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Acta Mathematica Scientia 2006,**26B**(2):358–370

Acta Mathematica Scientia
数学物理学报

www.wipm.ac.cn/publish/

ABSOLUTE CONTINUITIES OF EXIT MEASURES AND TOTAL WEIGHTED OCCUPATION TIME MEASURES FOR SUPER-α-STABLE PROCESSES*

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Abstract Suppose X is a super-α-stable process in $\mathbf{R}^d$, $(0<\alpha<2)$, whose branching rate function is dt, and branching mechanism is of the form $\Psi(z)=z^{1+\beta}$ $(0<\beta\le 1)$. Let X_τ and Y_τ denote the exit measure and the total weighted occupation time measure of X in a bounded smooth domain D, respectively. The absolute continuities of X_τ and Y_τ are discussed.

Key words Super-α-stable process, absolute continuity, exit measure, total weighted occupation time measure

2000 MR Subject Classification 60J80, 60J45

1 Introduction

For every Borel-measurable space $(E,\mathcal{B}(E))$, we denote by $\mathcal{M}(E)$ the set of all finite measures on $\mathcal{B}(E)$ endowed with the topology of weak convergence; denote by $\mathcal{M}_c(E)$ the set of all finite measures on $\mathcal{B}(E)$ with compact support; denote by $\mathcal{M}_0(E)$ the set of all finite measures on $\mathcal{B}(E)$ with finite points support. The expression $\langle f,\mu\rangle$ stands for the integral of f with respect to μ, that is, $\langle f,\mu\rangle=\int f(x)\mu(dx)$. We write $f\in\mathcal{B}(E)$ if f is a $\mathcal{B}(E)$-measurable function. Writing $f\in p\mathcal{B}(E)(b\mathcal{B}(E))$ means that, in addition, f is positive (bounded). We put $bp\mathcal{B}(E)=b\mathcal{B}(E)\cap p\mathcal{B}(E)$. If $E=\mathbf{R}^d$, we simply write $\mathcal{B}$ instead of $\mathcal{B}(\mathbf{R}^d)$ and $\mathcal{M}$ instead of $\mathcal{M}(\mathbf{R}^d)$.

Let $\xi=\{\xi_s,\Pi_x,s\ge 0,x\in\mathbf{R}^d\}$ denote a symmetric α- stable process $(0<\alpha<2)$. We denote by $\mathcal{T}$ the set of all exit times from open sets in $\mathbf{R}^d$. Set $\mathcal{F}_{\le r}=\sigma(\xi_s,s\le r)$; $\mathcal{F}_{>r}=\sigma(\xi_s,s>r)$ and $\mathcal{F}_\infty=\cup\{\mathcal{F}_{\le r},r\ge 0\}$. For $\tau\in\mathcal{T}$, we put $F\in\mathcal{F}_{\ge\tau}$ if $F\in\mathcal{F}_\infty$ and if for each $r,\{F,\tau>r\}\in\mathcal{F}_{>r}$.

Throughout this article, C denotes a constant which may change values from line to line.

*Received April 8, 2004. Supported by NNSF of China (10001020 and 10471003), Foundation for Authors Awarded Excellent Ph.D.Dissertation.

For $\beta \in (0,1]$, there exists a Markov process $X = (X_t, P_\mu)$ in $\mathcal{M}$ such that the following conditions are satisfied:

(1) If f is a bounded continuous function, then $\langle f, X_t \rangle$ is right continuous in t on $[0,\infty)$.

(2) For every $\mu \in \mathcal{M}$ and for every $f \in bp\mathcal{B}$,

$$P_\mu \exp\langle -f, X_t \rangle = \exp\langle -v_t, \mu \rangle, \tag{1}$$

where v is the unique solution of the integral equation

$$v_t(x) + \Pi_x \int_0^t (v_{t-s}(\xi_s))^{1+\beta} \mathrm{d}s = \Pi_x f(\xi_t). \tag{2}$$

Moreover, for every $\tau \in \mathcal{J}$, there are corresponding random measures X_τ and Y_τ on $\mathbf{R}^d$ associated with the first exit time τ such that, for $f, g \in bp\mathcal{B}$

$$P_\mu \exp\{-\langle f, X_\tau \rangle - \langle g, Y_\tau \rangle\} = \exp\langle -u, \mu \rangle, \tag{3}$$

where u is the unique solution of the integral equation

$$u(x) + \Pi_x \int_0^\tau (u(\xi_s))^{1+\beta} \mathrm{d}s = \Pi_x \Big[\int_0^\tau g(\xi_s) \mathrm{d}s + f(\xi_\tau) \Big]. \tag{4}$$

We call $X = \{X_t, X_\tau, Y_\tau, P_\mu\}$ the super-α-stable process with branching mechanism $z^{1+\beta}$. Throughout this paper τ denotes the first exit time of ξ from an open set D in $\mathbf{R}^d$, that is, $\tau \equiv \inf\{t > 0 : \xi_t \notin D\}$. And we call X_τ the exit measure of X in D, Y_τ the total weighted occupation time measure of X in D. From the properties of the super-α- stable process, we know that the support of X_τ is contained in $\overline{D}^c$, the support of Y_τ is contained in D. We will discuss the absolute continuity of X_τ and Y_τ.

2 Absolute Continuity of X_τ

From this point on, we always assume that D is a bounded smooth domain in $\mathbf{R}^d$. Let $K_D(x,z)$ denote the Poisson kernel of ξ in D. For $\nu \in \mathcal{M}(\overline{D}^c), f \in b\mathcal{B}(\overline{D}^c)$, define

$$H_D\nu(x) = \begin{cases} \displaystyle\int_{\overline{D}^c} K_D(x,z)\nu(\mathrm{d}z), & d=1, \\ \displaystyle A(d,\alpha) \int_{\overline{D}^c} K_D(x,z)\nu(\mathrm{d}z), & d \geq 2; \end{cases} \tag{5}$$

$$H_D f(x) = \Pi_x f(\xi_\tau) = \begin{cases} \displaystyle\int_{\overline{D}^c} K_D(x,z) f(z)\mathrm{d}z, & d=1, \\ \displaystyle A(d,\alpha) \int_{\overline{D}^c} K_D(x,z) f(z)\mathrm{d}z, & d \geq 2, \end{cases} \tag{6}$$

where

$$A(d,\alpha) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}.$$

Obviously, if $\nu(\mathrm{d}y) = f(y)\mathrm{d}y$, then $H_D f = H_D \nu$.

The study of the fundamental solutions of the following integral equation plays an important role in the investigation of the absolute continuity of X_τ,

$$u(x) + \Pi_x \int_0^\tau u^{1+\beta}(\xi_s)\mathrm{d}s = H_D\nu(x), \quad x \in D, \tag{7}$$

where $\nu \in \mathcal{M}_0(\overline{D}^c)$.

For $\nu = \sum\limits_{i=1}^{m} \lambda_i \delta_{z_i}$ with $z_1, z_2, \cdots, z_m \in \overline{D}^c$, $\lambda_i \in \mathbf{R}^+$, $i = 1, 2, \cdots, m$, let

$$\nu_n(\mathrm{d}z) = f_n(z)\mathrm{d}z, \tag{8}$$

where $\mathrm{d}z$ denotes the Lebesgue measure on $\overline{D}^c$ and

$$f_n(z) = \sum_{i=1}^{m} \lambda_i f_n^{z_i}(z), \tag{9}$$

$$f_n^{z_i}(z) = \begin{cases} \dfrac{1}{V(\overline{D}^c \cap B(z_i, 1/n))}, & z \in B(z_i, 1/n), \\ 0, & z \notin B(z_i, 1/n) \end{cases} \tag{10}$$

with $V(\overline{D}^c \cap B(z_i, 1/n))$ being the volume of $\overline{D}^c \cap B(z_i, 1/n)$. Clearly, as $n \to \infty$, $\nu_n \xrightarrow{w} \nu$. ν_n is called the regularization of ν.

We now give a theorem on the fundamental solutions of the integral equation (7).

Theorem 2.1 Suppose $D \subset \mathbf{R}^d$ is a bounded smooth domain, $\nu = \sum\limits_{i=1}^{m} \lambda_i \delta_{z_i}$, $z_1, z_2, \cdots, z_m \in \overline{D}^c, \lambda_i \in \mathbf{R}^+, i = 1, 2, \cdots, m$. Let ν_n be defined by (8), (9) and (10). Then we have

(1) (Existence and Uniqueness) There is exactly one measurable nonnegative function $U[\nu]$ defined in D which satisfies the equation (7).

(2) (Continuity of Regularization) The solution $U[\nu]$ is continuous with respect to the operation of regulation of ν in the following sense:

$$U[\nu_n](\cdot) \xrightarrow{bp} U[\nu](\cdot)(n \to \infty) \quad \text{in } D. \tag{11}$$

(3) (First Derivative with Respect to Small Parameter)

$$\lambda^{-1} U[\lambda\nu](\cdot) \xrightarrow{bp} H_D\nu(\cdot)(\lambda \to 0) \quad \text{in } D. \tag{12}$$

Using the above theorem we get the main result with respect to the absolute continuity of X_τ.

Theorem 2.2 Suppose $\mu \in \mathcal{M}_c(D)$, there exists a random measurable function x_D defined on $\overline{D}^c$ such that

$$P_\mu\{X_\tau(\mathrm{d}z) = x_D(z)\mathrm{d}z\} = 1.$$

that is, X_τ is P_μ-a.s. absolutely continuous with respect to the Lebesgue measure $\mathrm{d}z$ on $\overline{D}^c$.

3 Absolute Continuity of Y_τ

Let $G_D(x, y)$ denote the Green function of ξ in D. For $\nu \in \mathcal{M}(D), f \in b\mathcal{B}(D)$, define

$$G_D\nu(x) = \int_D G_D(x, y)\nu(\mathrm{d}y),$$

$$G_D f(x) = \Pi_x \int_0^\tau f(\xi_s)\mathrm{d}s = \int_D G_D(x,y)f(y)\mathrm{d}y. \tag{13}$$

Obviously, if $\nu(\mathrm{d}y) = f(y)\mathrm{d}y$, then $G_D f = G_D \nu$.

The study of the fundamental solutions of the following integral equation plays an important role in the investigation of the absolute continuity of Y_τ:

$$u(x) + \Pi_x \int_0^\tau u^{1+\beta}(\xi_s)\mathrm{d}s = G_D\nu(x), \quad x \in D \setminus N_\nu, \tag{14}$$

where $N_\nu = \{x : G_D\nu(x) = \infty\}, \nu \in \mathcal{M}_0(D)$.

For $\nu = \sum\limits_{i=1}^m \lambda_i \delta_{y_i}$ with $y_1, y_2, \cdots, y_m \in D$, $\lambda_i \in \mathbf{R}^+$, $i = 1, 2, \cdots, m$, let

$$\nu_n(\mathrm{d}y) = g_n(y)\mathrm{d}y, \tag{15}$$

where $\mathrm{d}y$ denotes the Lebesgue measure on $\bar{D}$ and

$$g_n(y) = \sum_{i=1}^m \lambda_i g_n^{y_i}(y), \tag{16}$$

$$g_n^{y_i}(z) = \begin{cases} \dfrac{1}{V(D \cap B(y_i, 1/n))}, & z \in B(y_i, 1/n), \\ 0, & y \notin B(y_i, 1/n). \end{cases} \tag{17}$$

ν_n is the regularization of ν.

Note that for $\nu = \sum\limits_{i=1}^m \lambda_i \delta_{y_i}$, $y_1, y_2, \cdots, y_m \in D$, $\lambda_i \in \mathbf{R}^+$, $i = 1, 2, \cdots, m$, we have $N_\nu = \mathrm{supp}(\nu) = \{y_1, y_2, \cdots, y_m\}$.

We now give a theorem on the fundamental solutions of the integral equation (14).

Theorem 3.1 Suppose $D \subset \mathbf{R}^d$ is a bounded smooth domain, $\nu = \sum\limits_{i=1}^m \lambda_i \delta_{y_i}$, $y_1, y_2, \cdots, y_m \in D, \lambda_i \in \mathbf{R}^+, i = 1, 2, \cdots, m$. Let ν_n be defined by (15), (16) and (17). Assume that there exists a sequence of bounded smooth domains $\{D_n\}_{n=1}^\infty$ satisfying $D_n \uparrow (D \setminus N_\nu)$ as $n \uparrow \infty$ and

$$\limsup_{n\to\infty} \Pi_x \int_{\tau_k}^\tau (G_D\nu_n(\xi_s))^{1+\beta}\mathrm{d}s \to 0 (k \to \infty), \quad \text{for } x \in D \setminus N_\nu, \tag{18}$$

where $\tau_k \equiv \inf\{t > 0 : \xi_t \notin D_k\}$. Then we have

(1) (Existence and Uniqueness) There is exactly one measurable nonnegative function $U[\nu]$ defined in D which satisfies the equation (14).

(2) (Continuity of Regularization) The solution $U[\nu]$ is continuous with respect to the operation of regulation of ν in the following sense:

$$U[\nu_n](\cdot) \xrightarrow{bp} U[\nu](\cdot)(n \to \infty) \quad \text{in each compact subsect } K \text{ of } D \setminus N_\nu. \tag{19}$$

(3) (First Derivative with Respect to Small Parameter)

$$\lambda^{-1}U[\lambda\nu](\cdot) \xrightarrow{bp} G_D\nu(\cdot)(\lambda \to 0) \ \text{ in each compact subsect } K \text{ of } D \setminus N_\nu. \tag{20}$$

Using the above theorem we get the main results with respect to the absolute continuity of Y_τ.

Theorem 3.2 Suppose there exists a Borel subset N of D with 0 Lebesgue measure such that for every $\nu \in M_0(D\setminus N)$ condition (18) holds. Then for $\mu \in \mathcal{M}_c(D)$, there exists a random measurable function y_D defined on D such that

$$P_\mu\{Y_\tau(\mathrm{d}y) = y_D(y)\mathrm{d}y\} = 1.$$

that is, Y_τ is P_μ-a.s. absolutely continuous with respect to the Lebesgue measure $\mathrm{d}y$ on D.

Theorem 3.3 Suppose $\mu \in \mathcal{M}_c(D)$. When $d < \alpha + \alpha/\beta$, Y_τ is P_μ-a.s. absolutely continuous with respect to the Lebesgue measure $\mathrm{d}y$ on D.

4 Proofs of Theorems in Sections 2 and 3

In the sequel we will use the following two lemmas. By the Fubini theorem and the Markov property of ξ, using an argument similar to that appearing in Lemma 2.1 in [1], we have the following lemma:

Lemma 4.1 Let $\tau \in \mathcal{T}$, $g \in bp\mathcal{B}$, C is a positive constant. Assume that $\omega \in \mathcal{B}$, $F \in \mathcal{F}_{\geq \tau}$ satisfy

$$\Pi_x \int_0^\tau |\omega(\xi_s)|\mathrm{d}s < \infty, \quad \Pi_x|F| < \infty, \quad x \in \mathbf{R}^d.$$

Then

$$g(x) = \Pi_x\left[\mathrm{e}^{-C\tau}F + \int_0^\tau \mathrm{e}^{-Cs}\omega(\xi_s)\mathrm{d}s\right]$$

if and only if

$$g(x) + \Pi_x \int_0^\tau Cg(\xi_s)\mathrm{d}s = \Pi_x\left[F + \int_0^\tau \omega(\xi_s)\mathrm{d}s\right].$$

We will also use another lemma which is a modification of Lemma 2.7.1 in [2].

Lemma 4.2 Let Y be a random measure defined on a probability space $(\Omega, \mathcal{B}(E), P)$ with values in $\mathcal{M}(E)$. Assume that

(1) there exists a Borel subset N of E of Lebesgue measure 0 such that for $\forall z \in E \setminus N$, there exists a sequence $\varepsilon_n(z) \to 0$ $(n \to \infty)$, and as $n \to \infty$

$$\frac{Y(O_{\varepsilon_n}(z))}{V(O_{\varepsilon_n}(z))} \xrightarrow{d} \eta(z),$$

where $O_\varepsilon(z) \equiv \{x : \|x - z\| < \varepsilon\}$, $\varepsilon > 0$, and $\eta(z)$ is a random variable with $P\eta(z) < \infty$.

(2) $P\langle f, Y\rangle = \int_E f(z)P\eta(z)\mathrm{d}z$ for all $f \in bp\mathcal{B}(E)$.

Then there exists a random measurable function y in E such that

$$P\{Y(\mathrm{d}z) = y(z)\mathrm{d}z\} = 1,$$

and for $\forall z \in E\setminus N$, the random variable $y(z)$ and $\eta(z)$ are identically distributed. In particular, Y is P-a.s. absolutely continuous with respect to $\mathrm{d}z$ in E.

Proof of Theorem 2.1 Assume that, $u_n, n = 1, 2, \cdots$ is a nonnegative solution of the equation (7) with ν replaced by ν_n, that is, u_n satisfies the following equation

$$u_n(x) = H_D\nu_n(x) - \Pi_x \int_0^\tau u_n^{1+\beta}(\xi_s)\mathrm{d}s = \Pi_x f_n(\xi_\tau) - \Pi_x \int_0^\tau u_n^{1+\beta}(\xi_s)\mathrm{d}s. \tag{21}$$

However, for $x \in D$,

$$\Pi_x f_n(\xi_\tau) = \begin{cases} \int_{\overline{D}^c} K_D(x,z)\nu_n(\mathrm{d}z), & d=1, \\ A(d,\alpha)\int_{\overline{D}^c} K_D(x,z)\nu_n(\mathrm{d}z), & d \ge 2. \end{cases}$$

Suppose $d=1$, without loss of generality, let $D=(a,b)$. From [3] we know for $x \in D, z \in \overline{D}^c$, the following estimate holds:

$$\begin{aligned} K_D(x,z) &\le C\frac{|x-a|^{\alpha/2}|x-b|^{\alpha/2}}{|z-a|^{\alpha/2}|z-b|^{\alpha/2}}\frac{1}{|x-z|} \\ &\le C\frac{|x-a|^{\alpha/2}|x-b|^{\alpha/2}}{|z-a|^{\alpha/2}|z-b|^{\alpha/2}}\frac{1}{\delta(z)} \\ &\le C\frac{1}{|z-a|^{\alpha/2}|z-b|^{\alpha/2}}\frac{1}{\delta(z)} \le C\frac{1}{\delta(z)^{1+\alpha}}, \end{aligned} \tag{22}$$

where $\delta(z) \equiv \min\{|z-a|,|z-b|\}$.

Suppose $d \ge 2$, from [4] we know for $x \in D, z \in \overline{D}^c$, the following estimate holds:

$$\begin{aligned} K_D(x,z) &\le \frac{C|x-z|^{\alpha/2}}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}}\frac{1}{|x-z|^d} \\ &= \frac{C}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}}\frac{1}{|x-z|^{d-\alpha/2}} \\ &\le \frac{C}{\delta(z)^{\alpha/2}(1+\delta(z))^{\alpha/2}}\frac{1}{\delta(z)^{d-\alpha/2}} \le \frac{C}{\delta(z)^d}, \end{aligned} \tag{23}$$

where $\delta(x)=d(x,\partial D)$ denotes the distance between x and ∂D.

Noticing that ν_n is only charged on $B(z_i,1/n), i=1,2,\cdots,m$, from (22) and (23), we conclude that there exists an n_0 such that, for $n \ge n_0$,

$$\Pi_x f_n(\xi_\tau) \le C\int_{\overline{D}^c}\nu_n(\mathrm{d}z) \le C.$$

Then from $\nu_n \xrightarrow{w} \nu$ it follows that

$$\Pi_x f_n(\xi_\tau) \xrightarrow{bp} H_D\nu(x), \quad n \to \infty, \quad x \in D. \tag{24}$$

Let

$$M = \Big(\sup_{x\in D, n\ge n_0} H_D f_n(x)\Big) \vee 1, \quad \eta = (1+\beta)M^\beta, \quad R(z) = \eta z - z^{1+\beta}.$$

From (21) it follows that for $x \in D$, $n \ge n_0$, $0 \le u_n(x) \le M$. Since $0 \le \frac{d(R(z))}{\mathrm{d}z} \le \eta$ for $z \in (0,M)$, we have $|R(z_1)-R(z_2)| \le \eta|z_1-z_2|$, $\quad 0 \le z_1, z_2 \le M$.

Then we get, for $x \in D$, $m,n \ge n_0$,

$$|R(u_m(x)) - R(u_n(x))| \le \eta|u_m(x)-u_n(x)|. \tag{25}$$

Using Lemma 4.1 with

$$g = u_n, \quad F = f_n(\xi_\tau), \quad \omega = \eta u_n - u_n^{1+\beta}, \quad C = \eta,$$

we get

$$u_n(x)=\Pi_x[\mathrm{e}^{-\eta\tau}f_n(\xi_\tau)]+\Pi_x\int_0^\tau \mathrm{e}^{-\eta s}R(u_n(\xi_s))\mathrm{d}s. \tag{26}$$

From (25) and (26) we have for $x\in D$, $m,n\geq n_0$,

$$|u_m(x)-u_n(x)|\leq \Pi_x\mathrm{e}^{-\eta\tau}|f_m(\xi_\tau)-f_n(\xi_\tau)|+\Pi_x\int_0^\tau \mathrm{e}^{-\eta s}\eta|u_m(\xi_s)-u_n(\xi_s)|\mathrm{d}s. \tag{27}$$

Iterating the inequality (27) $l\geq 1$ times, using the strong Markov property of ξ and the fact that $\int\cdots\int_{0<s_1<\cdots<s_l<s}=\frac{1}{l!}\int_0^s\cdots\int_0^s$, we get

$$|u_m(x)-u_n(x)|\leq \Pi_x|f_m(\xi_\tau)-f_n(\xi_\tau)|+2M\Pi_x\int_0^\tau \eta\mathrm{e}^{-\eta s}\frac{(\eta s)^l}{l!}\mathrm{d}s. \tag{28}$$

From (24) we have

$$\lim_{m,n\to\infty}\Pi_x|f_m(\xi_\tau)-f_n(\xi_\tau)|=0,\quad x\in D. \tag{29}$$

Noticing $\Pi_x\tau<\infty$, and by using the dominated convergence theorem it follows that

$$\lim_{l\to\infty}\Pi_x\int_0^\tau \eta\mathrm{e}^{-\eta s}\frac{(\eta s)^l}{l!}\mathrm{d}s=0. \tag{30}$$

Combing (28), (29) and (30), we have

$$\limsup_{m,n\to\infty}|u_m(x)-u_n(x)|=0,\quad x\in D.$$

Therefore there exists a nonnegative measurable function u in D such that,

$$u_n(x)\xrightarrow{bp}u(x),\quad n\to\infty,\quad x\in D.$$

By the dominated convergence theorem it follows that, u solves the equation (7). Repeating the procedure from the beginning with two different solutions of the equation (7) instead of u_n and u_m, respectively, we can conclude that u is uniquely determined by the equation. Summarizing the above , we now have proved the statements (1) and (2) of Theorem 2.1.

It remains to verify the asymptotic property (12). Let $U[\lambda\nu]$ be the nonnegative solution of the equation (7) with ν replaced by $\lambda\nu$, then

$$\begin{aligned}|\lambda^{-1}U[\lambda\nu]-H_D\nu|(x)&=\Pi_x\int_0^\tau\lambda^{-1}U^{1+\beta}[\lambda\nu](\xi_s)\mathrm{d}s\\&\leq\Pi_x\int_0^\tau\lambda^{-1}(\lambda H_D\nu(\xi_s))^{1+\beta}\mathrm{d}s\\&=\lambda^\beta\Pi_x\int_0^\tau(H_D\nu(\xi_s))^{1+\beta}\mathrm{d}s.\end{aligned} \tag{31}$$

Noticing that ν is only charged on finite points, we know that $H_D\nu(\cdot)$ is bounded in D, it follows that

$$\Pi_x\int_0^\tau(H_D\nu(\xi_s))^{1+\beta}\mathrm{d}s<\infty.$$

Let $\lambda\downarrow 0$ in (31), it follows that

$$\lim_{\lambda\downarrow 0}|\lambda^{-1}U[\lambda\nu]-H_D\nu|(x)\leq\lim_{\lambda\downarrow 0}\lambda^\beta\Pi_x\int_0^\tau(H_D\nu(\xi_s))^{1+\beta}\mathrm{d}s=0.$$

Therefore we conclude that

$$\lambda^{-1}U[\lambda\nu](\cdot) \xrightarrow{p} H_D\nu(\cdot)(\lambda \to 0) \quad \text{in } D.$$

But $\lambda^{-1}U[\lambda\nu](\cdot)$ are dominated by $H_D\nu(\cdot)$ and $H_D\nu(\cdot)$ is bounded in D. The statement (3) follows.

Proof of Theorem 2.2 (1) We choose $z_1, z_2, \cdots, z_m \in \overline{D}^c$ and let

$$\nu_n(\mathrm{d}z) = \sum_{i=1}^{m} \lambda_i f_n^{z_i}(z)\mathrm{d}z,$$

where $f_n^{z_i}(z)$ is given by (10), $\lambda_i \in \mathbf{R}^+, i = 1, 2, \cdots, m$. We have by (3)

$$P_\mu \exp\langle -\sum_{i=1}^{m} \lambda_i f_n^{z_i}, X_\tau\rangle = \exp\langle -U(\nu_n), \mu\rangle,$$

where $U[\nu_n]$ is the unique solution of (7) with ν replaced by ν_n. Clearly

$$\nu_n \xrightarrow{w} \nu \equiv \sum_{i=1}^{m} \lambda_i \delta_{z_i}, \quad n \to \infty.$$

By Theorem 2.1

$$U[\nu_n] \xrightarrow{bp} U[\nu](n \to \infty) \quad \text{in } D, \tag{32}$$

where $U[\nu]$ is the unique solution of (7). Then it follows that

$$\exp\langle -U[\nu_n], \mu\rangle \to \exp\langle -U[\nu], \mu\rangle, \quad n \to \infty. \tag{33}$$

Let $O_n(z_i) \equiv \{x \in \overline{D}^c : |x - z_i| < \frac{1}{n}\}$, then the left-hand side of (33) determines the Laplace transform of the random vector

$$\left(\frac{X_\tau(O_n(z_1))}{V(O_n(z_1))}, \frac{X_\tau(O_n(z_2))}{V(O_n(z_2))}, \cdots, \frac{X_\tau(O_n(z_m))}{V(O_n(z_m))}\right).$$

Note that

$$\langle U[\nu], \mu\rangle \le \langle H_D\nu, \mu\rangle \le \|\mu\| \sup_{x\in \mathrm{supp}(\mu)} H_D\nu(x) \le C|\lambda|,$$

where $|\lambda| = \max_i \lambda_i$. Therefore the right-hand side of (33) determines the Laplace transform of a random vector, we denote $(\eta(z_1), \eta(z_2), \cdots, \eta(z_m))$. Consequently as $n \to \infty$,

$$\left(\frac{X_\tau(O_n(z_1))}{V(O_n(z_1))}, \frac{X_\tau(O_n(z_2))}{V(O_n(z_2))}, \cdots, \frac{X_\tau(O_n(z_m))}{V(O_n(z_m))}\right) \xrightarrow{d} (\eta(z_1), \eta(z_2), \cdots, \eta(z_m)),$$

where

$$P_\mu \exp\Big\{ -\sum_{i=1}^{m} \lambda_i \eta(z_i)\Big\} = \exp\langle -U[\nu], \mu\rangle. \tag{34}$$

Assumption (1) of Lemma 4.2 is satisfied.

(2) Set $m = 1$, and write λ and z instead of λ_1 and z_1, respectively. Then

$$P_\mu \exp\{-\lambda\eta(z)\} = \exp\langle -U[\lambda\delta_z], \mu\rangle. \tag{35}$$

Differentiating with respect to λ at $\lambda = 0$ in (35), we get

$$P_\mu(-\eta(z)) = \exp\langle -U[\lambda\delta_z], \mu\rangle \langle -\frac{\mathrm{d}U[\lambda\delta_z]}{\mathrm{d}\lambda}, \mu\rangle|_{\lambda=0}.$$

Further by (12)

$$\begin{aligned} P_\mu(\eta(z)) &= \langle H_D\delta_z, \mu\rangle \\ &= \begin{cases} \langle K_D(\cdot, z), \mu\rangle = \int_{\overline{D}^c} K_D(x,z)\mu(dx), & d=1, \\ \langle A(d,\alpha)K_D(\cdot, z), \mu\rangle = \int_{\overline{D}^c} A(d,\alpha)K_D(x,z)\mu(dx), & d\geq 2. \end{cases} \end{aligned}$$

By (3) it follows that for $\forall f \in bp\mathcal{B}(\overline{D}^c)$

$$P_\mu \exp\langle -\lambda f, X_\tau\rangle = \exp\langle -v_\lambda, \mu\rangle, \tag{36}$$

where v_λ is the unique solution of the integral equation

$$v_\lambda(x) = \lambda\Pi_x f(\xi_\tau) - \Pi_x \int_0^\tau v_\lambda^{1+\beta}(\xi_s)\mathrm{d}s. \tag{37}$$

Differentiating with respect to λ at $\lambda = 0$ in (36), we get

$$\begin{aligned} P_\mu\langle f, X_\tau\rangle &= \langle \Pi_\cdot f(\xi_\tau), \mu\rangle = \int_D \Pi_x f(\xi_\tau)\mu(\mathrm{d}x) \\ &= \begin{cases} \int_D \mu(dx) \int_{\overline{D}^c} K_D(x,z) f(z)\mathrm{d}z, & d=1 \\ \int_D \mu(dx) \int_{\overline{D}^c} A(d,\alpha)K_D(x,z)f(z)\mathrm{d}z, & d\geq 2 \end{cases} \\ &= \begin{cases} \int_{\overline{D}^c} f(z) \int_D K_D(x,z)\mu(dx)\mathrm{d}z, & d=1 \\ \int_{\overline{D}^c} f(z) \int_D A(d,\alpha)K_D(x,z)\mu(dx)\mathrm{d}z, & d\geq 2 \end{cases} \\ &= \int_{\overline{D}^c} f(z)P_\mu(\eta(z))\mathrm{d}z. \end{aligned}$$

Assumption (2) of Lemma 4.2 is satisfied.

Therefore the statements of Theorem 2.2 follow from Lemma 4.2.

Proof of Theorem 3.1 Assume $u_n, n = 1, 2, \cdots$ is a nonnegative solution of the equation (14) with ν replaced by ν_n, that is, u_n satisfies the following equation

$$u_n(x) = G_D\nu_n(x) - \Pi_x \int_0^\tau u_n^{1+\beta}(\xi_s)\mathrm{d}s = \int_D G_D(x,y)g_n(y)\mathrm{d}y - \Pi_x \int_0^\tau u_n^{1+\beta}(\xi_s)\mathrm{d}s. \tag{38}$$

From [3] and [4] we know that for $x, y \in D$ the following estimate holds: for $d = 1$

$$G_D(x,y) \leq \begin{cases} C|\ln \frac{1}{|x-y|}|, & \alpha = 1, \\ C|x-y|^{\alpha-1}, & \alpha \neq 1; \end{cases} \tag{39}$$

for $d \geq 2$

$$G_D(x,y) \leq C\frac{1}{|x-y|^{d-\alpha}}. \tag{40}$$

Let K be any fixed compact subset of $D\backslash N_\nu$. Noticing that g_n is only non-zero on $B(y_i, 1/n), i = 1, 2, \cdots, m$, we have there exists an n_0 such that, for $n \geq n_0$, $g_n = 0$ in a neighborhood of K. Therefore, there exists constant $C > 0$ such that

$$G_D(x,y) \leq C, \quad \forall x \in K, y \in B(y_i, 1/n)(n \geq n_0, i = 1, 2, \cdots, m).$$

Hence we have

$$0 \leq u_n(x) \leq \int_D G_D(x,y) g_n(y) \mathrm{d}y \leq C \int_D g_n(y) \mathrm{d}y = C\nu_n(D) \leq C, \quad x \in K, \quad n \geq n_0. \tag{41}$$

Therefore for $\forall k \geq 1$, there exists an integer n_k such that, for $n \geq n_k$, $g_n = 0$ and $G_D g_n$ are uniformly bounded in D_k. Let

$$M_k = \Big(\sup_{x \in D_k, n \geq n_k} G_D g_n(x) \Big) \vee 1, \quad \eta_k = (1+\beta) M_k^\beta, \quad R_k(z) = \eta_k z - z^{1+\beta}.$$

From (38) it follows that for $x \in D_k, n \geq n_k$, $0 \leq u_n(x) \leq M_k$. Since $0 \leq \frac{d(R_k(z))}{\mathrm{d}z} \leq \eta_k$ for $z \in (0, M_k)$, we have

$$|R_k(z_1) - R_k(z_2)| \leq \eta_k |z_1 - z_2|, \quad 0 \leq z_1, z_2 \leq M_k.$$

Then we get, for $x \in D_k, m, n \geq n_k$,

$$|R(u_m(x)) - R(u_n(x))| \leq \eta_k |u_m(x) - u_n(x)|. \tag{42}$$

Using Lemma 4.1 with

$$g = u_n, \quad F = \int_{\tau_k}^{\tau} g_n(\xi_s) \mathrm{d}s - \int_{\tau_k}^{\tau} u_n^{1+\beta}(\xi_s) \mathrm{d}s, \quad \omega = \eta_k u_n + g_n - u_n^{1+\beta}, \quad C = \eta_k,$$

and noticing that, for all $n \geq n_k$, $g_n = 0$ in D_k, we get

$$u_n(x) = \Pi_x \Big[\mathrm{e}^{-\eta_k \tau_k} \int_{\tau_k}^{\tau} (g_n(\xi_s) - u_n^{1+\beta}(\xi_s)) \mathrm{d}s \Big] + \Pi_x \int_0^{\tau_k} \mathrm{e}^{-\eta_k s} R(u_n(\xi_s)) \mathrm{d}s. \tag{43}$$

From (42) and (43), using the strong Markov property of ξ and noticing that for $x \in D$, $u_m(x) \leq G_D g_m(x)$, $u_n(x) \leq G_D g_n(x)$. We have for $x \in D \backslash N_\nu$, $m, n \geq n_k$ and sufficiently large k (satisfying $x \in D_k$),

$$\begin{aligned} |u_m(x) - u_n(x)| \leq\; & \Pi_x \mathrm{e}^{-\eta_k \tau_k} \Big| \Pi_{\xi_{\tau_k}} \int_0^{\tau} (g_m(\xi_s) - g_n(\xi_s)) \mathrm{d}s \Big| \\ & + \Pi_x \mathrm{e}^{-\eta_k \tau_k} \int_{\tau_k}^{\tau} [(G_D g_n(\xi_s))^{1+\beta} + (G_D g_m(\xi_s))^{1+\beta})] \mathrm{d}s \\ & + \Pi_x \int_0^{\tau_k} \mathrm{e}^{-\eta_k s} \eta_k |u_m(\xi_s) - u_n(\xi_s)| \mathrm{d}s. \end{aligned} \tag{44}$$

Using Fubini theorem and the Markov property of ξ, iterating the inequality (44) $l \geq 1$ times yields

$$\begin{aligned} |u_m(x) - u_n(x)| \leq\; & \Pi_x \Big| \Pi_{\xi_{\tau_k}} \int_0^{\tau} (g_m(\xi_s) - g_n(\xi_s)) \mathrm{d}s \Big| \\ & + \Pi_x \int_{\tau_k}^{\tau} [(G_D g_m(\xi_s))^{1+\beta} + (G_D g_n(\xi_s))^{1+\beta}] \mathrm{d}s \\ & + 2M_k \Pi_x \int_0^{\tau_k} \eta_k \mathrm{e}^{-\eta_k s} \frac{(\eta_k s)^l}{l!} \mathrm{d}s. \end{aligned} \tag{45}$$

From $\nu_n \xrightarrow{w} \nu$, it follows that for fixed k,

$$\Pi_x \int_0^{\tau} g_n(\xi_s)\mathrm{d}s = G_D g_n(x) = G_D \nu_n(x) \xrightarrow{bp} G_D \nu(x), \quad n \to \infty, \quad x \in \bar{D}_k.$$

From the dominated convergence theorem, we obtain

$$\lim_{m,n\to\infty} \Pi_x \left| \Pi_{\xi_{\tau_k}} \int_0^{\tau} (g_m(\xi_s) - g_n(\xi_s))\mathrm{d}s \right| = 0. \tag{46}$$

Noticing $\Pi_x \tau_k < \infty$, and using the dominated convergence theorem, it follows that

$$\lim_{l\to\infty} \Pi_x \int_0^{\tau_k} \eta_k \mathrm{e}^{-\eta_k s} \frac{(\eta_k s)^l}{l!} \mathrm{d}s = 0. \tag{47}$$

Combining (45), (46), (47) and the condition (18), we have

$$\limsup_{m,n\to\infty} |u_m(x) - u_n(x)| = 0, \quad x \in D \setminus N_\nu.$$

Therefore there exists a nonnegative measurable function u in $D\setminus N_\nu$ such that, for each compact subset $K \subset D \setminus N_\nu$,

$$u_n(x) \xrightarrow{bp} u(x), \quad n \to \infty, \quad x \in K.$$

Repeating the procedure from the beginning with this u instead of u_m we conclude that u solves the equation (14). By similar arguments we conclude that u is uniquely determined by the equation. Summarizing the above, we now have proved the statements (1) and (2) of Theorem 3.1.

It remains to verify the asymptotic property (20). Let $U[\lambda\nu]$ be the nonnegative solution of the equation (14) with ν replaced by $\lambda\nu$, then

$$\begin{aligned} |\lambda^{-1}U[\lambda\nu] - G_D\nu|(x) &= \Pi_x \int_0^{\tau} \lambda^{-1} U^{1+\beta}[\lambda\nu](\xi_s)\mathrm{d}s \\ &\leq \Pi_x \int_0^{\tau} \lambda^{-1}(\lambda G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s \\ &= \lambda^\beta \Pi_x \int_0^{\tau} (G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s. \end{aligned} \tag{48}$$

Condition (18) and Fatou Lemma imply that

$$\Pi_x \int_{\tau_k}^{\tau} (G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s \to 0, \quad k \to \infty.$$

Noticing that for $n \geq n_k$, $G_D\nu_n$ is uniformly bounded in D_k, then it follows that

$$\Pi_x \int_0^{\tau} (G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s < \infty.$$

Let $\lambda \downarrow 0$ in (48), it follows that

$$\limsup_{\lambda\downarrow 0} |\lambda^{-1}U[\lambda\nu] - G_D\nu|(x) \leq \lim_{\lambda\downarrow 0} \lambda^\beta \Pi_x \int_0^{\tau} (G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s = 0.$$

Therefore we conclude that

$$\lambda^{-1}U[\lambda\nu](\cdot) \xrightarrow{p} G_D\nu(\cdot)(\lambda \to 0) \quad \text{in} \quad D.$$

But $\lambda^{-1}U[\lambda\nu](\cdot)$ is dominated by $G_D\nu(\cdot)$ and $G_D\nu(\cdot)$ is bounded in any compact subset K of $D\setminus N_\nu$, the statement (3) follows.

Proof of Theorem 3.2 Using an argument similar to that of the proof of Theorem 2.2, we can prove the results of Theorem 3.2. hold. We omit the details.

Proof of Theorem 3.3 We need only to prove that for every $\nu \in M_0(D)$, condition (18) holds. For $x \in D\setminus N_\nu$ and any integer k, from the proof of Theorem 3.1 we know that, there exists an integer n_k such that, for $n \geq n_k$, $G_D\nu_n(\cdot)$ is uniformly bounded in D_k. Then

$$\lim_{n\to\infty} \Pi_x \int_0^{\tau_k} (G_D\nu_n(\xi_s))^{1+\beta}\mathrm{d}s < \infty.$$

Consequently, condition (18) is satisfied if

$$\lim_{n\to\infty} \Pi_x \int_0^{\tau} (G_D\nu_n(\xi_s))^{1+\beta}\mathrm{d}s = \Pi_x \int_0^{\tau} (G_D\nu(\xi_s))^{1+\beta}\mathrm{d}s < \infty,$$

that is,

$$\lim_{n\to\infty} \int_D G_D(x,y)(G_D\nu_n(y))^{1+\beta}\mathrm{d}y = \int_D G_D(x,y)(G_D\nu(y))^{1+\beta}\mathrm{d}y < \infty. \tag{49}$$

From the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{D_k} G_D(x,y)(G_D\nu_n(y))^{1+\beta}\mathrm{d}y = \int_{D_k} G_D(x,y)(G_D\nu(y))^{1+\beta}\mathrm{d}y < \infty.$$

It is sufficient to prove

$$\sup_{n\geq 1} \int_{D\setminus D_k} G_D(x,y)(G_D\nu_n(y))^{1+\beta}\mathrm{d}y \to 0, \quad \text{as } k\to\infty. \tag{50}$$

Without loss of generality, we can assume that $x \in D_k, k\geq 1$. Then there exists a constant C such that $G_D(x,y) \leq C, y \in D\setminus D_k$. Hence it is sufficient to show

$$\sup_{n\geq 1} \int_{D\setminus D_k} (G_D\nu_n(y))^{1+\beta}\mathrm{d}y \to 0, \quad k\to\infty. \tag{51}$$

Put $g_n(x) = G_D\nu_n(x), \alpha_n(\lambda) = \int_{D\cap(G_D\nu_n>\lambda)}\mathrm{d}y$. For $M>0$, we have

$$\int_{D\setminus D_k} (G_D\nu_n(y))^{1+\beta}\mathrm{d}y \leq M^{1+\beta}\int_{D\setminus D_k}\mathrm{d}y + \int_{D\cap(g_n>M)} g_n^{1+\beta}(y)\mathrm{d}y, \tag{52}$$

and

$$\int_{D\cap(g_n>M)} g_n^{1+\beta}(y)\mathrm{d}y = -\int_M^\infty \lambda^{1+\beta}\mathrm{d}\alpha_n(\lambda). \tag{53}$$

Then we have the estimate

$$\begin{aligned}\lambda\alpha_n(\lambda) &\leq \int_{D\cap(g_n>\lambda)} g_n(y)\mathrm{d}y = \int_D \nu_n(\mathrm{d}y_1)\int_{D\cap(g_n>\lambda)} G_D(y,y_1)\mathrm{d}y \\ &\leq C\sup_{y_1\in D}\int_{D\cap(g_n>\lambda)} G_D(y,y_1)\mathrm{d}y.\end{aligned} \tag{54}$$

Choose $a > 1+\beta$, from the Hölder inequality, we have

$$\int_{D\cap(g_n>\lambda)} G_D(y,y_1)\mathrm{d}y \leq (B(y_1))^{\frac{1}{a}}(\alpha_n(\lambda))^{\frac{1}{b}},$$

where $B(y_1) = \int_D G_D(y, y_1)^a \mathrm{d}y$, $\frac{1}{a} + \frac{1}{b} = 1$. Then from (54), it follows that

$$\lambda\alpha_n(\lambda) \le C \sup_{y_1 \in D} (B(y_1))^{\frac{1}{a}} (\alpha_n(\lambda))^{\frac{1}{b}}.$$

Using the estimate of the Green function in [3] and [4], we can conclude that when $d > \alpha$,

$$B(y_1) = \int_D G_D(y, y_1)^a \mathrm{d}y \le \int_D C \left(\frac{1}{|y - y_1|^{d-\alpha}} \right)^a \mathrm{d}y \le \int_0^{\mathrm{diam}D} C r^{a(\alpha-d)+d-1} \mathrm{d}r,$$

where $\mathrm{diam}D$ is the diameter of D. Since $d < \alpha + \alpha/\beta$, we can choose $a > 1 + \beta$ such that $\int_0^{\mathrm{diam}D} C r^{a(\alpha-d)+d-1} \mathrm{d}r < \infty$. When $d = \alpha = 1$,

$$\begin{aligned} B(y_1) &= \int_D G_D(y, y_1)^a \mathrm{d}y \le \int_D C \left| \ln \frac{1}{|y - y_1|} \right|^a \mathrm{d}y \le \int_0^{\mathrm{diam}D} C |\ln r|^a \mathrm{d}r \\ &= \int_0^1 C(-\ln r)^a \mathrm{d}r + \int_1^{\mathrm{diam}D} C(\ln r)^a \mathrm{d}r < \infty. \end{aligned}$$

When $\alpha > d = 1$,

$$B(y_1) = \int_D G_D(y, y_1)^a \mathrm{d}y \le \int_D C|y - y_1|^{a(\alpha-1)} \mathrm{d}y \le \int_0^{\mathrm{diam}D} C r^{a(\alpha-1)} \mathrm{d}r < \infty.$$

Thus we conclude that in any dimension $d < \alpha + \alpha/\beta$,

$$\lambda\alpha_n(\lambda) \le C[\alpha_n(\lambda)]^{\frac{1}{b}}, \qquad \lambda > 0, \quad n \ge 1,$$

that is,

$$\alpha_n(\lambda) \le C\lambda^{-a}, \qquad \lambda > 0, \quad n \ge 1. \tag{55}$$

Since $a > 1 + \beta$, by integration by parts we have

$$-\int_M^\infty \lambda^{1+\beta} \mathrm{d}\alpha_n(\lambda) = M^{1+\beta}\alpha_n(M) + (1+\beta) \int_M^\infty \alpha_n(\lambda)\lambda^\beta \mathrm{d}\lambda \le CM^{1+\beta-a}. \tag{56}$$

Combining (52), (53) and (56), we have

$$\int_{D \setminus D_k} (G_D \nu_n(y))^{1+\beta} \mathrm{d}y \le M^{1+\beta} \int_{D \setminus D_k} \mathrm{d}y + CM^{1+\beta-a}. \tag{57}$$

Letting $k \to \infty, M \to \infty$ in the above inequality, we conclude that (51) holds.

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