \psi(W_s, u_{(t-s)}(W_s))A(\mathrm{d}s) = \Pi_x f(W_t), \quad x \in \mathbb{R}^{d+1}. \tag{1.4}$$

(2) *For every bounded open set D in $\mathbb{R}^{d+1}$, there exists a random exit measure X_D such that for every $f \in bp\mathscr{B}$ and $\mu \in M(\mathbb{R}^{d+1})$*

$$P_\mu \exp\{-\langle f, X_D\rangle\} = \exp\{-\langle u, \mu\rangle\}, \tag{1.5}$$

where u is the unique bounded solution of (1.1).

Moreover for $n \geqslant 2$, the joint probability distribution of $X_{D_1}, \cdots, X_{D_n}$ is described as follows. Let

$$I = \{1, 2, \cdots, n\}, \quad \tau_I = \min\{\tau_{D_1}, \cdots, \tau_{D_n}\}, \quad \lambda = \min\{i : \tau_{D_i} = \tau_I\}.$$

Then

$$P_\mu \exp\left\{-\sum_{i=1}^{n} \langle f_i, X_{D_i}\rangle\right\} = \exp\langle -u_I, \mu\rangle, \tag{1.6}$$

where the functions u_I are determined recursively by the integral equations

$$u_I(x) + \Pi_x\int_0^{\tau_I} \psi(W_t, u_I(W_t))A(\mathrm{d}t) = \Pi_x G_I \tag{1.7}$$

with

$$G_I = [f_\lambda + u_{I-\lambda}]\left(W_{\tau_{D_\lambda}}\right)$$

(note that (1.6) and (1.7) with $n=1$ and $u_\varnothing=0$ coincide with (1.5) and (1.1)). Following Dynkin, we call $X=(X_t, X_D, P_\mu)$ the super-Brownian motion with parameters (ψ, A).

The first part of the above theorem is proved by Dynkin[5]. The proof of the second part is given in sec.2. The approach given in this paper is a modification of Theorem 1.1 in ref.[2] in which the additive functions are of the form $A(\mathrm{d}t)=\xi(t, W_t)\mathrm{d}t$, where $\xi(r,x)$ is a positive measurable function on $[0,\infty)\times\mathbb{R}^{d+1}$ satisfying: For every $a\in\mathbb{R}^+$, there exits a constant C_a such that $\xi(r,x)\leqslant C_a$ for all $r\in[0,a)$, $x\in\mathbb{R}^{d+1}$.

We write $\mu\in M_c(D)$ if $\mu\in M(D)$ and μ has a compact support in D.

Theorem 1.2. *Suppose D is a bounded smooth domain in $\mathbb{R}^{d+1}$ and ψ is given by (0.4) with $a\equiv 0$. Then*

(1) *if $\mu\in M_c(D)$, there exists a random measurable function x_D on ∂D such that*

$$P_\mu\{X_D(\mathrm{d}z)=x_D(z)S(\mathrm{d}z)\}=1;$$

(2) *for each finite collection $z_1,\cdots,z_m$ of points in $\partial D\setminus l$, the Laplace function of the random vector $[x_D(z_1),\cdots,x_D(z_m)]$ with respect to P_μ is given by*

$$P_\mu\exp\left[-\sum_{i=1}^m\lambda_i x_D(z_i)\right]=\exp\langle -u,\mu,\rangle,\lambda_1,\cdots,\lambda_m\geqslant 0,$$

where $\mu\in M_c(D)$, $l=\{(x_d,0);x_d\in\mathbb{R}^d\}$ and u is the unique (fundamental) solution of (1.2) with $\nu=\sum_{i=1}^m\lambda_i\delta_{z_i}$.

2 Construction of superprocesses

Throughout this section D is a bounded open set. Let us first state some lemmas on the integral equation (1.1). For $c\in p\mathscr{B}$ and $f\in\mathscr{B}$, put

$$H^c_{D,A}f=\Pi.\left[f(W_{\tau_D})\exp\left(-\int_0^{\tau_D}c(W_s)A(\mathrm{d}s)\right)\right];$$

$$G^c_{D,A}f=\Pi.\int_0^{\tau_D}f(W_t)\exp\left(-\int_0^t c(W_s)A(\mathrm{d}s)\right)A(\mathrm{d}t);$$

$$G_{D,A}f=G^0_{D,A}f;\qquad H_{D,A}f=H^0_{D,A}f.$$

Lemma 2.1. *Suppose c belongs to $bp\mathscr{B}$. For every $f\in bp\mathscr{B}$. $u\in bp\mathscr{B}$ is a solution of (1.1) if and only if $u\in bp\mathscr{B}$ is a solution of the following integral equation:*

$$u=H^c_{D,A}f+G^c_{D,A}(cu-\psi(u)),\tag{2.1}$$

where $\psi(u)(x)=\psi(x,u(x)),x\in\mathbb{R}^{d+1}$.

Proof. Since $A(\mathrm{d}t)=L^0(\mathrm{d}t)$, where L^0 is the Brownian local time of W_t^1 at zero. We have

$$G_{D,A}1=\Pi_x\int_0^{\tau_D}A(\mathrm{d}t)\leqslant\|\Pi_x\tau_D\|_\infty<\infty.\tag{2.2}$$

By the Markov property and Fubini's theorem, it is easy to check that if $f,g\in b\mathscr{B}$, then

$$G_{D,A}g=G^c_{D,A}(cG_{D,A}g)+G^c_{D,A}g=G_{D,A}(cG^c_{D,A}g)+G^c_{D,A}g,\tag{2.3}$$

$$H_{D,A}f=H^c_{D,A}f+G^c_{D,A}(cH_{D,A}f)=H^c_{D,A}f+G_{D,A}(cH^c_{D,A}f).\tag{2.4}$$

If u is a positive bounded solution of (1.1), we have

$$u=H_{D,A}f-G_{D,A}\psi(u)$$

$$
\begin{aligned}
&= H^c_{D,A}f - G^c_{D,A}\psi(u) + G^c_{D,A}[c(H_{D,A}f - G_{D,A}\psi(u))] \\
&= H^c_{D,A}f - G^c_{D,A}\psi(u) + G^c_{D,A}(cu) \\
&= H^c_{D,A}f + G^c_{D,A}(cu - \psi(u)),
\end{aligned}
$$

which is (2.1).

Conversely if u is a positive bounded solution of (2.1), using (2.3) and (2.4) with $g = cu - \psi(u)$, we obtain

$$
\begin{aligned}
G_{D,A}(cu) &= G_{D,A}[cH^c_{D,A}f + cG^c_{D,A}(cu - \psi(u))] \\
&= H_{D,A}f - H^c_{D,A}f + G_{D,A}(cu - \psi(u)) - G^c_{D,A}(cu - \psi(u)) \\
&= H_{D,A}f + G_{D,A}(cu - \psi(u)) - u.
\end{aligned}
$$

Therefore $u = H_{D,A}f + G_{D,A}\psi(u)$.

Lemma 2.2. *Let c, f and Λ belong to $p\mathscr{B}$. If $h_n \in p\mathscr{B}$ satisfy the following conditions:*

$$G_{D,A}(\Lambda h_0) < \infty,$$

$$h_n(x) \leqslant H^{c+\Lambda}_{D,A}f + qG^{c+\Lambda}_{D,A}1 + G^{c+\Lambda}_{D,A}(\Lambda h_{n-1}), \quad \text{for } x \in \mathbb{R}^{d+1},\ n \in \mathbb{N},$$

where M and q are positive constants, then

$$
\begin{aligned}
h_n(x) \leqslant{} & H^c_{D,A}f + qG^c_{D,A}1 \\
& + \Pi_x\int_0^{\tau_D} A(\mathrm{d}t)\exp\left(-\int_0^t (c+\Lambda)(W_s)A(\mathrm{d}s)\right) \\
& \cdot \frac{\left(\int_0^t \Lambda(W_s)A(\mathrm{d}s)\right)^{n-1}}{(n-1)!}\Lambda(W_t)h_0(W_t).
\end{aligned} \tag{2.5}
$$

In particular, if $h_0 = 0$ or if h_n does not depend on n, then

$$h_n(x) \leqslant H^c_{D,A}f(x) + qG^c_{D,A}1, \text{ for } x \in \mathbb{R}^{d+1}. \tag{2.6}$$

Proof. By induction in n, we get

$$
\begin{aligned}
h_n(x) \leqslant{} & \Pi_x\left[\exp\left(-\int_0^{\tau_D}(c+\Lambda)(W_s)A(\mathrm{d}s)\right)\cdot\sum_{i=0}^{n-1}\frac{\left(\int_0^{\tau_D}\Lambda(W_s)A(\mathrm{d}s)\right)^i}{i!}f(W_{\tau_D})\right] \\
& + q\Pi_x\int_0^{\tau_D}A(\mathrm{d}t)\exp\left(-\int_0^t(c+\Lambda)(W_s)A(\mathrm{d}s)\right)\cdot\sum_{i=0}^{n-1}\frac{\left(\int_0^t\Lambda(W_s)A(\mathrm{d}s)\right)^i}{i!} \\
& + \Pi_x\int_0^{\tau_D}A(\mathrm{d}t)\exp\left(-\int_0^t(c+\Lambda)(W_s)A(\mathrm{d}s)\right) \\
& \cdot\frac{\left(\int_0^t\Lambda(W_s)A(\mathrm{d}s)\right)^{n-1}}{(n-1)!}\Lambda(W_t)h_0(W_t).
\end{aligned}
$$

Clearly, this implies (2.5). Letting $n \to \infty$ in inequality (2.5), by the dominated convergence theorem, we get (2.6).

For $0 < \beta < 1$, let

$$
\begin{aligned}
\psi_\beta(x,z) = {} & a(x)z + \int_{(\beta,\infty)}(\mathrm{e}^{-uz} - 1 + uz)n_x(\mathrm{d}u) \\
& + 2b(x)\beta^{-2}[\mathrm{e}^{-\beta z} - 1 + \beta z];
\end{aligned} \tag{2.7}
$$

$$\Lambda_\beta(x) = 2b(x)\beta^{-1} + \int_{(\beta,\infty)}un_x(\mathrm{d}u), \tag{2.8}$$

$$R_\beta(x,z) = [a(x) + \Lambda_\beta(x)]z - \psi_\beta(x,z)$$
$$= \int_{(\beta,\infty)} (1 - e^{-uz}) n_x(du) + 2b(x)\beta^{-2}(1 - e^{-\beta z}). \tag{2.9}$$

Then Λ_β is bounded in $\mathbb{R}^{d+1}$, $R_\beta(x,z)$ is increasing about z and

$$0 \leqslant R_\beta(x,z) \leqslant \Lambda_\beta(x)z, \quad \text{for } x \in \mathbb{R}^{d+1},\ z \in [0,\infty). \tag{2.10}$$

Theorem 2.1. *For every $f \in bp\mathscr{B}$, there exists a positive solution $u(\beta, f)$ of (1.1) with ψ replaced by ψ_β.*

Proof. By Lemma 2.1, we only need to prove that (2.11) has a positive bounded solution

$$u = H_{D,A}^{a+\Lambda_\beta} f + G_{D,A}^{a+\Lambda_\beta} R_\beta(u). \tag{2.11}$$

Define a sequence $u_n(\beta, f)$ by recursive formulae:

$$u_0(\beta,f) = 0;$$
$$u_n(\beta,f) = H_{D,A}^{a+\Lambda_\beta} f + G_{D,A}^{a+\Lambda_\beta} R_\beta(u_{n-1}(\beta,f)). \tag{2.12}$$

By Lemma 2.2

$$0 \leqslant u_n(\beta,f) \leqslant H_{D,A}^{a} f \leqslant \| f \|_\infty,$$

where $\| f \|_\infty = \sup_x |f(x)|$. Since $R_\beta(x,z)$ is increasing about z, there exists $u(\beta,f) \in bp\mathscr{B}$ such that for all $x \in \mathbb{R}^{d+1}$, $u_n(\beta,f)(x) \uparrow u(\beta,f)(x)$.

Using the monotone convergence theorem, letting $n \to \infty$ in (2.12), we know $u(\beta,f)$ is a positive bounded solution of (2.11) and therefore a solution of (1.1).

Theorem 2.2. *Let ψ be given by (0.4). The following results hold.*

(1) (*Existence and uniqueness*). *For every $f \in bp\mathscr{B}(\partial D)$, there is exactly one measurable nonnegative function $U(f)$ defined on $\mathbb{R}^{d+1}$ which solves (1.1).*

(2) (*Continuity*). *$U(f)$ as a map of $bp\mathscr{B}(\partial D) \to bp\mathscr{B}$ is continuous.*

(3) (*First derivative with respect to a small parameter*).

$$\lambda^{-1} U(\lambda f) \xrightarrow[\lambda \to 0^+]{bp} H_{D,A}^{a} f, \quad f \in bp\mathscr{B}(\partial D).$$

Proof. (1) Let $\psi_\beta(x,z)$, $0 < \beta < 1$ be given by (2.7), and let $u(\beta,f)$ be the solution of (1.1) with ψ replaced by ψ_β constructed in Theorem 2.1. Note that

$$|\psi(x,z) - \psi_\beta(x,z)| \leqslant \int_{[0,\beta]} (e^{-uz} - 1 + uz) n_x(du)$$
$$+ 2b(x)\beta^{-2} \left| \frac{1}{2} z^2 \beta^2 - e^{-\beta z} + 1 - \beta z \right|$$
$$\leqslant \int_{[0,\beta]} \frac{1}{2} u^2 z^2 n_x(du) + \frac{1}{3} b(x) \beta z^3.$$

So, for every $C \in (0,\infty)$, there exist constants $\alpha(\beta, C) \to 0$ as $\beta \to 0$ such that

$$\| \psi(x,z) - \psi_\beta(x,z) \|_\infty \leqslant \alpha(\beta, C), \quad \text{for all } \beta \in (0,1),\ 0 \leqslant z \leqslant C. \tag{2.13}$$

Let $L \geqslant 1$ be a constant such that $\| f \|_\infty \leqslant L$, and let

$$\Lambda(x) = [2b(x) + \int_0^\infty u \wedge u^2 n_x(du)]L; \tag{2.14}$$
$$R(x,z) = (a(x) + \Lambda(x))z - \psi(x,z). \tag{2.15}$$

Then

$$[R(x,z)]'_z = \Lambda(x) - \left[2b(x) + \int_0^\infty u(1 - e^{-uz}) n_x(du)\right] z$$

$$= 2b(x)(L - z) + \int_0^1 u(Lu - 1 + e^{-uz}) n_x(du)$$
$$+ \int_1^\infty u(L - 1 + e^{-uz}) n_x(du),$$

which means for all $x \in \mathbb{R}^{d+1}$, $0 \leqslant z \leqslant L$. $0 \leqslant [R(x,z)]_z^1 \leqslant \Lambda(x)$.

Consequently

$$|R(x,z_1) - R(x,z_2)| \leqslant \Lambda(x)|z_1 - z_2|, \quad \text{for all } x \in \mathbb{R}^{d+1},\ 0 \leqslant z_1, z_2 \leqslant L. \tag{2.16}$$

Combining (2.13) and (2.16) we get

$$|R_\beta(x,z_1) - R_{\beta'}(x,z_2)| \leqslant \alpha(\beta,L) + \alpha(\beta',L) + \Lambda(x)|z_1 - z_2|,$$
$$\text{for } x \in \mathbb{R}^{d+1},\ 0 \leqslant z_1,\ z_2 \leqslant L,\ \beta,\ \beta' \in (0,1), \tag{2.17}$$

where $\alpha(\beta, L)$ and $\beta \in (0,1)$ are constants satisfying $\alpha(\beta,L) \to 0$ as $\beta \to 0$. By Lemma 2.1, for every $\beta \in (0,1)$, $u(\beta,f)$ satisfies

$$u(\beta,f) = H_{D,A}^{a+A} f + G_{D,A}^{a+A} R(u(\beta,f)).$$

Let $h_{\beta,\beta'} = |u(\beta,f) - u(\beta',f)|$. By (2.17) and the above equality we have

$$h_{\beta,\beta'} \leqslant q G_{D,A}^{c+A} 1 + G_{D,A}^{c+A}(\Lambda h_{\beta,\beta'}),$$

where $q = \alpha(\beta,L) + \alpha(\beta',L)$. By Lemma 2.2, $h_{\beta,\beta'} \leqslant q G_{D,A}^a 1$. By (2.2), $\| G_{D,A} 1 \|_\infty < \infty$. So there exists a function $U(f) \in bp\mathscr{B}$ such that

$$u(\beta,f) \xrightarrow{bp} U(f).$$

The dominated convergence theorem implies $U(f)$ is a bounded positive solution of (1.1).

The uniqueness can be proved similarly as above. In fact, assume that $u_1, u_2 \in bp\mathscr{B}$ are two solutions of (1.1), and $L \geqslant 1$ is a constant such that $0 \leqslant u_1, u_2 \leqslant L$. Let Λ and R be defined as in (2.14) and (2.15), respectively. Then similarly we get

$$|u_1 - u_2| \leqslant G_{D,A}^{a+A}(\Lambda(|u_1 - u_2|)).$$

By Lemma 2.2, $u_1 \equiv u_2$.

(2) Let Λ, R be given by (2.14) and (2.15), respectively with constant L satisfying $L \geqslant 1 \vee \|f_1\|_\infty \vee \|f_2\|_\infty$. Then

$$|U(f_1) - U(f_2)| \leqslant H_{D,A}^{a+A}|f_1 - f_2| + G_{D,A}^{a+A}(\Lambda |U(f_1) - U(f_2)|).$$

By Lemma 2.2, $|U(f_1) - U(f_2|) \leqslant H_{D,A}^a |f_1 - f_2| \leqslant \|f_1 - f_2\|_\infty$, which means statement (2) is valid.

(3) By Lemma (2.1) $U(\lambda f)$ satisfies

$$U(\lambda f) = \lambda H_{D,A}^a f - G_{D,A}^a \Phi(U(\lambda f)), \tag{2.18}$$

where $\Phi(x,z) = \psi(x,z) - a(x)z$. Consequently, $\Phi(x,z)$ is increasing about $z \geqslant 0$ and for any constant $C > 0$, $\Phi(x, C\lambda)/\lambda \to 0$ as $\lambda \to 0^+$. Letting $n \to \infty$ in (2.18), by (2.2) and the dominated convergence theorem we have $\frac{1}{\lambda} U(\lambda f) \xrightarrow{bp} H_{D,A}^a f$ as $\lambda \to 0^+$.

A real-valued function u on the Abelian semigroup $G = bp\mathscr{B}$ is called negative definite if

$$\sum_{i,j=1}^n \lambda_i \lambda_j u(g_i + g_j) \leqslant 0.$$

For every $n \geqslant 2$, all $g_1, \cdots, g_n \in G$ and all $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ such that $\sum \lambda_i = 0$. It is known that if u is negative definite, then $L(f) = e^{-u(f)}$ is positive definite[2].

Proof of Theorem 1.1. By Theorem 2.2, u_1 satisfying (1.7) exists and is unique, and G_I

and u_I are positive if $f_i \geqslant 0$. We consider G_I and u_I as functions of $(f_1, \cdots f_n) \in (bp\mathscr{B})^n$. By induction on n and the construction process of u_I given by Theorems 2.1 and 2.2, we can prove that u_I is negative definite and vanishes if $f_1 = \cdots = f_n = 0$ (For details see e.g. Dynkin[2]). Let

$$L_I(f_1, \cdots, f_n) = \exp\langle - u_I, \mu \rangle.$$

Then L_I is positive definite. By Theorem 2.2 L_I is continuous. It follows from Lemma 1.4 and sec. 1.6 in ref. [2] that there exists a unique probability measure P_μ on $M(\mathbb{R}^{d+1})^n$ such that (1.6) and (1.7) hold. It is easy to see that $L_I(f_1, \cdots, f_n) = L_J(f_1, \cdots, f_{n-1})$ if $J = \{1, \cdots, n-1\}$ and $f_n = 0$. Therefore the existence of the stochastic process (X_D, P_μ) subject to statement (2) of Theorem 1.1 follows from Kolmogorov's theorem.

3 Absolute continuous states of X_D

The purpose of this section is devoted to the proof of Theorem 1.2. Throughout this section, D is a bounded smooth domain. Notation C always denotes a constant which may change values from line to line. For $y \in \mathbb{R}^{d+1}$ we denote $y = [y_d, y_1]$ with $y_d \in \mathbb{R}^d$, $y_1 = \mathbb{R}$. For any set $\Gamma \subset \mathbb{R}^{d+1}$, let $\Gamma_0 = \{y_d \in \mathbb{R}^d$ such that $[y_d, 0] \in \Gamma\}$. We replace (1.2) by an equivalent integral equation

$$u(x) + \int_{D_0} g(x, [y_d, 0])\psi(u)([y_d, 0])\mathrm{d}y_d = \int_{\partial D} k(x, z)\nu(\mathrm{d}z), \quad x \in D, \tag{3.1}$$

where $g(x, y)$ is the Green function of Brownian motion in D, and $k(x, z)$ is the Possion kernel. Note that there is a constant C depending only on D such that

$$k(x, z) \leqslant C \| x - z \|^{-d}, \quad x \in D, \ z \in \partial D. \tag{3.2}$$

Theorem 3.1. *Let $\nu_n \in M(\in \partial D)$ be a sequence of measures such that there exists a compact set B satisfying* $\mathrm{supp}(\nu_n) \subset B$ *and $B \cap l = \emptyset$, where $l = \{(x_d, 0), x_d \in \mathbb{R}^d\}$. Suppose for each n, u_n is a solution of (3.1) with ν replaced by ν_n. If ν_n converges weakly to ν in $M(\partial D)$, then there exists a subsequence $n_k \to \infty$ (as $k \to \infty$) and a measurable function u in D such that $u_{n_k} \xrightarrow{bp} u$ in every compact set $K \subset D$ and u satisfies (3.1).*

Proof. *Step* 1. We show that the family $\psi(u_n)([y_d, 0])$ is relatively weakly compact in $L^1(D_0, \mathrm{d}y_d)$. By the Dunford-Pettis theorem (see e.g. IV. 8, Corollary 11 of ref. [6]) we only need to prove that for any $\varepsilon > 0$, it is possible to find $\sigma > 0$ such that for any n and any measurable set $E \subset D_0$,

$$\mathrm{m}(E) < \sigma \text{ implies } \int_E \psi(u_n)([y_d, 0])\mathrm{d}y_d < \varepsilon, \tag{3.3}$$

where m is the Lebesgue measure in $\mathbb{R}^d$.

Note that

$$\begin{aligned} \psi(x, z) &= a(x)z + b(x)z^2 + \int_0^1 (uz - 1 + \mathrm{e}^{-uz})n_x(\mathrm{d}u) + \int_1^\infty (uz - 1 + \mathrm{e}^{-uz})n_x(\mathrm{d}u) \\ &\leqslant a(x)z + b(x)z^2 + \left(\frac{1}{2}\int_0^1 u^2 n_x(\mathrm{d}u)\right)z^2 \\ &\quad + \left(\int_1^\infty u n_x(\mathrm{d}u)\right)z \leqslant C(z + z^2). \end{aligned} \tag{3.4}$$

For every $E \subset D_0$ satisfying $\mathrm{m}(E) \leqslant 1$ and $M > 0$, we have

$$\int_E \psi(u_n)([y_d, 0])\mathrm{d}y_d \leqslant C\int_E (u_n + u_n^2)([y_d, 0])\mathrm{d}y_d \leqslant C\int_E u_n^2([y_d, 0])\mathrm{d}y_d$$

$$\leqslant C\left[M^2\int_E \mathrm{d}y_d + \int_{D_0\cap(u_n>M)} u_n^2([y_d,0])\mathrm{d}y_d\right]$$
$$= C\left[M^2\mathrm{m}(E) - \int_M^\infty \lambda^2 \mathrm{d}\beta_n(\lambda)\right], \tag{3.5}$$

where $\beta_n(\lambda) = \int_{D_0\cap(u_n>\lambda)} \mathrm{d}y_d$, $\lambda > 0$. Set $h_n(x) = \int_{\partial D} k(x,z)\nu_n(\mathrm{d}z)$.

Clearly $u_n \leqslant h_n$ and therefore $\beta_n(\lambda) \leqslant \gamma_n(\lambda)$, where $\gamma_n(\lambda) = \int_{D_0\cap(h_n>\lambda)} \mathrm{d}y_d$. By the assumption of supp $(\nu_n) \subset B$ and noticing that $\sup_n \mu_n(\partial D) < \infty$, we have

$$\lambda\gamma_n(\lambda) \leqslant \int_{D_0\cap(h_n>\lambda)} h_n([y_d,0])\mathrm{d}y_d = \int_{\partial D}\mu_n(\mathrm{d}z)\int_{D_0\cap(h_n>\lambda)} k([y_d,0],z)\mathrm{d}y_d$$
$$\leqslant C\sup_{z\in B}\int_{D_0\cap(h_n>\lambda)} k([y_d,0],z)\mathrm{d}y_d. \tag{3.6}$$

Choosing $\alpha > 2$, by Hölder's inequality, we have for $z \in B$,

$$\int_{D_0\cap(h_n>\lambda)} k([y_d,0],z)\mathrm{d}y_d \leqslant (\gamma_n(\lambda))^{\frac{1}{\alpha'}} \cdot F(z)^{\frac{1}{\alpha}}, \tag{3.7}$$

where $\frac{1}{\alpha'} + \frac{1}{\alpha} = 1$ and $F(z) = \int_{D_0} k([y_d,0],z)^\alpha \mathrm{d}_{y_d}$. By (3.2) $F(Z) \leqslant C\int_{D_0} \| [y_d,0] - z \|^{-\alpha d}\mathrm{d}y_d$. Since D_0 is bounded and $B \cap l = \emptyset$, $\sup_{z\in B} F(z) < \infty$. Combining (3.6) and (3.7) we get $\lambda\gamma_n(\lambda) \leqslant C\gamma_n(\lambda)^{\frac{1}{\alpha'}}$, and so

$$\beta_n(\lambda) \leqslant \gamma_n(\lambda) \leqslant C\lambda^{-\alpha} \quad \text{for all } \lambda > 0. \tag{3.8}$$

By integration by parts and (3.8),

$$-\int_M^\infty \lambda^2 \mathrm{d}\beta_n(\lambda) = M^2\beta_n(M) + 2\int_M^\infty \beta_n(\lambda)\lambda \mathrm{d}\lambda$$
$$\leqslant C\left[M^{2-\alpha} + 2\int_M^\infty \lambda^{1-\alpha}\mathrm{d}\lambda\right] \leqslant CM^{2-\alpha}. \tag{3.9}$$

Condition (3.3) follows easily from (3.5) and (3.9), and therefore $\{\psi(u_n)([y_d,0])\}$ is relatively weakly compact.

Step 2. Choose a sequence $n_k \to \infty$ such that $\{\psi(u_{n_k})([y_d,0])\}$ converges weakly to ω in $L^1(D_0, \mathrm{d}y_d)$. Fix $x \in D$ and let $B_m = \left\{y; \| y - x \| \leqslant \frac{1}{m}\right\}$. Then for sufficiently large m, $B_m \subset D$ and

$$\int_{D_0} g(x,[y_d,0])\psi(u_{n_k})([y_d,0])\mathrm{d}y_d = I + J, \tag{3.10}$$

where

$$I = \int_{(B_m)_0} g(x,[y_d,0])\psi(u_{n_k})([y_d,0])\mathrm{d}y_d \tag{3.11}$$

and

$$J = \int_{D_0\setminus(B_m)_0} g(x,[y_d,0])\psi(u_{n_k})([y_d,0])\mathrm{d}y_d. \tag{3.12}$$

Since $g(x,[y_d,0])$ is bounded in $D_0 \setminus (B_m)_0$,

$$J \to \int_{D_0 \setminus (B_m)_0} g(x,[y_d,0])\omega(y_d)\mathrm{d}y_d, \quad \text{as } k \to \infty. \tag{3.13}$$

For any compact set $K \subset D$, u_n is uniformly bounded in K by the following domination:

$$u_n(y) \leqslant h_n(y) \leqslant C\int_{\partial D} \| y - z \|^{-d} \nu_n(\mathrm{d}z).$$

Hance by (2.2) and the dominated convergence theorem

$$I \leqslant C\Pi_x \int_0^{\tau_D} I_{B_m}(W_s)A(\mathrm{d}s) \to 0, \quad \text{as } m \to \infty. \tag{3.14}$$

Letting $k \to \infty$ and letting $m \to \infty$ in (3.10), by (3.11)—(3.14) we obtain

$$\int_{D_0} g(x,[y_d,0])\psi(u_n)([y_d,0])\mathrm{d}y_d \to \int_{D_0} g(x,[y_d,0])\omega(y_d)\mathrm{d}y_d. \tag{3.15}$$

Since ν_n converges weakly to ν, $h_n(x)$ converges to $h(x)$, where

$$h(x) = \int_{\partial D} k(x,z)\nu(\mathrm{d}z).$$

Passing to a limit in (3.1) with $u = u_{n_k}$ and $\nu = \nu_{n_k}$ we obtain, by (3.15), $u_{n_k} \to u(x)$ pointwisely and

$$u(x) + \int_{D_0} g(x,[y_d,0])\omega(y_d)\mathrm{d}y_d = h(x).$$

Since u_{n_k} is uniformly bounded in any compact set $K \subset D$, $u_{n_k} \xrightarrow{bp} u$ in any compact set $K \subset D$. It remains to prove that $\psi(u)([y_d,0]) = \omega(y_d)$, $y_d \in D_0$. To do this it suffices to show that $\psi(u_{n_k})([y_d,0])$ converges weakly in $L^1(D_0,\mathrm{d}y_d)$ to $\psi(u)([y_d,0])$. Let $f \in L^\infty(D_0,\mathrm{d}y_d)$ and K be an arbitrary compact set in D. Then

$$\left|\int_{D_0} \psi(u_{n_k})([y_d,0])f(y_d)\mathrm{d}y_d - \int_{D_0} \psi(u)([y_d,0])f(y_d)\mathrm{d}y_d\right| \leqslant I' + J', \tag{3.16}$$

where

$$I' = \left|\int_{K_0} [\psi(u_{n_k}) - \psi(u)]([y_d,0])f(y_d)\mathrm{d}y_d\right|,$$

$$J' = \left|\int_{D_0 \setminus K_0} [\psi(u_{n_k}) - \psi(u)]([y_d,0])f(y_d)\mathrm{d}y_d\right|.$$

The bounded convergence theorem implies that $I' \to 0$ as $k \to \infty$. By Fatou's lemma, $J' \leqslant C \sup_k C\int_{D_0 \setminus K_0} \psi(u_{n_k})([y_d,0])\, d_{y_d}$. Condition (3.3) implies that $J' \to 0$ as $K \uparrow D$. Letting $k \to \infty$ and then $K \uparrow D$ in (3.16) we get

$$\int_{D_0} \psi(u_{n_k})([y_d,0])f(y_d)\mathrm{d}y_d \to \int_{D_0} \psi(u)([y_d,0])f(y_d)\mathrm{d}y_d,$$

which completes the proof of Theorem 3.1.

Theorem 3.2 (Fundamental solutions). *Suppose that z_i, $i = 1, \cdots, m$ are points belonging to $\partial D \setminus l$. The following statements hold.*

(1) (*Existence and uniqueness*). *There is exactly one measurable nonnegative function $U(\nu)$ defined in $\mathbb{R}^{d+1}$ which solves* (1.2) *iwth $\nu = \sum_{i=1}^{m} \lambda_i \delta_{z_i}$, $\lambda_i \in \mathbb{R}^+$, $i = 1, \cdots, m$.*

(2) (*First derivative with respect to small parameter*). *If φ is given by* (0.4) *with $a \equiv 0$*

and ν *has a finite support satisfying* $\operatorname{supp}(\nu)\cap l=\emptyset$, *then*

$$\lambda^{-1}U(\lambda\nu)\xrightarrow[\lambda\to 0^+]{bp}\int_{\partial D}k(x,z)\nu(\mathrm{d}z) \tag{3.17}$$

in any compact set $K\subset D$.

Proof. (1) *Existence*. Without loss of generality, assume $\nu=\lambda_1\delta_{z_1}$, $z_1\in\partial D\setminus l$, $\lambda_1\in\mathbb{R}^+$. Let

$$O_n=\left\{x;x\in\partial D,\ \|x-z_1\|<\frac{1}{n}\right\};$$

$$f_n(z)=\begin{cases}1/S(O_n), & z\in O_n,\\ 0, & z\notin O_n;\end{cases} \tag{3.18}$$

$$v_n=\lambda_1 fn(z)S(\mathrm{d}z). \tag{3.19}$$

Clearly as $n\to\infty$, ν_n converges weakly to ν. Let $U(\nu_n)$ be a solution of (3.1) with ν replaced by ν_n. By Theorem 3.1, there exists a sequence $n_k\to\infty$ such that $U(\nu_{u_k})\to U(\nu)$ in D and $U(\nu)$ satisfies (3.1) which is equivalent to (1.2).

Uniqueness. Let u_1, u_2 be two solutions of (1.2). Then

$$u_1-u_2+G_{D,A}(\psi(u_1)-\psi(u_2))=0 \tag{3.20}$$

and

$$0\leqslant u_1,u_2\leqslant h(x), \tag{3.21}$$

where $h(x)=\int_{\partial D}k(x,z)\nu(\mathrm{d}z)$.

Note that (2.3) is also valid for $c\in p\mathscr{B}$ and $g\in\mathscr{B}$ satisying $G_{D,A}|g|<\infty$. Therefore, using (2.3) with $g=\psi(u_1)-\psi(u_2)$, (3.20) can be rewritten as

$$u_1-u_2=G_{D,A}^{a+A}[R(u_1)-R(u_2)], \tag{3.22}$$

where

$$A(x)=\left[2b(x)+\int_0^\infty u\wedge u^2n_x(\mathrm{d}u)\right]L(x);\ L(x)=h(x)+1,$$

$$R(x,z)=(a+A(x))z-\psi(x,z).$$

As in the proof of Theorem 2.2, for all $x\in D$, $0\leqslant z_1,z_2\leqslant L(x)$

$$|R(x,z_1)-R(x,z_2)|\leqslant A(x)|z_1-z_2|. \tag{3.23}$$

Consequently, by (3.21), (3.22) and (3.23)

$$|u_1-u_2|\leqslant G_{D,A}^{a+A}(A|u_1-u_2|). \tag{3.24}$$

By (3.2) and the assumption on ν, we get

$$M:=\sup_{y_d\in D_0}h([y_d,0])\leqslant\int_{\partial D}\sup_{y_d\in D_0}\|[y_d,0]-z\|^{-d}\nu(\mathrm{d}z)<\infty. \tag{3.25}$$

Since $\lambda|u_i|\leqslant C(h(x)+1)h(x)$, $i=1,2$, by (2.2), we have

$$G_{D,A}(\lambda|u_i|)\leqslant CM(M+1)\int_{D_0}g(x,[y_d,0])\mathrm{d}y_d$$

$$=CM(M+1)\Pi_x\int_0^{\tau_D}A(\mathrm{d}t)<\infty,\ i=1,2.$$

Thus by Lemma 2.2, $u_1\equiv u_2$.

(2) Note that

$$U(\lambda\nu)/\lambda = \int_{\partial D} k(x,z)\nu(dz) - \frac{1}{\lambda}\int_{D_0} g(x,[y_d,0])\psi(U(\lambda\nu))([y_d,0])dy_d,$$

and $\int_{\partial D} k(x,z)\nu(dz) = h(x)$ is locally bounded in D. To prove the desired result, it suffices to prove that for $x \in D$,

$$A_\lambda(x) := \frac{1}{\lambda}\int_{D_0} g(x,[y_d,0])\psi(U(\lambda\nu))([y_d,0])dy_d \to 0, \ x \in D, \text{ as } \lambda \to 0.$$

By (3.25)

$$U(\lambda\nu)([y_d,0]) \leqslant \lambda h([y_d,0]) \leqslant \lambda M, \ y_d \in D_0.$$

Therefore

$$A_\lambda(x) \leqslant \int_{D_0} g(x,[y_d,0])\psi([y_d,0],\lambda M)/\lambda \, dy_d.$$

Since $\int_{D_0} g(x,[y_d,0])dy_d = G_{D,A}1(x) < \infty$, letting $n \to \infty$ in the above inequality, using (3.4) and the dominated convergence theorem, we get $A_\lambda(x) \to 0$ as $\lambda \to 0^+$. This finishes the proof of Theorem 3.2.

To prove Theorem 1.2, we first state a lemma which is a modification of Lemma 2.7.1 in reference [1].

Lemma 3.1. *Let Y be a random measure defined on a probability space $(\Omega, \mathscr{B}, P)$ with valuees in $M(\partial D)$. Assume that*

(a) *there exists a Borel subset $N \subset \partial D$ of surface zero such that for each $z \in \partial D \setminus N$, there is a sequence $\varepsilon_n(z) \to 0$ and as $n \to \infty$,*

$$\frac{Y(O_{\varepsilon_n}(z))}{S(O_{\varepsilon_n}(z))} \xrightarrow[n\to\infty]{} \eta(z) \ \textit{in law},$$

where $O_\varepsilon(z) = \{x;\ \|x-z\| < \varepsilon\}$, $\varepsilon > 0$ and $\eta(z)$ is a random variable with $P\eta(z) < \infty$.

(b) $P(f, Y) = \int_{\partial D} f(z) \cdot P\eta(z) S(dz)$ *for all $f \in C(\partial D)$.*

Then there exists a random measurable function y in ∂D such that $P\{Y(dz) = y(x)S(dz)\} = 1$, and for each $z \in \partial D \setminus N$, the random variable $y(z)$ and $\eta(z)$ are identically distributed. In particular, Y is an absolutely continuous measure (with respect to $S(dz)$) on ∂D.

Morover, if (a) even holds for vectors, i.e. there is an exceptional set N such that for each choice of finitely many points $z_1, \cdots, z_m$ in $\partial D \setminus N$ there is a sequence $\varepsilon_n(z_1, \cdots, z_m) \to 0$ and as $n \to \infty$

$$\left(\frac{Y(O_{\varepsilon_n}(z_1))}{S(O_{\varepsilon_n}(z_1))}, \cdots, \frac{Y(O_{\varepsilon_n}(z_m))}{S(O_{\varepsilon_n}(z_m))}\right) \xrightarrow[n\to\infty]{} \textit{some } (\eta(z_1)), \cdots, \eta(z_m)) \ \textit{in law},$$

then $(y(z_1), \cdots, y(z_m)) = (\eta(z_1), \cdots, \eta(z_m))$ in distribution.

Proof of Theorem 1.2. (1) (assumption (a) of Lemma 3.1). We choose $z_1, \cdots z_m \in \partial D \setminus l$ and let

$$\nu_n(dz) = \sum_{i=1}^{m} \lambda_i f_n(z_i) S(dz),$$

where $f_n(z)$ is given by (3.18). We have by (1.5)

$$P_\mu \exp\left\{-\left\langle\sum_{i=1}^m \lambda_i f_n(z_i), X_D\right\rangle\right\} = \exp\{-\langle U(\nu_n), \mu\rangle\},$$

where $U(\nu_n)$ is the unique solution of (1.2) with ν replaced by ν_n. Clearly as $n \to \infty$, ν_n converges weakly to $\nu =: \sum_{i=1}^m \lambda_i \delta_{z_i}$. By Theorem 3.1, there exists a sequence $n_k \to \infty$ such that $u(\nu_{n_k}) \xrightarrow{bp} U(\nu)$ in all compact subset $K \subset D$ and $U(\nu)$ is the unique solution of (1.2). The bounded convergence theorem implies that

$$P_\mu \exp\{-\langle U(\nu_{n_k}), \mu\rangle\} \to \exp\{-\langle U(\nu), \mu\rangle\}. \tag{3.26}$$

The left-hand side of (3.26) determines the Laplace transform of the random vector $[X_D(O_n(z_1)), \cdots, X_D(O_n(z_m))]$. Note that

$$\langle U(\nu), \mu\rangle \leqslant \langle H_{D,A}\nu, \mu\rangle \leqslant \|\mu\| \sup_{x \in \sup(\mu)} H_{D,A}\nu(x) \leqslant C \mid \lambda \mid,$$

where $\mid \lambda \mid = \max_i \lambda_i$. Therefore the right-hand side of (3.26) determines the Laplace transform of a random vector, we denote $[\eta(z_1), \cdots, \eta(z_m)]$. Consequently,

$$[X_D(O_n(z_1)), \cdots, X_D(O_n(z_m))] \to [\eta(z_1), \cdots, \eta(z_m)] \text{ in law},$$

where

$$P_\mu \exp\left|-\sum_{i=1}^m \lambda_i \eta(z_i)\right| = \exp\{-\langle U(\nu), \mu\rangle\}. \tag{3.27}$$

(2) (Assumption (b) of Lemma 3.1). Set $m = 1$ and write z instead of z_1. By (3.27) and (3.17) we have

$$P_\mu(\eta(z)) = \lim_{\lambda \to 0}\left\langle \frac{U(\lambda\delta_z)}{\lambda}, \mu\right\rangle = (k(\cdot, z), \mu) = \int_D k(x, z)\mu(\mathrm{d}x).$$

Thus, for every $f \in C(\partial D)$, by the above equality and (1.5),

$$P_\mu\langle f, X_D\rangle = \int_D \mu(\mathrm{d}x)\int_{\partial D} k(x, z) f(z) S(\mathrm{d}z) = \int_{\partial D} f(z) \cdot P_\mu(\eta(z)) S(\mathrm{d}z).$$

Therefore the statements of Theorem 1.2 follow from Lemma 3.1.

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