S-polar sets of super-Brownian motions and solutions of nonlinear differential equations

LI Qiuyue & REN Yanxia

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
Correspondence should be addressed to Ren Yanxia (email: yxren@math.pku.edu.cn)
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Abstract This paper gives probabilistic expressions of the minimal and maximal positive solutions of the partial differential equation $-\frac{1}{2}\Delta v(x)+\gamma(x)v(x)^{\alpha}=0$ in D, where D is a regular domain in $\mathbb{R}^d(d\geqslant 3)$ such that its complement D^c is compact, $\gamma(x)$ is a positive bounded integrable function in D, and $1<\alpha\leqslant 2$. As an application, some necessary and sufficient conditions for a compact set to be S-polar are presented.

Keywords: super-Brownian motion, nonlinear differential equation, minimal positive solutions, maximal positive solutions, S-polar sets.

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1 Introduction

We consider the positive solutions of the differential equation:

$$-\frac{1}{2}\Delta v(x)+\gamma(x)v(x)^{\alpha}=\rho(x)\quad \text{for}\ \ x\in D, \tag{1}$$

where Δ is the Laplace operator, $1<\alpha\leqslant 2$, D is a domain in $\mathbb{R}^d(d\geqslant 3)$ such that its complement D^c is compact. Here $\gamma(x)$ and $\rho(x)$ satisfy the following condition:

$*\ \gamma(x),\rho(x)\in C^{0,\lambda}(D)$ are positive bounded integrable functions in D, where $C^{0,\lambda}(D)$ denotes the Hölder continuous functions in D with exponent $\lambda\in(0,1]$.

The differential operator $\frac{1}{2}\Delta$ is the generator of a Brownian motion $\xi=(\xi_t,\Pi_x)$ in $\mathbb{R}^d$. Let $\mathcal{R}^d$ be the Borel σ-algebra in $\mathbb{R}^d$, M be the set of all finite measures on $\mathbb{R}^d$ and $\mathcal{M}$ be the σ-algebra in M generated by the functions $f_B(\mu)=\mu(B)$ with $B\in\mathcal{R}^d$. There exists a measure-valued Markov process $X=(X_t,P_\mu)$ in $(M,\mathcal{M})$ such that:

(a) If f is a bounded continuous function in $\mathbb{R}^d$, then $\langle f,X_t\rangle$ is right-continuous in t on $\mathbb{R}^+$ (writing $\langle v,\mu\rangle$ means the integral of v with respect to μ);

(b) for every $\mu\in M$,

$$P_\mu\exp\{-\langle f,X_t\rangle\}=\exp\langle -v_t,\mu\rangle,$$

where v_t is the unique solution of the integral equation

$$v_t(x)+\Pi_x\int_0^t\gamma(\xi_s)v_{t-s}(\xi_s)^{\alpha}ds=\Pi_x f(\xi_t).$$

Moreover, for every open set $D \in \mathbb{R}^d$, there exist correspondingly the random measures X_τ and Y_τ on $\mathbb{R}^d$ associated with the first exit time $\tau = \inf\{t : \xi_t \notin D\}$ from D, such that:

$$P_\mu \exp\{-\langle \rho, Y_\tau \rangle - \langle f, X_\tau \rangle\} = \exp\langle -v, \mu \rangle, \tag{2}$$

where

$$v(x) + \Pi_x \int_0^\tau \gamma(\xi_s) v(\xi_s)^\alpha ds = \Pi_x \left[\int_0^\tau \rho(\xi_s) ds + f(\xi_\tau) 1_{(\tau<\infty)} \right]. \tag{3}$$

We call $X=(X_t, X_\tau, Y_\tau; P_\mu)$ the super-Browian motion with parameters $(\frac{1}{2}\Delta, \gamma(x)z^\alpha)$.

For every $\epsilon \geqslant 0$, we denote by $\mathcal{R}_\epsilon$ the minimal closed set which contains the supports S_t of X_t for all $t \geqslant \epsilon$. And the set $\mathcal{R} = \mathcal{R}_0$ is called the range of X.

We say that B is S-polar if, for every $\mu \in M$ and every $\epsilon > 0$, there exists an analytic set $A \supset B$ such that $P_\mu(\mathcal{R}_\epsilon \cap A \neq \emptyset) = 0$. Dynkin[1] proved that an analytic set B is S-polar if and only if

$$P_{\delta_x}(\mathcal{R} \cap B \neq \emptyset) = 0 \quad \text{for all} \quad x \notin B. \tag{4}$$

Suppose D is a regular Greenian domain, and γ and ρ satisfy the condition $*$. Let φ be a positive bounded continuous function on ∂D and has limit c at infinity if ∂D is unbounded. Consider the boundary condition

$$v(x) \to \varphi(a) \quad \text{as} \quad x \to a \in \partial D, x \in D; \tag{5}$$

$$v(x) \to c \quad \text{as} \quad \|x\| \to \infty, x \in D. \tag{6}$$

Let $\{D_n\}$ be a sequence of bounded domains such that $\overline{D_n} \subset D_{n+1}$ and $D_n \uparrow D$, here the sign $\overline{D_n}$ denotes the closure of set D_n. Let τ_n denote the first exit time from D_n.

Ren, Wu and Yang[2] proved that there is a unique bounded solution of (1), (5) and (6):

$$v(x) = -\log P_{\delta_x} \exp\{-\langle \rho, Y_\tau \rangle - \langle \varphi, X_\tau \rangle - cZ_D\}, \tag{7}$$

where

$$Z_D = \lim_{n\to\infty} \langle \Pi_\cdot(\tau = \infty), X_{\tau_n} \rangle. \tag{8}$$

Dynkin[2] studied some analytic properties of the range of X and S-polar sets, and obtained some connections between S-polar sets and the solutions of partial differential equations. He assumed there that D is bounded and $\gamma(x)$ satisfies the condition: $\inf_x \gamma(x) > 0$. More generally, this paper arrives at the similar results to Dynkin's by using more relaxed confinements on D and $\gamma(x)$.

We organize this paper as follows. In Section 2, we obtain the minimal and maximal positive solutions of the problem

$$\begin{cases} \dfrac{1}{2}\Delta v(x) = \gamma(x) v(x)^\alpha & \text{in} \quad D, \\ v(x) \to +\infty & \text{as} \quad D \ni x \to a \in \partial D, \\ v(x) \to c & \text{as} \quad \|x\| \to \infty. \end{cases} \tag{E1}$$

Using the results obtained in Section 2, we give several necessary and sufficient conditions for a compact set Γ to be S-polar in different cases in Section 3.

2 Minimal positive and maximal positive solutions

Lemma 1. Suppose D is a regular domain. γ is a positive bounded integrable function in D such that for every bounded subset D_0 satisfying $\overline{D_0} \subset D$,

$$\inf_{x\in D_0} \gamma(x) > 0. \tag{9}$$

For each $x_0 \in D$, let $U = \{x : |x - x_0| < R\}$ with R small enough such that $\overline{U} \subset D$. Put

$$u(x) = \lambda(R^2 - r^2)^{-\frac{2}{\alpha-1}},$$

where $r = |x - x_0|$ and

$$\lambda = cR^{\frac{2}{\alpha-1}} \tag{10}$$

with c being a constant depending only on α, the dimension d and the lower bound for γ in U. Then we have

$$\frac{1}{2}\Delta u - \gamma u^\alpha \leqslant 0 \quad \text{in} \ \ U, \tag{11}$$

and $\lim\limits_{x\to a, x\in U} u(x) = \infty$ for all $a \in \partial U$.

Proof. By a direct computation we get

$$\frac{1}{2}\Delta u - \gamma u^\alpha = \lambda(R^2 - r^2)^{-\frac{2\alpha}{\alpha-1}}[c_1r^2 + c_2d(R^2 - r^2) - \gamma\lambda^{\alpha-1}], \tag{12}$$

where $c_1 = 4(\alpha + 1)(\alpha - 1)^{-2}$, $c_2 = 2(\alpha - 1)^{-1}$. Clearly (12) implies (11) if

$$c_1r^2 + c_2d(R^2 - r^2) - \gamma\lambda^{\alpha-1} \leqslant 0 \quad \text{for all} \ \ 0 \leqslant r \leqslant R. \tag{13}$$

Let $A = \inf\limits_{U} \gamma(x)$. The condition (9) implies $A > 0$. So (13) holds if $\lambda^{\alpha-1} \geqslant (\frac{c_1}{A} + \frac{c_2}{A}d)R^2$, which is true for λ given by (10).

Lemma 2. Suppose D is a regular domain satisfying D^c being compact. Let $C^2(D)$ denote the class of all functions which are twice differentiable in D and all their partial derivatives are continuous in D. If u and v belong to $C^2(D)$ and satisfy

$$\frac{1}{2}\Delta u(x) - \gamma(x)u(x)^\alpha \geqslant \frac{1}{2}\Delta v(x) - \gamma(x)v(x)^\alpha \quad \text{for all} \ \ x \in D, \tag{14}$$

and

$$\limsup_{\|x\|\to\infty}[u(x) - v(x)] \leqslant 0; \tag{15}$$

if ∂D is not empty, and u, v also satisfy

$$\limsup_{x\to a, x\in D}[u(x) - v(x)] \leqslant 0 \quad \text{for all} \ \ a \in \partial D, \tag{16}$$

then $u(x) \leqslant v(x)$ in D.

Proof. Suppose ∂D is not empty. The case when $D = \mathbb{R}^d$ is similar. Let $w = u - v$. If the statement is false, then $\widetilde{D} := \{x \in D : w(x) > 0\}$ is not empty. Clearly $\widetilde{D}$ is open and not loss of generality, we assume that $\widetilde{D}$ is connected. If we can prove that $\widetilde{D}$ is bounded, then by Theorem 0.5 in Dynkin[1], $w(x) \leqslant 0$ in $\widetilde{D}$, which contradicts the definition of $\widetilde{D}$.

So we are left to prove that $\widetilde{D}$ is bounded. If not, choose a point $x_0 \in \widetilde{D}$. Since ∂D is bounded, there exists a constant $A > 0$ such that for every $r > A$, we have $\widetilde{D} \cap \partial B(x_0, r) \neq \emptyset$. By (15), $\limsup_{\|x\| \to \infty} w(x) \leqslant 0$. Then there exists a constant $R > A$, such that for every $x \in B(x_0, R)^c$, $w(x) \leqslant \frac{1}{2} w(x_0)$. Let $D_1 = \widetilde{D} \bigcap B(x_0, R)$. By (14),

$$\frac{1}{2}\Delta w(x) = \frac{1}{2}\Delta u(x) - \frac{1}{2}\Delta v(x) \geqslant \gamma(x) u(x)^\alpha - \gamma(x) v(x)^\alpha \geqslant 0 \quad \text{in} \ D_1.$$

By (16), $\limsup\limits_{x \to a, x \in \widetilde{D}} w(x) \leqslant 0$ for all $a \in \partial \widetilde{D} \setminus \partial B(x_0, R)$, and notice that $w(x) \leqslant \frac{1}{2} w(x_0)$ on $\partial B(x_0, R)$. Then the maximum can not be reached on the boundary of D_1. This contradicts the maximum principle for linear elliptic equations in D_1 (see, Theorem 2.7.19 in ref. [3]). Now we complete the proof.

Assume hereafter that $\gamma(x)$ satisfies the condition (9), D is a regular domain satisfying D^c being compact, and τ denotes the first exit time from D.

Theorem 1. $v_{\infty,\, c,\, D}(x) := -\log P_{\delta_x}\{1_{(X_\tau = 0)} \exp(-cZ_D)\}$ is the minimal positive solution of the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x) v(x)^\alpha & \text{in} \ \ D, \\ v(x) \to +\infty & \text{as} \ \ D \ni x \to a \in \partial D, \\ v(x) \to c & \text{as} \ \ \|x\| \to \infty. \end{cases} \tag{E1}$$

Proof. By (7), $v_k(x) = -\log P_{\delta_x}\{\exp(-\langle k, X_\tau \rangle - cZ_D)\}$ is a solution of the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x) v(x)^\alpha & \text{in} \ \ D, \\ v(x) \to k & \text{as} \ \ D \ni x \to a \in \partial D, \\ v(x) \to c & \text{as} \ \ \|x\| \to \infty. \end{cases}$$

Then

$$P_{\delta_x}\{\exp(-\langle k, X_\tau \rangle - cZ_D)\} \downarrow P_{\delta_x}\{1_{(X_\tau = 0)} \exp(-cZ_D)\}, \quad \text{as} \ \ k \uparrow \infty,$$

so we get $v_k(x) \uparrow v_{\infty,\, c,\, D}(x)$.

For each $x_0 \in D$, let $U = B(x_0, \frac{r}{2})$, $\widetilde{U} = B(x_0, r)$. Choose r being small enough such that $\overline{\widetilde{U}} \subset D$, then $u(x) = -\log P_{\delta_x} \exp\langle -v_k, X_{\tau_U} \rangle$ is the solution of the problem

$$\begin{cases} \frac{1}{2}\Delta u(x) = \gamma(x) u(x)^\alpha & \text{in} \ \ U, \\ u|_{\partial U} = v_k. \end{cases}$$

By the uniqueness of the above problem, we get

$$u(x) = v_k(x) = -\log P_{\delta_x} \exp\langle -v_k, X_{\tau_U} \rangle \quad \text{in} \ \ U.$$

And by Lemma 1, there exists a function $w(x)$ such that

$$\begin{cases} \dfrac{1}{2}\Delta w(x) \leqslant \gamma(x)w(x)^\alpha, & x \in \widetilde{U}, \\ \lim\limits_{x\to a, x\in\widetilde{U}} w(x) = \infty, & a \in \partial\widetilde{U}. \end{cases}$$

By the maximum principle, $v_k \leqslant w$ in $\widetilde{U}$. Then we have for all positive integers k,

$$v_k(x) \leqslant \max_{x\in U} w(x) := M < \infty, \quad \forall x \in U. \tag{17}$$

By the dominated convergence theorem,

$$\begin{aligned} v_{\infty,\,c,\,D}(x) &= \lim_{k\to\infty} v_k(x) \\ &= \lim_{k\to\infty} (-\log P_{\delta_x} \exp\langle -v_k, X_{\tau_U}\rangle) \\ &= -\log P_{\delta_x} \exp\langle -v_{\infty,\,c,\,D}, X_{\tau_U}\rangle, \quad \forall x \in U. \end{aligned}$$

Then $v_{\infty,\,c,\,D}$ also satisfies

$$\begin{cases} \dfrac{1}{2}\Delta u(x) = \gamma(x)u(x)^\alpha \quad \text{in } U, \\ u|_{\partial U} = v_{\infty,\,c,\,D}. \end{cases}$$

Since $x_0 \in D$ is arbitrary, we have $\frac{1}{2}\Delta v_{\infty,\,c,\,D} = \gamma v_{\infty,\,c,\,D}^\alpha$ in D.

Notice that $v_k \leqslant v_{\infty,\,c,\,D}$ for all k, we get for every $a \in \partial D$,

$$\lim_{x\in D, x\to a} v_{\infty,\,c,\,D}(x) = \infty.$$

And by $\lim\limits_{\|x\|\to\infty} v_k(x) = c$, we have

$$\liminf_{\|x\|\to\infty} v_{\infty,\,c,\,D}(x) \geqslant c. \tag{18}$$

Let D_1 be a regular domain such that D_1^c is compact and $\overline{D_1} \subset D$. Since every open covering of ∂D_1 has a finite sub-covering, by (17), we have $A := \sup\limits_{x\in\partial D_1} \sup\limits_k v_k(x) < \infty$, and then

$$\begin{aligned} v_k(x) &= -\log P_{\delta_x}\{\exp(-\langle k, X_\tau\rangle - cZ_D)\} \\ &= -\log P_{\delta_x}\{\exp(-\langle v_k, X_{\tau_{D_1}}\rangle - cZ_{D_1})\} \\ &\leqslant -\log P_{\delta_x}\{\exp(-\langle A, X_{\tau_{D_1}}\rangle - cZ_{D_1})\} := u(x), \quad \text{for each } x \in D_1. \end{aligned}$$

Letting $k \to \infty$, we get $v_{\infty,\,c,\,D}(x) \leqslant u(x)$ for each $x \in D_1$. However, $u(x)$ has the limit c at infinity, so

$$\limsup_{\|x\|\to\infty} v_{\infty,\,c,\,D}(x) \leqslant c. \tag{19}$$

Combining (18) and (19) we get $\lim\limits_{\|x\|\to\infty} v_{\infty,\,c,\,D}(x) = c$. Thus $v_{\infty,\,c,\,D}$ is a solution of (E1).

Let $u \geqslant 0$ be any solution to problem (E1), then by Lemma 2, $v_k \leqslant u$ in D and therefore $v_{\infty,\,c,\,D} \leqslant u$, which says that $v_{\infty,\,c,\,D}$ is the minimal solution to problem (E1).

Lemma 3. Let σ_k be the first exit time from $B_k = B(0,k)$ and $\tau_k = \sigma_k \wedge \tau$, then $\lim\limits_{k\to\infty} \langle \Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle$ exists P_{δ_x}-a.s. for all $x \in D$.

Proof. We claim that, $\{\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle; \mathcal{F}_{\tau_k}, P_{\delta_x}\}$ is a submartingale.

In fact,

$$\begin{aligned} &P_{\delta_x}(\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle / \mathcal{F}_{\tau_{k-1}}) \\ =\ &P_{X_{\tau_{k-1}}}(\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle) \\ =\ &\exp\langle -v, X_{\tau_{k-1}}\rangle, \quad x\in D, \end{aligned} \tag{20}$$

where $v(x)$ satisfies

$$v(x) + \Pi_x\int_0^{\tau_k}\gamma(\xi_s)v(\xi_s)^\alpha ds = \Pi_x(\Pi_{\xi_{\tau_k}}(\tau=\infty)).$$

Since $\Pi_\cdot(\tau=\infty)=0$ on ∂D, we have

$$\begin{aligned} &\Pi_x(\Pi_{\xi_{\tau_k}}(\tau=\infty)) \\ =\ &\Pi_x(\Pi_{\xi_{\sigma_k\wedge\tau}}(\tau=\infty)) = \Pi_x(\Pi_{\xi_{\sigma_k}}(\tau=\infty), \tau>\sigma_k) \\ =\ &\Pi_x(\tau=\infty, \tau>\sigma_k) \leqslant \Pi_x(\tau=\infty), \quad x\in D, \end{aligned} \tag{21}$$

then $v(x)\leqslant \Pi_x(\tau=\infty)$. So, by (20),

$$P_{\delta_x}(\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle / \mathcal{F}_{\tau_{k-1}}) \geqslant \exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_{k-1}}\rangle,$$

and therefore $\{\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle; \mathcal{F}_{\tau_k}, P_{\delta_x}\}$ is a submartingale.

By the convergence theorem of bounded submartingale, $\lim\limits_{k\to\infty}\exp\langle -\Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle$ exists P_{δ_x}-a.s., and then $\lim\limits_{k\to\infty}\langle \Pi_\cdot(\tau=\infty), X_{\tau_k}\rangle$ exists P_{δ_x}-a.s., $x\in D$.

Lemma 4. Suppose $\{D_k\}$ is a sequence of regular domains such that $\overline{D_k}\subset D_{k+1}$, D_k^c is compact and $D_k\uparrow D$. Let σ_k, τ_k be the first exit time from $B_k=B(0,k)$ and D_k respectively. Then for every $x\in D$,

$$\liminf_{k\to\infty}\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}|_{\partial D_k}\rangle = 0 \quad P_{\delta_x}\text{-a.s.}$$

Proof. By Fatou's Lemma,

$$\begin{aligned} &P_{\delta_x}(\liminf\nolimits_{k\to\infty}\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}|_{\partial D_k}\rangle) \\ \leqslant\ &\liminf\nolimits_{k\to\infty} P_{\delta_x}(\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}|_{\partial D_k}\rangle) \\ =\ &\liminf\nolimits_{k\to\infty}\Pi_x(\Pi_{\xi_{\tau_k\wedge\sigma_k}}(\tau=\infty); \xi_{\tau_k\wedge\sigma_k}\in\partial D_k). \end{aligned}$$

Notice that $\Pi_\cdot(\tau=\infty)|_{\partial D}=0$, then

$$\liminf_{k\to\infty}\Pi_x(\Pi_{\xi_{\tau_k\wedge\sigma_k}}(\tau=\infty); \xi_{\tau_k\wedge\sigma_k}\in\partial D_k) = 0.$$

So we have

$$\liminf_{k\to\infty}\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}|_{\partial D_k}\rangle = 0 \quad P_{\delta_x}\text{-a.s.}$$

Theorem 2. $V_{\infty,c,D}(x) := -\log P_{\delta_x}\{\exp(-cZ_D); \mathcal{R}\subset D\}$ is the maximal solution of the problem (E1).

Proof. Let $\{D_k\}$ be a sequence of regular domains such that $\overline{D_k}\subset D_{k+1}$, D_k^c is compact and $D_k\uparrow D$. Let τ_k be the first exit time from D_k, and σ_k the first exit time from

B_k. Then $\tau_k \wedge \sigma_k$ is the first exit time from $B_k \cap D_k$. Suppose k is large enough such that $\partial D \subset B_k$. Then for $x \in D$,

$$\begin{aligned} V_{\infty,\,c,\,D}(x) &:= -\log P_{\delta_x}\{\exp(-cZ_D); \mathcal{R} \subset D\} \\ &= -\log P_{\delta_x}\{\exp(-c\lim_{k\to\infty}\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}\rangle); \cup_{k=1}^{\infty}(X_{\tau_k}=0)\} \\ &= \lim_{k\to\infty} -\log P_{\delta_x}\{\exp(-c\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}\rangle); X_{\tau_k}=0\}. \end{aligned}$$

By the special Markov property,

$$\begin{aligned} V_{\infty,\,c,\,D}(x) &= \lim_{k\to\infty} -\log P_{\delta_x}\{\exp(-c\langle \Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}\rangle)\cdot P_{X_{\tau_k\wedge\sigma_k}}(X_{\tau_k}=0)\} \\ &= \lim_{k\to\infty} -\log P_{\delta_x}\{\exp(-\langle c\Pi_\cdot(\tau=\infty), X_{\tau_k\wedge\sigma_k}\rangle)\cdot \exp\langle -v_k, X_{\tau_k\wedge\sigma_k}\rangle\} \\ &= \lim_{k\to\infty} -\log P_{\delta_x}\{\exp(-c\langle \Pi_\cdot(\tau=\infty)+v_k, X_{\tau_k\wedge\sigma_k}\rangle)\} \\ &= \lim_{k\to\infty} u_k(x), \quad x\in D, \end{aligned}$$

where $v_k(x)$ is the minimal positive solution of the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x)v(x)^{\alpha} & \text{in } \ D_k, \\ v(x)\to+\infty & \text{as } \ D_k \ni x \to a \in \partial D_k, \\ v(x)\to 0 & \text{as } \ \|x\|\to\infty, \end{cases}$$

and $u_k(x)$ is the solution of the problem

$$\begin{cases} \frac{1}{2}\Delta u(x) = \gamma(x)u(x)^{\alpha} \quad \text{in } \ D_k\cap B_k, \\ u|_{\partial(D_k\cap B_k)} = c\Pi_\cdot(\tau=\infty)+v_k. \end{cases}$$

Then as the same argument in Theorem 1, $V_{\infty,\,c,\,D}(x)$ satisfies

$$\frac{1}{2}\Delta V_{\infty,\,c,\,D}(x) = \gamma(x)V_{\infty,\,c,\,D}(x)^{\alpha}, \quad x\in D.$$

Notice $\{\mathcal{R}\subset D\}\subset\{X_\tau=0\}$, then

$$\begin{aligned} V_{\infty,\,c,\,D}(x) &= -\log P_{\delta_x}\{\exp(-cZ_D); \mathcal{R}\subset D\} \\ &\geqslant -\log P_{\delta_x}\{\exp(-cZ_D); X_\tau=0\} = v_{\infty,\,c,\,D}(x), \end{aligned}$$

where $v_{\infty,\,c,\,D}$ is the minimal positive solution of the problem (E1). Therefore $V_{\infty,\,c,\,D}|_{\partial D}=\infty$.

Now we prove that

$$\lim_{\|x\|\to\infty} V_{\infty,\,c,\,D}(x) = c.$$

First, by $V_{\infty,\,c,\,D}(x)\geqslant v_{\infty,\,c,\,D}(x)$ and $\lim\limits_{\|x\|\to\infty} v_{\infty,\,c,\,D}(x)=c$, we have

$$\liminf_{\|x\|\to\infty} V_{\infty,\,c,\,D}(x)\geqslant c. \tag{22}$$

Put

$$v_n(x) = -\log P_{\delta_x}\{\exp(-cZ_{D_n}); X_{\tau_n}=0\}.$$

By Lemma 2 and Theorem 1, $V_{\infty,\,c,\,D}(x)\leqslant v_n(x)$ in D_n, and therefore

$$V_{\infty,\,c,\,D}(x)\leqslant \liminf_{n\to\infty} v_n(x) \ \text{ in } \ D. \tag{23}$$

Suppose n is large enough, let $\{D_k^{(n)}\}$ be a sequence of regular domains such that $D_k^{(n)}\uparrow D_n$, $\overline{D_k^{(n)}}\subset D_{k+1}^{(n)}$ and $(D_k^{(n)})^c$ is compact. Let $\tau_k^{(n)}$ be the first exit time from

$D_k^{(n)}$, σ_k be the first exit time of B_k and $\sigma_k^{(n)} := \tau_k^{(n)} \wedge \sigma_k$. By the definition of Z_D (see (8)),

$$\begin{aligned} Z_{D_n} &= \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\sigma_k^{(n)}} \rangle \\ &= \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\sigma_k^{(n)}}|_{\partial B_k} \rangle + \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\sigma_k^{(n)}}|_{\partial D_k^{(n)}} \rangle \\ &\leqslant \liminf_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k} \rangle + \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\sigma_k^{(n)}}|_{\partial D_k^{(n)}} \rangle, \end{aligned}$$

Using Lemma 3 and Lemma 4 continues the above domination:

$$Z_{D_n} \leqslant \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k} \rangle \ P_{\delta_x}\text{-a.s.}, \ \forall x \in D.$$

Then we have

$$\begin{aligned} v_n(x) &= -\log P_{\delta_x}\{\exp(-cZ_{D_n}); X_{\tau_n} = 0\} \\ &\leqslant -\log P_{\delta_x}\{\exp(-c \lim_{k\to\infty} \langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k} \rangle); X_{\tau_n} = 0\} \\ &= \lim_{k\to\infty} -\log P_{\delta_x}\{\exp(-\langle c\Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k} \rangle); X_{\tau_n} = 0\}. \end{aligned} \tag{24}$$

Notice that $X_{\tau_n\wedge\sigma_k}|_{\partial D_n} \leqslant X_{\tau_n}$ and

$$P_{\delta_x} \langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n} \rangle = \Pi_x(\Pi_{\xi_{\tau_n}}(\tau_n = \infty); \tau_n < \infty) = 0.$$

Then

$$\langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k} \rangle = \langle \Pi_\cdot(\tau_n = \infty), X_{\tau_n\wedge\sigma_k}|_{\partial B_k} \rangle, \quad P_{\delta_x}\text{-a.s.} \tag{25}$$

If k is large enough,

$$X_{\tau_n\wedge\sigma_k}|_{\partial B_k} \leqslant X_{\tau_k\wedge\sigma_k}|_{\partial B_k}. \tag{26}$$

Then by (24), (25) and (26), we have

$$\begin{aligned} v_n(x) &\leqslant \liminf_{k\to\infty} -\log P_{\delta_x}\{\exp(-\langle c\Pi_\cdot(\tau_n = \infty), X_{\tau_k\wedge\sigma_k}|_{\partial B_k} \rangle); X_{\tau_n} = 0\} \\ &\leqslant \liminf_{k\to\infty} -\log P_{\delta_x}\{\exp(-\langle c\Pi_\cdot(\tau_n = \infty), X_{\tau_k\wedge\sigma_k} \rangle); X_{\tau_n} = 0\} \\ &= -\log P_{\delta_x}\{\exp(-cZ_D); X_{\tau_n} = 0\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} v_n(x) \leqslant V_{\infty,c,D}(x) \quad \text{in } D. \tag{27}$$

By (23), (27) and $\{v_n(x)\}$ being decreasing for all $x \in D$,

$$v_n(x) \downarrow V_{\infty,c,D}(x) \quad \text{as } n \to \infty, \quad \text{for all } x \in D.$$

For each $\varepsilon > 0$, there exists $R > 0$ such that $v_1(x) < c + \varepsilon, \ \forall x \in B(0,R)^c$. Since $\{v_n(x)\}$ is decreasing, $v_n(x) \leqslant c + \varepsilon$, $\forall x \in B(0,R)^c$. And therefore $V_{\infty,c,D}(x) < c + \varepsilon$ for $x \in B(0,R)^c$, then $\limsup_{\|x\|\to\infty} V_{\infty,c,D}(x) \leqslant c + \varepsilon$. Since ϵ is arbitrary, we get

$$\limsup_{\|x\|\to\infty} V_{\infty,c,D}(x) \leqslant c. \tag{28}$$

Combining (22) with (28), we get $\lim_{\|x\|\to\infty} V_{\infty,c,D}(x) = c$. Thus $V_{\infty,c,D}$ is a solution to problem (E1).

By (27) and Lemma 2, we know that $V_{\infty,c,D}(x)$ is the maximal solution of the problem (E1). Now we complete the proof.

Corollary 1. $V_{\infty,\infty,D}(x) := -\log P_{\delta_x}(\mathcal{R} \subset\subset D)$ is the maximal solution of the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } D, \\ v(x) \to +\infty & \text{as } D \ni x \to a \in \partial D, \\ v(x) \to +\infty & \text{as } \|x\| \to \infty, \end{cases} \tag{E2}$$

where $\{\mathcal{R} \subset\subset D\}$ denotes the union of the sets $\{\mathcal{R} \subset \Gamma\}$ over all compact sets $\Gamma \subset D$.

Proof. Let $\{D_n\}$ be a sequence of bounded domains such that $\overline{D_n} \subset D_{n+1}$ and $D_n \uparrow D$. Let τ_n denote the first exit time from D_n. Then by Theorem 1.2 in Dynkin[1], $v_n(x) = -\log P_{\delta_x}(X_{\tau_n} = 0)$ is the minimal positive solution of the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } D_n, \\ v(x) \to +\infty & \text{as } D_n \ni x \to a \in \partial D_n. \end{cases}$$

Clearly, $v_n(x) = -\log P_{\delta_x}(X_{\tau_n} = 0) \downarrow V_{\infty,\infty,D}(x) = -\log P_{\delta_x}(\mathcal{R} \subset\subset D)$. As the same argument in Theorem 1, $\frac{1}{2}\Delta V_{\infty,\infty,D}(x) = \gamma(x)V_{\infty,\infty,D}(x)^\alpha$ in D.

By the maximum principle, we have $V_{\infty,c,D}(x) \leqslant v_n(x)$ in D_n. Then $V_{\infty,c,D}(x) \leqslant V_{\infty,\infty,D}(x)$ in D. For $x \in \partial D$,

$$\infty = \lim_{x \to a, x \in D} V_{\infty,c,D}(x) \leqslant \lim_{x \to a, x \in D} V_{\infty,\infty,D}(x).$$

And

$$c = \lim_{\|x\| \to \infty} V_{\infty,c,D}(x) \leqslant \lim_{\|x\| \to \infty} V_{\infty,\infty,D}(x).$$

Letting $c \to \infty$, we have

$$\lim_{\|x\| \to \infty} V_{\infty,\infty,D}(x) = \infty.$$

Thus $V_{\infty,\infty,D}$ is a solution of (E2).

Suppose $u \geqslant 0$ is a solution to problem (E2), then by the maximum principle, $u \leqslant v_n$ in D_n and therefore $u \leqslant V_{\infty,\infty,D}$ in D. Therefore, $V_{\infty,\infty,D}$ is the maximal solution of (E2).

3 S-polar sets

Theorem 3. Each of the following conditions is necessary and sufficient for a compact set Γ to be S-polar:

A. If $v \geqslant 0$ satisfies the problem

$$\begin{cases} \frac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } D = \Gamma^c, \\ v(x) \to 0 & \text{as } \|x\| \to \infty, \end{cases} \tag{E3}$$

then $v = 0$.

B. The maximal solution of the problem (E3) in D is bounded.

Proof. First by Dynkin[1], Γ is S-polar if and only if $P_{\delta_x}(\mathcal{R} \cap \Gamma = \emptyset) = 1$ for all $x \in D$. Then by Theorem 2, Γ is S-polar if and only if the maximal solution $V_{\infty,0,D}(x) =$

$-\log P_{\delta_x}(\mathcal{R} \subset D)$ of problem (E3) is equal to zero. So any nonnegative solution of the problem (E3) can only be zero.

Clearly A implies B. If B holds, let $\{D_n\}$ be a sequence of bounded domains such that $\overline{D_n} \subset D_{n+1}$ and $D_n \uparrow D$, and let τ_n denote the first exit time from D_n. For $V_{\infty,0,D}$ is bounded, by (2.22) in Dynkin[1], we have for every $x \in D$, $\langle V_{\infty,0,D}, X_{\tau_n}\rangle \to 0$, P_{δ_x}-a.s. Also, for each n, $V_{\infty,0,D}(x) = -\log P_{\delta_x} \exp\{-\langle V_{\infty,0,D}, X_{\tau_n}\rangle\}$ in D_n. Letting $n \to \infty$, we get $V_{\infty,0,D}(x) \equiv 0$. So B also implies A.

Theorem 4. Suppose Γ is a compact set. If Γ is S-polar, then the maximal solution $V_{\infty,\infty,D}$ of the equation $\frac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha$ in $D = \Gamma^c$ coincides, in D, with the maximal solution $V_{\infty,\infty,\mathbb{R}^d}$ of this equation in $\mathbb{R}^d$; conversely, if $V_{\infty,\infty,D}$ is bounded near Γ, then Γ is S-polar.

Proof. By Corollary 1,

$$V_{\infty,\infty,D}(x) = -\log P_{\delta_x}(\mathcal{R} \subset\subset D), \quad x \in D,$$

and

$$V_{\infty,\infty,\mathbb{R}^d}(x) = -\log P_{\delta_x}(\mathcal{R} \subset\subset \mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

If Γ is S-polar, then $P_{\delta_x}(\mathcal{R} \cap \Gamma = \emptyset) = 1$. So we get $V_{\infty,\infty,D}(x) = -\log P_{\delta_x}(\mathcal{R} \subset\subset \mathbb{R}^d) = V_{\infty,\infty,\mathbb{R}^d}(x)$ for $x \in D$.

Now suppose $V_{\infty,\infty,D}$ is bounded near Γ. Clearly

$$V_{\infty,0,D} = -\log P_{\delta_x}(\mathcal{R} \cap \Gamma = \emptyset) \leqslant V_{\infty,\infty,D},$$

where $V_{\infty,0,D}$ is the maximal solution of the problem (E3), then $V_{\infty,0,D}$ is bounded near Γ. By Theorem 3B, Γ is S-polar.

Moreover, we have

Theorem 5. Each of the following conditions is necessary and sufficient for a compact set Γ to be S-polar:

A. For every $0 \leqslant c < \infty$, the solution of the problem

$$\begin{cases} \dfrac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } D = \Gamma^c, \\ v(x) \to c & \text{as } \|x\| \to \infty \end{cases} \tag{E4}$$

is unique.

B. There exists $0 \leqslant c < \infty$ such that the maximal solution of the above problem in D is bounded.

Proof. By Theorem 2, the maximal solution of problem (E4) is

$$V_{\infty,c,D}(x) = -\log P_{\delta_x}\{\exp(-cZ_D); \mathcal{R} \cap \Gamma = \emptyset\}.$$

By (7) and Lemma 2, the minimal solution of problem (E4) is

$$v_{0,c,D}(x) = -\log P_{\delta_x}\{\exp(-cZ_D)\}.$$

So we get Γ is S-polar if and only if

$$V_{\infty,c,D} = v_{0,c,D},$$

which is equivalent to say the solution of the problem (E4) is unique.

Clearly A implies B. Now suppose there exists $c \geqslant 0$ such that $V_{\infty, c, D}$ is bounded, we get that $V_{\infty, 0, D}$ is also bounded, where $V_{\infty, 0, D}$ is the maximal solution of problem (E3). Then by Theorem 3B, Γ is S-polar.

We say that an analytic set B is B-polar if

$$\Pi_x\{\xi_t \notin B \text{ for all } t > 0\} = 1, \quad \forall x \in B^c.$$

It is easy to see that B is B-polar if and only if

$$\Pi_x\{\sigma = \infty\} = 1, \quad \forall x \in B^c,$$

where $\sigma = \inf\{t : \xi_t \in B\}$.

Lemma 5. Suppose a compact set K is contained in a domain D. Let $\widetilde{D} = D \setminus K$. If K is B-polar, then $Z_D = Z_{\widetilde{D}}$ P_{δ_x}-a.s., for all $x \in \widetilde{D}$.

Proof. Let $\{D_n\}$ be a sequence of bounded regular Greenian domains such that $\overline{D_n} \subset D_{n+1}$ and $D_n \uparrow D$, $\{K_n\}$ be a sequence of compact sets such that $K_n \downarrow K$. Let $\widetilde{D}_n = D_n \setminus K_n$, then $\widetilde{D}_n \uparrow \widetilde{D}$. Let σ_n, σ be the first hitting times of K_n and K separately, and let τ_n be the first exit time from D_n. Put $\widetilde{\tau}_n = \tau_n \wedge \sigma_n$, $\widetilde{\tau} = \tau \wedge \sigma$, then $\widetilde{\tau}_n$ is the first exit time from $\widetilde{D}_n$ and $\widetilde{\tau}$ is the first exit time from $\widetilde{D}$. So

$$\begin{aligned}&\langle \Pi_\cdot(\widetilde{\tau} = \infty), X_{\widetilde{\tau}_n}\rangle \\ &= \langle \Pi_\cdot(\widetilde{\tau} = \infty), X_{\widetilde{\tau}_n}|_{\partial D_n}\rangle + \langle \Pi_\cdot(\widetilde{\tau} = \infty), X_{\widetilde{\tau}_n}|_{\partial K_n}\rangle \\ &\leqslant \langle \Pi_\cdot(\tau = \infty), X_{\tau_n}\rangle + \langle \Pi_\cdot(\widetilde{\tau} = \infty), X_{\widetilde{\tau}_n}|_{\partial K_n}\rangle.\end{aligned} \tag{29}$$

As the same argument in Lemma 4, $\liminf\limits_{n\to\infty} < \Pi_\cdot(\widetilde{\tau} = \infty), X_{\widetilde{\tau}_n}|_{\partial K_n}\rangle = 0$, P_{δ_x}-a.s. Letting $n \to \infty$ in (29), we get

$$Z_{\widetilde{D}} \leqslant Z_D \quad P_{\delta_x}\text{-a.s.}, \ \forall x \in \widetilde{D}. \tag{30}$$

Assume that we have proved

$$P_{\delta_x}(\exp(-Z_D)) \geqslant P_{\delta_x}(\exp(-Z_{\widetilde{D}})), \ \forall x \in \widetilde{D}, \tag{31}$$

then together with (30), we have $Z_D = Z_{\widetilde{D}}$ P_{δ_x}-a.s. for all $x \in \widetilde{D}$. Now we are left to prove (31).

Since K is B-polar, we have $\Pi_x\{\sigma = \infty\} = 1$, $\forall x \in K^c$. Then $\Pi_x(\tau = \infty) = \Pi_x(\widetilde{\tau} = \infty)$, $\forall x \in K^c$, and

$$\begin{aligned}Z_D &= \lim_{n\to\infty} \langle \Pi_\cdot(\tau = \infty), X_{\tau_n}\rangle \\ &= \lim_{n\to\infty} \langle \Pi_\cdot(\widetilde{\tau} = \infty), X_{\tau_n}\rangle.\end{aligned} \tag{32}$$

Notice that

$$P_\mu\{\exp\langle -\Pi_\cdot(\widetilde{\tau} = \infty), X_{\tau_n}\rangle\} = \exp\langle -v_n, \mu\rangle,$$

where v_n satisfies

$$v_n(x) + \Pi_x \int_0^{\tau_n} \gamma(\xi_s) v_n(\xi_s)^\alpha ds = \Pi_x(\Pi_{\xi_{\tau_n}}(\widetilde{\tau} = \infty)).$$

And

$$P_\mu\{\exp\langle -\Pi_\cdot(\widetilde{\tau} = \infty), X_{\tilde{\tau}_n}\rangle\} = \exp\langle -\widetilde{v}_n, \mu\rangle,$$

where $\widetilde{v}_n$ satisfies

$$\widetilde{v}_n(x) + \Pi_x \int_0^{\widetilde{\tau}_n} \gamma(\xi_s)\widetilde{v}_n(\xi_s)^\alpha ds = \Pi_x(\widetilde{\tau} = \infty).$$

Notice that

$$\begin{aligned} v_n(x) &\leqslant \Pi_x(\Pi_{\xi_{\tau_n}}(\widetilde{\tau} = \infty) \leqslant \Pi_x(\Pi_{\xi_{\tau_n}}(\tau = \infty)) \\ &= \Pi_x(\tau = \infty) = \Pi_x(\widetilde{\tau} = \infty), \quad \forall x \in K^c. \end{aligned}$$

Since $\widetilde{\tau}_n \leqslant \tau_n$, by the special Markov property,

$$\begin{aligned} &P_{\delta_x}(\exp\langle -\Pi_\cdot(\widetilde{\tau} = \infty), X_{\tau_n}\rangle / \mathcal{F}_{\tilde{\tau}_n}) \\ &= P_{X_{\tilde{\tau}_n}}(\exp\langle -\Pi_\cdot(\tilde{\tau} = \infty), X_{\tau_n}\rangle) = \exp\langle -v_n, X_{\tilde{\tau}_n}\rangle \\ &\geqslant \exp\langle -\Pi_\cdot(\tilde{\tau} = \infty), X_{\tilde{\tau}_n}\rangle, \quad x \in \widetilde{D}. \end{aligned} \tag{33}$$

Letting $n \to \infty$, by Theorem 2.2.4 in Renvez[4] and by noticing (32), we have, for $x \in \widetilde{D}$,

$$P_{\delta_x}(\exp(-Z_D)/\bigcup_n \mathcal{F}_{\tilde{\tau}_n}) \geqslant \exp(-Z_{\widetilde{D}}) \quad P_{\delta_x}\text{-a.s.}$$

Taking the expectation on both sides of the above inequality, we get

$$P_{\delta_x}(\exp(-Z_D)) \geqslant P_{\delta_x}(\exp(-Z_{\widetilde{D}})), \quad x \in \widetilde{D}.$$

This completes the proof.

Theorem 6. Suppose a compact set Γ is contained in a domain D. If Γ is B-polar and S-polar, then for every $0 \leqslant c < \infty$, the maximal solution $V_{\infty, c, \widetilde{D}}$ of the problem

$$\begin{cases} \dfrac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } \widetilde{D} = D \setminus \Gamma, \\ v(x) \to c & \text{as } \|x\| \to \infty \end{cases}$$

coincides, in $\widetilde{D}$, with the maximal solution $V_{\infty, c, D}$ of the problem

$$\begin{cases} \dfrac{1}{2}\Delta v(x) = \gamma(x)v(x)^\alpha & \text{in } D, \\ v(x) \to c & \text{as } \|x\| \to \infty, \end{cases}$$

and hence $V_{\infty, c, \widetilde{D}}$ is bounded in a neighborhood of Γ.

Conversely, if there exists a constant $0 \leqslant c < \infty$ such that the maximal solution $V_{\infty, c, \widetilde{D}}$ is bounded near Γ, then Γ is S-polar.

Proof. By Theorem 2,

$$V_{\infty, c, \widetilde{D}}(x) = -\log P_{\delta_x}\{\exp(-cZ_{\widetilde{D}}), \mathcal{R} \subset \widetilde{D}\}, \quad x \in \widetilde{D},$$

and

$$V_{\infty, c, D}(x) = -\log P_{\delta_x}\{\exp(-cZ_D), R \subset D\}, \quad x \in D.$$

If Γ is S-polar, then $P_{\delta_x}\{\mathcal{R} \subset \Gamma^c\} = 1$, $\forall x \in \Gamma^c$. By Lemma 5, $Z_{\widetilde{D}} = Z_D$, so we get $V_{\infty, c, \widetilde{D}} = V_{\infty, c, D}$ in $\widetilde{D}$, and hence $V_{\infty, c, \widetilde{D}}$ is bounded in a neighborhood of Γ.

Conversely, suppose that $V_{\infty, c, \widetilde{D}}$ is bounded near Γ. Notice that $-\log P_{\delta_x}(\mathcal{R} \cap \Gamma = \emptyset) \leqslant -\log P_{\delta_x}(\mathcal{R} \subset \widetilde{D}) \leqslant -\log P_{\delta_x}\{\exp(-cZ_{\widetilde{D}}), \mathcal{R} \subset \widetilde{D}\} = V_{\infty, c, \widetilde{D}}(x)$, so $-\log P_{\delta_x}(\mathcal{R} \cap \Gamma = \emptyset)$ is bounded near Γ and it is the maximal solution of the problem (E3). By Theorem 3B, Γ is S-polar.

Remark. If Γ is B-polar and S-polar, $V_{\infty,\,c,\,\mathbb{R}^d}(x) = -\log P_{\delta_x}\{\exp(-cZ_{\mathbb{R}^d})\}$ is the unique solution to problem (E4). In fact, Theorem 5 says that the solution is unique, and by Theorem 6, the maximal solution $V_{\infty,\,c,\,D}$ coincides, in $D = \Gamma^c$ with $V_{\infty,\,c,\,\mathbb{R}^d}(x) = -\log P_{\delta_x}\{\exp(-cZ_{\mathbb{R}^d})\}$, the maximal solution of problem (E4) with D replaced by $\mathbb{R}^d$.

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