Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®
ELSEVIER
Stochastic Processes and their Applications 110 (2004) 19–44
stochastic processes and their applications
www.elsevier.com/locate/spa

Support properties of super-Brownian motions with spatially dependent branching rate

Yan-Xia Ren*,[1]

LMAM School of Mathematical Sciences, Peking University, Beijing 100871, PR China

Received 10 January 2003; received in revised form 22 September 2003; accepted 23 September 2003

Abstract

We consider a critical finite measure-valued super-Brownian motion $X = (X_t, P_\mu)$ in $\mathbb{R}^d$, whose log-Laplace equation is associated with the semilinear equation $(\partial/\partial t)u = \frac{1}{2}\Delta u - ku^2$, where the coefficient $k(x) > 0$ for the branching rate varies in space, and is continuous and bounded. Suppose that $\operatorname{supp}\mu$ is compact. We say that X has the compact support property, if $P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) = 1$ for every $t > 0$, and we say that the global support of X is compact if $P_\mu\left(\bigcup_{0\leqslant s<\infty} \operatorname{supp} X_s \text{ is bounded}\right) = 1$. We prove criteria for the compact support property and the compactness of the global support. If there exists a constant $M > 0$ such that $k(x) \geqslant \exp(-M\|x\|^2)$ as $\|x\| \to \infty$ then X possesses the compact support property, whereas if there exist constant $\beta > 2$ such that $k(x) \leqslant \exp(-\|x\|^\beta)$ as $\|x\| \to \infty$ then X does not have the compact support property. For the global support, we prove that if $k(x) = \|x\|^{-\beta}$ $(0 \leqslant \beta < \infty)$ for sufficiently large $\|x\|$, then the maximum decay order of k for the global support being compact is different for $d = 1$, $d = 2$ and $d \geqslant 3$: it is $O(\|x\|^{-3})$ in dimension one, $O(\|x\|^{-2}(\log\|x\|)^{-3})$ in dimension two, and $O(\|x\|^{-2})$ in dimensions three or above.
© 2003 Elsevier B.V. All rights reserved.

MSC: primary 60J80; secondary 60G57; 60J45

Keywords: Super-Brownian motion; Compact support property; Global support; Finite time extinction

1. Introduction and main results

Consider a measure-valued super-Brownian motion $X = (X_t, P_\mu)$ in $\mathbb{R}^d$, which is related to the semilinear equation $(\partial/\partial t)u = \frac{1}{2}\Delta u - ku^2$ through its log-Laplace functional.

* Tel.: +86-10-62756663; fax: +86-10-62751801.
E-mail address: yxren@math.pku.edu.cn (Y.-X. Ren).
[1] Supported by NNSF of China (Grant No. 10001020), Foundation for Authors Awarded Excellent Ph.D. Dissertation and an NSERC Canada Research Grant.

0304-4149/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.spa.2003.09.009

Let $M_F(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$, equipped with the topology of weak convergence. $M_c(\mathbb{R}^d)$ refers to the subset of all compactly support $\mu \in M_F(\mathbb{R}^d)$. For $0 \neq \mu \in M_c(\mathbb{R}^d)$, we say that the measure-valued process corresponding to P_μ possesses the compact support property if

$$P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) = 1 \quad \text{for all } t \geqslant 0. \tag{1}$$

The global support of X, $\operatorname{Gsupp}(X)$, is defined as the closure of $\bigcup_{t\geqslant 0} \operatorname{supp} X_t$. We say X becomes extinct (survives) in finite time if

$$P_\mu(X_t = 0, \exists t > 0) = 1 \ (< 1). \tag{2}$$

It is well known that if $\inf_{x\in\mathbb{R}^d} k(x) > 0$, then X has the compact support property, and the global support of X is compact. But if $k(x)$ tends to 0 at infinity, the situation is less clear. Abstract conditions have been derived in Engländer and Pinsky (1999) for the compact support property and the finite time extinction property of supercritical superdiffusions (including the critical case). These abstract theorems describe the correspondence between these properties and problems of nonlinear partial differential equations. But for a concrete branching mechanism, these theorems do not tell us when a super-Brownian motion becomes extinct and when it possesses the compact support property. We collect their main results for super-Brownian motion corresponding to the equation $(\partial/\partial t)u = \frac{1}{2}\Delta u - ku^2$ in Lemma 1.1 below.

Lemma 1.1. *Suppose that* $\mu \in M_c(\mathbb{R}^d)$.

(1) $P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) = \exp\left(-\int u(t,x)\mu(\mathrm{d}x)\right)$, *where* u *is the maximum solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u - ku^2 & \text{in } \mathbb{R}^d, \\ \lim\limits_{t\to 0} u(t,x) = 0. \end{cases} \tag{3}$$

(2) *There exists a nonnegative function* $\omega(x)$, *which solves the equation* $\frac{1}{2}\Delta u = ku^2$ *on* $\mathbb{R}^d$, *such that*

$$P_\mu(X_t = 0,\ \exists t > 0) = \exp\left(-\int_{\mathbb{R}^d} \omega(x)\mu(\mathrm{d}x)\right). \tag{4}$$

Furthermore, ω *is either identically zero or positive everywhere in* $\mathbb{R}^d$.

(3) *There exists a maximal nonnegative solution to* $\frac{1}{2}\Delta u = ku^2$ *on* $\mathbb{R}^d$, $\omega_{\max}$, *such that*

$$P_\mu(\operatorname{Gsupp}(X) \text{ is bounded}) = \exp\left(-\int_{\mathbb{R}^d} \omega_{\max}\,\mu(\mathrm{d}x)\right). \tag{5}$$

(4) *If the compact support property holds, then* $\omega \equiv \omega_{\max}$.

Later in Engländer (2000), a concrete criteria for the compact support property is given for the superprocess related to $(\partial/\partial t)u = \rho(x)\Delta u - u^2$ with $\rho > 0$. For this particular superprocess, the compact support property is equivalent to the global support being compact. It is proved that the maximal order of $\rho(x)$ as $x \to \infty$ for X having the compact support property is $O(x^3)$ in dimension one and $O(x^2)$ in higher dimensions. In Engländer and Fleischmann (2000), concrete criteria for the finite time extinction property of super-Brownian motions with spatially dependent mass production are given.

Dawson et al. (2000) investigated the finite time extinction of catalytic super-Brownian motions in one dimension. Here "catalytic" means that branching of the reactant X is only possible in the presence of some catalyst. They gave an abstract sufficient criterion for finite time extinction based on the concept of good and bad paths. Using this abstract criterion they showed that if $k(x) = \|x\|^q \wedge 1$ $(q > 0)$, the super-Brownian motion X dies in finite time.

In this article, we investigate support properties of super-Brownian motion X in the case $k(x) \to 0$ as $x \to \infty$. Throughout this paper, we suppose k is a bounded continuous function and $k(x) > 0$ for every $x \in \mathbb{R}^d$.

In the first part of this paper, we are going to derive concrete criteria for the super-Brownian motion X with spatially dependent branching rate k to have the compact support property, which, according to Lemma 1.1(1), is equivalent to the condition that (3) has no positive solution. It turns out that if k decays at infinity no faster than $\exp(-M\|x\|^2)$ for some constant $M > 0$ then X possesses the compact support property, whereas if k decays at infinity no slower than $\exp(-\|x\|^\beta)$ for some constant $\beta > 2$ then X does not possess the compact support property. Here is the precise statement of the result.

Theorem 1.1. *Suppose $X = (X_t, P_\mu)$ is a super-Brownian motion with branching rate k starting at $\mu \in M_c(\mathbb{R}^d)$ at time 0.*

(1) *If there exist a constant $\beta > 2$ such that $k(x) \leqslant \exp(-\|x\|^\beta)$ for sufficiently large $x \in \mathbb{R}^d$, then X does not possess the compact support property.*
(2) *If there exist a constant $M > 0$ such that $k(x) \geqslant \exp(-M\|x\|^2)$ for sufficiently large $x \in \mathbb{R}^d$, then X possesses the compact support property.*

Remark. According to Lemma 1.1(1), the above result can be translated to result on the PDE problem (3): If there exist a constant $\beta > 2$ such that $k(x) \leqslant \exp(-\|x\|^\beta)$ for sufficiently large $x \in \mathbb{R}^d$, then there isn't positive solution to problem (3); if there exist a constant $M > 0$ such that $k(x) \geqslant \exp(-M\|x\|^2)$ for sufficiently large $x \in \mathbb{R}^d$, then (3) possesses positive solution.

The second part of this paper is devoted to investigate conditions which guarantee that X has compact global support, which, according to Engländer and Pinsky's results (see Lemma 1.1(3)), is equivalent to the condition that the differential equation

$$\tfrac{1}{2}\Delta u(x) = k(x)u^2(x), \quad x \in \mathbb{R}^d \tag{6}$$

has no positive solution. Therefore PDE results concerning (6) are helpful for us to get a condition for X to have compact global support. In Cheng and Ni (1992) it is proved that if $k(x) = k(\|x\|)$ is a radial function and satisfies

$$\begin{cases} \text{(a)} & \displaystyle\int_{-\infty}^{\infty} r^2 k(r)\,\mathrm{d}r < \infty, \quad d = 1, \\ \text{(b)} & \displaystyle\int_{1}^{\infty} r(\ln r)^2 k(r)\,\mathrm{d}r < \infty, \quad d = 2, \\ \text{(c)} & \displaystyle\int_{0}^{\infty} rk(r)\,\mathrm{d}r < \infty, \quad d \geqslant 3, \end{cases} \tag{7}$$

where $r = \|x\|$, then (6) has infinitely many positive solutions. It is also proved that if k satisfies

$$k(x) \geqslant \begin{cases} M[\|x\|^{-3}], & d = 1, \\ M[\|x\|^{-2}(\log \|x\|)^{-3}], & d = 2, \\ M\|x\|^{-2}, & d \geqslant 3, \end{cases} \tag{8}$$

for sufficiently large $\|x\|$ and some $M > 0$, then (6) has no positive solution. I would like to mention here that, using analytic methods, Engländer and Pinsky (2003) obtained the same result on (6) for radial symmetric k in $d = 1$ and $d \geqslant 3$, while their result is a little bit weaker in $d = 2$. So, from these analytic results we know that condition (8) is sufficient for X to have compact global support and under condition (7), X does not have global compact support. Our purpose is to give a probabilistic approach. Concerning the probabilistic approach to (6), Sheu (1995) proved that for dimension $d \geqslant 3$ condition (c) in (7) is sufficient for (6) to have a positive solution. Recently Ren (2003) proved that conditions (a) and (b) in (7) are sufficient for (6) to have positive solutions for $d = 1$ and 2, respectively. In this paper we are going to present a new and probabilistic proof for the fact that (8) is a sufficient condition for X to have compact global support, which is equivalent to (6) having no positive solution. And we also give another probabilistic proof of the fact that under condition (7) the global support of X isn't compact with positive probability (see Theorem 1.2(2) below, which is presented for general k, not just for radial function $k(r)$).

Theorem 1.2. *Let $X = (X_t, P_\mu)$ be a super-Brownian motion with branching mechanism k starting at $\mu \in M_c(\mathbb{R}^d)$ at time 0.*

(1) *If k satisfies (8), then the global support of X is compact, i.e.,*

$$P_\mu(\mathrm{Gsupp}\,(X) \text{ is bounded}) = 1. \tag{9}$$

(2) *If there exists a constant $M > 0$ such that*

$$\Pi_x \int_0^{\tau_{B^c(0,M)}} k(B_s)h^2(B_s)\,\mathrm{d}s < \infty \quad \text{for } x \in B^c(0, M), \tag{10}$$

where $B=(B_t,\Pi_x)$ *is a Brownian motion starting at* $x\in\mathbb{R}^d$ *and* $\tau_{B^c(0,M)}$ *denotes the first exit time of* B *from* $B^c(0,M)=\{x\in\mathbb{R}^d,\|x\|\geqslant M\}$, *and*

$$h(x)=\begin{cases}|x|, & d=1,\\ \ln\|x\|, & d=2,\\ 1, & d\geqslant 3,\end{cases} \tag{11}$$

then the global support of X *is not compact, i.e.,*

$$P_\mu(\mathrm{Gsupp}\,(X)\ \textit{is bounded})<1. \tag{12}$$

Remark. (1) In the theorem above, h was chosen to be a positive harmonic function for $\frac{1}{2}\Delta h=0$ on D with boundary condition $h=0$ on ∂D, where

$$D_d:=\begin{cases}(0,\infty), & d=1,\\ \mathbb{R}^d\setminus\overline{B(0,1)}, & d=2,\\ \mathbb{R}^d, & d\geqslant 3.\end{cases}$$

The choice of the domain reflects that Brownian motion is point recurrent in $d=1$, is Harris recurrent in $d=2$ and is transient in $d\geqslant 3$. Intuitively, the global support of X is compact if and only if the global support of the super-process $\tilde{X}$ with underlying motion being conditioned on not hitting the boundary $\partial D\setminus\{\infty\}$ is compact. $\tilde{X}$ is known as the h-transform of X in Engländer and Pinsky (1999),

(2) If

$$k(x)\leqslant\begin{cases}M[|x|^{-3-\varepsilon}], & d=1,\\ M[\|x\|^{-2}(\log\|x\|)^{-3-\varepsilon}], & d=2,\\ M\|x\|^{-2-\varepsilon}, & d\geqslant 3\end{cases}$$

for sufficiently large $\|x\|$ and some ε, $M>0$, then (10) holds, and therefore the global support of X isn't compact.

(3) According to Lemma 1.1(3), the result of Theorem 1.2 can be translated to a result on the PDE problem (6): If k satisfies (8), then (6) has no positive solution; if k satisfies (10), then (6) has at least one positive solution.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need to use historical super-Brownian motion. Some useful facts about historical super-Brownian motion are collected below. For details, please see Dawson and Perkins (1991), Perkins (2002) and Dynkin (1991b).

Let us introduce some notations first. E will stand for a Polish space. Let $C(E)=C(\mathbb{R}^+,\mathbf{E})$ be the Polish space of continuous E-valued functions with the topology of uniform convergence on compact sets of $\mathbb{R}^+$. Let $\mathscr{B}(E)$ denote the Borel subset of E.

$C(\mathbb{R}^d)$ will be simply denoted by C, and $\mathscr{B}(C)$ as $\mathscr{C}$. Let $bp\mathscr{B}(E)$ denote the set of all bounded positive $\mathscr{B}$-measurable functions on E. We use $M_F(E)$ to denote the space of finite measures on $M_F(E)$, equipped with the topology of weak convergence. We use $\langle \mu, f\rangle$ to denote the integral of f with respect to μ. Suppose $B = \{B_t, \mathscr{C}_t, \Pi_x\}$ is the time homogeneous coordinate Brownian motion on $(C, \mathscr{C})$, where $\mathscr{C}_\Delta = \sigma\{B_s; s \in \Delta\}$ for every $\Delta \subseteq \mathbb{R}^+$ and $\mathscr{C}_t = \mathscr{C}_{[0,t]}$. For every $A \in \mathscr{C}$ and $\mu \in M_F(\mathbb{R}^d)$, put

$$\Pi_\mu(A) = \int_{\mathbb{R}^d} \Pi_x(A)\, \mathrm{d}\mu. \tag{13}$$

To each $\omega \in C$ and $t \geqslant 0$, we associate the stopped path ω^t defined by $\omega^t(s) := \omega(s \wedge t)$, $s \geqslant 0$. Write C^t for the closed subspace of C of all those paths which are constant after t, i.e., $C^t = \{y \in C, y = y^t\}$. Note that C^0 could be regarded as $\mathbb{R}^d$. For $y, \omega \in C$ and $s \geqslant 0$, let

$$(y/s/\omega)(t) = \begin{cases} y(t), & t < s, \\ \omega(t-s), & t \geqslant s. \end{cases} \tag{14}$$

The historical process for B is a path valued process $\hat{B} = \{\hat{B}_t, \hat{\mathscr{C}}_t, (\hat{\Pi}_{r,y})_{y \in C^r}\}$ such that, for every $\omega \in C$, $\hat{B}_t(\omega) = \omega(\cdot \wedge t)$, $\hat{\mathscr{C}}_t = \mathscr{C}_t$ and

$$\hat{\Pi}_{r,y}(\hat{B}_\cdot(\omega) \in A) = \Pi_{y(r)}(\{\omega\colon\ y/r/\omega \in A\}), \quad y \in C^r,\ A \in \mathscr{C}. \tag{15}$$

Put

$$\hat{C}^t = \{t\} \times C^t = \{(t, y^t), y \in C\}$$

and

$$\hat{C} = \bigcup_{t \geqslant 0} \hat{C}^t = \{(t, y^t); t \geqslant 0, y \in C\},$$

equipped with the subspace topology it inherits from $\mathbb{R}^+ \times C$, and let $\hat{\mathscr{C}} = \mathscr{B}(\hat{C})$.

For every finite measure $\hat{\mu}$ on $(\hat{C}, \hat{\mathscr{C}})$, we set

$$\hat{\Pi}_{\hat{\mu}}(\cdot) = \int_{\hat{C}} \hat{\Pi}_{r,y}(\cdot)\, \mathrm{d}\hat{\mu}. \tag{16}$$

In particular, for $r \geqslant 0$ and $y \in C^r$, $\hat{\Pi}_{\delta_{r,y}} = \hat{\Pi}_{r,y}$ where $\delta_{r,y}$ is the Dirac's measure at point (r, y).

Throughout this paper we put

$$\hat{k}(r, y) = k(y(r)), \quad r \geqslant 0,\ y \in C^r. \tag{17}$$

Suppose $\hat{X} = \{\hat{X}_t, \hat{P}_{s,\hat{\mu}}, s \geqslant 0, \mu \in M_F(C^s)\}$ is a historical super-Brownian motion with branching mechanism $k(x)z^2$. More precisely, $\hat{X}$ is a strong Markov process with state $\hat{X}_t \in M_F(\hat{C}^t)$, $t \geqslant s$, and the Laplace functional

$$\hat{P}_{s,\hat{\mu}} \exp\langle \hat{X}_t, -\hat{f}\rangle = \exp\langle \hat{\mu}, -v^t(s, \cdot)\rangle, \quad 0 \leqslant s \leqslant t,\ \hat{\mu} \in M_F(C^s),\ \hat{f} \in bp\hat{\mathscr{C}}. \tag{18}$$

where $v^t(s,y)$ is the unique solution of the integral equation

$$v^t(s,y)+\hat{\Pi}_{s,y}\int_s^t [v^t(r,\hat{B}_r)]^2\hat{k}(r,\hat{B}_r)\,\mathrm{d}r=\hat{\Pi}_{s,y}\hat{f}(t,\hat{B}_t),\quad 0\leqslant s\leqslant t,\ \ y\in C^s. \tag{19}$$

Moreover, to each $\hat{\mu}\in M_F(\hat{C})$ there is a Markov process $(\hat{X}_t,\hat{P}_{\hat{\mu}})$ with states $\hat{X}_t\in M_F(\hat{C})$ and such that

$$\hat{P}_{\hat{\mu}}\exp\langle\hat{X}_t,-\hat{f}\rangle=\exp\langle\hat{\mu},-v^t(\cdot,\cdot)\rangle,\quad t\geqslant 0,\ \ \hat{f}\in bp\hat{\mathscr{C}}, \tag{20}$$

where $v^t(s,\cdot)$ is given by (19) if $0\leqslant s\leqslant t$, and $v^t(s,\cdot)=0$ otherwise. We call $\hat{X}$ the historical super-Brownian motion with branching rate k.

$(\hat{X}_t,\hat{P}_{\hat{\mu}})$ has finite moments of all orders. In particular, for $\hat{\mu}\in M_F(\hat{C})$ and $\hat{f}\in b\hat{\mathscr{C}}$,

$$\hat{P}_{\hat{\mu}}\langle\hat{X}_t,\hat{f}\rangle=\hat{\Pi}_{\hat{\mu}}\hat{f}(t,\hat{B}_t)\quad t\geqslant 0. \tag{21}$$

$$\hat{\mathrm{V}}\mathrm{ar}_{\hat{\mu}}\langle\hat{X}_t,\hat{f}\rangle=2\hat{\Pi}_{\hat{\mu}}\int_0^t \mathrm{d}r\hat{k}(r,\hat{B}_r)[\hat{\Pi}_{r,\hat{B}_r}\hat{f}(t,\hat{B}_t)]^2,\quad t\geqslant 0. \tag{22}$$

We will also use Dynkin's "stopped" historical superprocesses. For every stopping time τ of the Brownian motion, there corresponds a random measure in $M_F(\hat{C})$, which measures paths stopped at the moment τ. For example, if τ is the first exit time of Brownian motion from an open set D, we stop every path at the moment that the path first hitting the boundary of D. $\hat{X}_\tau$ measures all these stopped paths. More precisely, for $y\in C$, let

$$\tau(y)=\inf\{t\geqslant 0, y(t)\notin D\},$$

$$\hat{C}^\tau=\{(\tau(y),y^{\tau(y)});\ y\in C\}.$$

$\hat{X}_\tau$ takes value in $M_F(\hat{C}^\tau)$ and is defined by the following integration:

$$\hat{X}_\tau=\int\langle\mathrm{d}\eta_\tau,\hat{X}_t\rangle.$$

where $\eta(B)$ is defined by

$$\eta_\tau(B)=I_B((\tau,y^\tau))I_{(\tau<\infty)},\quad B\in\hat{\mathscr{C}}.$$

(See Section 1.9 in Dynkin, 1991b.) The Laplace functional is described by the following: for every $\hat{\mu}\in M_F(\hat{C})$, and $\hat{f}\in bp\hat{\mathscr{C}}$,

$$\hat{P}_{\hat{\mu}}\exp\langle\hat{X}_\tau,-\hat{f}\rangle=\exp\langle\hat{\mu},-v\rangle, \tag{23}$$

where v solves

$$v(s,y)+\hat{\Pi}_{s,y}I_{s\leqslant\tau}\int_s^\tau \hat{k}(r,\hat{B}_r)v^2(r,\hat{B}_r)\,\mathrm{d}r=\hat{\Pi}_{s,y}I_{s\leqslant\tau}\hat{f}(\tau,\hat{B}_\tau),\quad y\in C^s. \tag{24}$$

The following first two moment formulas hold:

$$\hat{P}_{\hat{\mu}}\langle \hat{X}_\tau, \hat{f}\rangle = \int \hat{\Pi}_{s,y} I_{s\leqslant\tau} \hat{f}(\tau, \hat{B}_\tau)\hat{\mu}(\mathrm{d}r, \mathrm{d}y), \tag{25}$$

$$\hat{\mathrm{V}}\mathrm{ar}_{\hat{\mu}}\langle \hat{X}_\tau, \hat{f}\rangle = 2\hat{\Pi}_{\hat{\mu}} \int \mathrm{d}r\hat{k}(r, \hat{B}_r) I_{r\leqslant\tau}[\hat{\Pi}_{r,\hat{B}_r} \hat{f}(\tau, \hat{B}_\tau)]^2, \tag{26}$$

where $\hat{\mu} \in M_{\mathrm{F}}(\hat{C})$, $\hat{f} \in b\hat{\mathscr{C}}$.

Similarly, since $t \wedge \tau$ is a stopping time, there corresponds a random measure $\hat{X}_{t\wedge\tau}$ defined by

$$\hat{X}_{t\wedge\tau} = \int \langle \mathrm{d}\eta_{t\wedge\tau}, \hat{X}_t\rangle$$

with $\eta_{t\wedge\tau}$ defined by

$$\eta_{t\wedge\tau}(B) = I_B((t \wedge \tau), y^{t\wedge\tau}) I(\tau < \infty), \quad B \in \hat{\mathscr{C}}.$$

The Laplace function of $\hat{X}_{t\wedge\tau}$ is given by (23) and (24) with τ replaced by $t \wedge \tau$. The following first two moment formulas hold:

$$\hat{P}_{\hat{\mu}}\langle \hat{X}_{t\wedge\tau}, \hat{f}\rangle = \int \hat{\Pi}_{s,y} I_{s\leqslant\tau} \hat{f}(t \wedge \tau, \hat{B}_{t\wedge\tau})\hat{\mu}(\mathrm{d}r, \mathrm{d}y), \tag{27}$$

$$\hat{\mathrm{V}}\mathrm{ar}_{\hat{\mu}}\langle \hat{X}_{t\wedge\tau}, \hat{f}\rangle = 2\hat{\Pi}_{\hat{\mu}} \int \mathrm{d}r\hat{k}(r, \hat{B}_r) I_{r\leqslant t\wedge\tau}[\hat{\Pi}_{r,\hat{B}_r} \hat{f}(t \wedge \tau, \hat{B}_{t\wedge\tau})]^2, \tag{28}$$

where $\hat{\mu} \in M_{\mathrm{F}}(\hat{C})$, $\hat{f} \in b\hat{\mathscr{C}}$.

Let l be the mapping from $\hat{C}$ to $\mathbb{R}^d$ defined by the formula

$$l((t, y^t)) = y^t(t), \quad y^t \in C^t. \tag{29}$$

For $\mu \in M_{\mathrm{F}}(\hat{C})$, define $l(\hat{\mu}) \in M_{\mathrm{F}}(\mathbb{R}^d)$ by

$$l(\hat{\mu})(A) = \hat{\mu}(l^{-1}(A)), \quad A \in \mathscr{B}(\mathbb{R}^d). \tag{30}$$

Put

$$X_t := l(\hat{X}_t),\ X_\tau := l(\hat{X}_\tau),\ X_{t\wedge\tau} := l(\hat{X}_{t\wedge\tau}), \tag{31}$$

i.e.,

$$X_t(\cdot) = \hat{X}_t((t, y^t); y \in C, y(t) \in \cdot),$$

$$X_\tau(\cdot) = \hat{X}_\tau((\tau, y^\tau); y \in C, y(\tau) \in \cdot),$$

$$X_{t\wedge\tau}(\cdot) = \hat{X}_{t\wedge\tau}((t \wedge \tau, y^{t\wedge\tau}); y \in C, y(t \wedge \tau) \in \cdot). \tag{32}$$

Note that $\hat{X}_t$, X_t, $\hat{X}_\tau$, X_τ, $\hat{X}_{t\wedge\tau}$ and $X_{t\wedge\tau}$ are all defined on the same sample space. We denote it by Ω. For $\Delta \subseteq \mathbb{R}^+$, define $\mathscr{F}(\Delta)$ as the σ-algebra in Ω generated by X_t, $t \in \Delta$. Let $\mathscr{F} = \mathscr{F}(\mathbb{R}^+)$. Define

$$P_{l(\hat{\mu})}(F) = \hat{P}_{\hat{\mu}}(F), \quad \hat{\mu} \in M_{\mathrm{F}}(\hat{C}),\ F \in \mathscr{F}. \tag{33}$$

$X = \{X_t, P_\mu, t \geqslant 0, \mu \in M_F(\mathbb{R}^d)\}$ is a time-homogeneous super-Brownian motion with states $X_t \in M_F(\mathbb{R}^d)$, $t \geqslant 0$, and the Laplace functional

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle \mu, -v(t,\cdot)\rangle \tag{34}$$

with $v \in M_F(\mathbb{R}^d)$, $f \in bp\mathscr{B}(\mathbb{R}^d)$, where v is the unique solution of the integral equation

$$v(t,x) + \Pi_x \int_0^t k(B_s)v^2(t-s, B_s)\,\mathrm{d}s = \Pi_x f(B_t). \tag{35}$$

In this paper, we assume the branching rate k is bounded and continuous. So, the above integral equation is equivalent to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \dfrac{1}{2}\Delta v(t,x) - k(x)v^2(x), \quad t \geqslant 0,\ x \in \mathbb{R}^d, \\ v(0,x) = f(x). \end{cases} \tag{36}$$

The first two moment formulas for X_t are given by the following:

$$P_\mu\langle X_t, f\rangle = \Pi_\mu f(B_t), \quad t \geqslant 0; \tag{37}$$

$$\mathrm{Var}_\mu\langle X_t, f\rangle = 2\Pi_\mu \int_0^t k(B_r)[\Pi_{B_r} f(B_{t-r})]^2, \quad t \geqslant 0, \tag{38}$$

where $f \in b\mathscr{B}(\mathbb{R}^d)$.

The Laplace functional of X_τ is given by

$$P_\mu \exp\langle X_\tau, -f\rangle = \exp\langle \mu, -v\rangle, \quad \mu \in M,\ f \in bp\mathscr{B}(\mathbb{R}^d), \tag{39}$$

where v is the unique solution of the integral equation

$$v(x) + \Pi_x \int_0^\tau k(B_s)v^2(B_s)\,\mathrm{d}s = \Pi_x f(B_\tau). \tag{40}$$

The Dirichlet problem equivalent to the above integral equation is

$$\begin{cases} \frac{1}{2}\Delta v(x) = kv^2(x),\ x \in D, \\ v|_{\partial D} = f. \end{cases} \tag{41}$$

The first two moment formulas for X_τ are given by the following:

$$P_\mu\langle X_\tau, f\rangle = \Pi_\mu f(B_\tau), \tag{42}$$

$$\mathrm{Var}_\mu\langle X_\tau, f\rangle = 2\Pi_\mu \int_0^\tau k(B_r)[\Pi_{B_r} f(B_\tau)]^2, \tag{43}$$

where $f \in b\mathscr{B}(\partial D)$.

The Laplace functional of $X_{t\wedge\tau}$ is given by (39) and (40) with τ replaced by $t\wedge\tau$. And the first two moment formulas for $X_{t\wedge\tau}$ are given by (42) and (43) with τ replaced by $t\wedge\tau$.

In the remainder of this paper, for simplicity, we use c to denote a positive constant depending at most on d, and use $c_{\cdot}$ to denote constant depending also on the variable "$\cdot$" in addition to d; the values of the constants may change from one line to another. Let $B(x,r)$ denote an open ball of radius r centered at $x\in\mathbb{R}^d$.

Before we prove the main results we establish three lemmas first. The first two of them are analytic results.

Lemma 2.1. *Consider the Cauchy problem*:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\Delta u(t,x), \quad 0\leqslant t\leqslant T,\ x\in\mathbb{R}^d, \\ u(0,x)=0. \end{cases} \tag{44}$$

where $T>0$ is a constant. There exists a solution $u(t,x)\neq 0$ to problem (44) such that, for every $\varepsilon>0$,

$$|u(t,x)|\leqslant c_{T,\varepsilon}\exp(c_{T,\varepsilon}|x|^{2+\varepsilon}), \quad 0\leqslant t\leqslant T. \tag{45}$$

Proof. For dimension $d=1$, nonzero solution $u(t,x)$ of (44) satisfying (45) was constructed at the end of Chapter 1 of Friedman (1964). For $d\geqslant 2$, let

$$u(t,x) := \sum_{i=1}^{d} u(t,x_i), \quad x=(x_1,x_2,\ldots,x_d)\in\mathbb{R}^d. \tag{46}$$

Then $u(t,x)$ is a solution of (44) and (45). □

The following Lemma 2.2 was proved in Dynkin (1991a) for general diffusion operator L. Since we use it so often, we include his proof for the particular case $L=\frac{1}{2}\Delta$ here.

Lemma 2.2. *Consider the problem*:

$$\begin{cases} \frac{1}{2}\Delta u(x)\leqslant Ku^2(x), \quad x\in B(x_0,R), \\ u|_{\partial B(x_0,R)}=\infty, \end{cases} \tag{47}$$

where $K>0$ is a constant. There exists a constant $c>0$ such that

$$u(x) := cK^{-1}R^2(R^2-r^2)^{-2}, \quad r=\|x-x_0\|. \tag{48}$$

is a solution of (47).

Proof. Put

$$\lambda = cK^{-1}R^2.$$

An easy calculation shows that

$$\begin{aligned}\frac{1}{2}\Delta u &= \frac{1}{2}\left(u''(r)+\frac{d-1}{r}u'(r)\right)\\ &= \lambda(R^2-r^2)^{-4}(2(R^2-r^2)+12r^2+2(d-1)(R^2-r^2))\\ &\leqslant \lambda(R^2-r^2)^{-4}(2R^2+10r^2+2(d-1)R^2)\\ &\leqslant \lambda(R^2-r^2)^{-4}(10+2d)R^2.\end{aligned}$$

So, $\frac{1}{2}\Delta u \leqslant Ku^2$ holds if $(10+2d)R^2 \leqslant K\lambda(=cR^2)$, which is true if we choose $c \geqslant 10+2d$. The boundary condition $u|_{\partial B(x_0,R)}=\infty$ is obvious. □

Lemma 2.3. *For d-dimensional Brownian motion B, we have for every* $b\in\mathbb{R}^+$

$$\begin{aligned}\Pi_0\left(\max_{0\leqslant s\leqslant t}\|B_s\|>b\right) &\leqslant \frac{4d}{\sqrt{2\pi}}\int_{b/d\sqrt{t}}^{\infty}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\\ &\sim \frac{4d^2\sqrt{t}}{\sqrt{2\pi}b}\exp\left(-\frac{b^2}{2d^2t}\right)\quad (as\ b\to\infty).\end{aligned}\tag{49}$$

($f\sim g$ *as* $x\to\infty$ *means* $\lim_{x\to\infty}(f(x)/g(x))=1$.)

Proof. For $d=1$,

$$\begin{aligned}\Pi_0\left(\max_{0\leqslant s\leqslant t}|B_s|>b\right) &\leqslant 2\Pi_0\left(\max_{0\leqslant s\leqslant t}B_s>b\right)\\ &= \frac{4}{\sqrt{2\pi}}\int_{b/\sqrt{t}}^{\infty}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\\ &\sim \frac{4\sqrt{t}}{\sqrt{2\pi}b}\exp\left(-\frac{b^2}{2t}\right)\quad (\text{as } b\to\infty).\end{aligned}$$

For $d\geqslant 2$, put $B_s=(B_s^1,\ldots,B_s^d)$. Then we have

$$\begin{aligned}\Pi_0\left(\max_{0\leqslant s\leqslant t}\|B_s\|>b\right) &\leqslant \Pi_0\left(\bigcup_{i=1}^{d}\left(\max_{0\leqslant s\leqslant t}|B_s^i|>\frac{b}{d}\right)\right)\\ &\leqslant d\Pi_0\left(\max_{0\leqslant s\leqslant t}|B_s^i|>\frac{b}{d}\right)\\ &\leqslant \frac{4d}{\sqrt{2\pi}}\int_{b/d\sqrt{t}}^{\infty}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\\ &\sim \frac{4d^2\sqrt{t}}{\sqrt{2\pi}b}\exp\left(-\frac{b^2}{2d^2t}\right)\quad (\text{as } b\to\infty).\end{aligned}$$

□

Proof of Theorem 1.1(1). Assume

$$k(x) \leqslant \exp(-\|x\|^{\beta}) \quad (\beta > 2), \tag{50}$$

for sufficiently large $x \in \mathbb{R}^d$. Let $h(t,x)$ be the solution to (44) and (45). Since $h(t,x) \neq 0$, there exist $\mu \in M_c(\mathbb{R}^d)$ such that $\langle h(t,\cdot), \mu\rangle \neq 0$. We will use this μ in the remainder of the proof.

First note that, for fixed t,

$$\begin{aligned} &P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) \\ &\quad= \lim_{n\to\infty} \hat{P}_{0,\mu}\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} l(\hat{X}_s) \subset B(0,n)\right) \\ &\quad\leqslant \lim_{n\to\infty} \hat{P}_{0,\mu}(\hat{X}_s(\hat{y}; l(\hat{y}) \in B^c(0,n)) = 0, \forall 0 \leqslant s \leqslant t), \end{aligned}$$

where $l(\cdot)$ is defined by (29) and (30). By Theorem 1 in Addendum to Dynkin (1991b),

$$\begin{aligned} &\hat{P}_{0,\mu}(\hat{X}_s(\hat{y}; l(\hat{y}) \in B^c(0,n)) = 0, \forall 0 \leqslant s \leqslant t) \\ &\quad\leqslant \hat{P}_{0,\mu}(\hat{X}_{t\wedge\tau_n}(\hat{y}; l(\hat{y}) \in B^c(0,n)) = 0) \\ &\quad= \hat{P}_{0,\mu}(\hat{X}_{t\wedge\tau_n}(\hat{y}; \tau_n(\hat{y}) \leqslant t) = 0), \end{aligned}$$

where τ_n is the first exit time from $B(0,n)$. Therefore,

$$P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) \leqslant \limsup_{n\to\infty} \hat{P}_{0,\mu}(\hat{X}_{t\wedge\tau_n}(\hat{y}; \tau_n(\hat{y}) \leqslant t) = 0). \tag{51}$$

If we can prove there exists $t > 0$ and $\mu \in M_c(\mathbb{R}^d)$ such that

$$\limsup_{n\to\infty} \hat{P}_{0,\mu}(\hat{X}_{t\wedge\tau_n}(\hat{y}; \tau_n(\hat{y}) \leqslant t) > 0) > 0, \tag{52}$$

then, by (51), $P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) < 1$, and therefore X does not possess the compact support property. Note that

$$\hat{P}_{0,\mu}(\hat{X}_{t\wedge\tau_n}(\hat{y}; \tau_n(\hat{y}) \leqslant t) > 0) \geqslant \hat{P}_{0,\mu}(|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle| > 0),$$

where we set

$$\hat{h}(t-s, y) = h(t-s, y(s)), \quad 0 \leqslant s \leqslant t, \ y \in C^s \tag{53}$$

with the $h(t,x)$ constructed in Lemma 2.1. To prove (52), we only need to prove

$$\limsup_{n\to\infty} \hat{P}_{0,\mu}(|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle| > 0) > 0. \tag{54}$$

To obtain (54), we fix $n \geqslant 1$ for the moment. We are going to use the elementary inequality

$$\hat{P}_{0,\mu}(|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle| > 0)$$
$$\geqslant \frac{(\hat{P}_{0,\mu}|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle|)^2}{\hat{P}_{0,\mu}(\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle)^2}. \tag{55}$$

By the expectation formula (27) and using the fact that $h(0,x) = 0$ we get

$$\hat{P}_{0,\mu}|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle|$$
$$\geqslant |\hat{P}_{0,\mu}\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle|$$
$$= |\hat{\Pi}_{0,\mu}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n}); \tau_n \leqslant t)|$$
$$= |\hat{\Pi}_{0,\mu}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n}))|$$
$$= |\Pi_{0,\mu}(h(t-t\wedge\tau_n, B_{t\wedge\tau_n}))|. \tag{56}$$

By the covariance formula (28) and using the fact that $h(0,x) = 0$ we get

$$\hat{P}_{0,\mu}(\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t), \hat{h}(t-\cdot,\cdot)\rangle)^2$$
$$= (\hat{\Pi}_{0,\mu}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n}); \tau_n \leqslant t))^2$$
$$+ 2\hat{\Pi}_{0,\mu}\int_0^{t\wedge\tau_n} \hat{k}(s,\hat{B}_s)(\hat{\Pi}_{s,\hat{B}_s}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n}); \tau_n \leqslant t))^2\,\mathrm{d}s$$
$$= (\hat{\Pi}_{0,\mu}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n})))^2$$
$$+ 2\hat{\Pi}_{0,\mu}\int_0^{t\wedge\tau_n} \hat{k}(s,\hat{B}_s)(\hat{\Pi}_{s,\hat{B}_s}(\hat{h}(t-t\wedge\tau_n, \hat{B}_{t\wedge\tau_n})))^2\,\mathrm{d}s$$
$$= (\Pi_{0,\mu}(h(t-t\wedge\tau_n, B_{t\wedge\tau_n})))^2$$
$$+ 2\Pi_{0,\mu}\int_0^{t\wedge\tau_n} k(B_s)(\Pi_{s,B_s}(h(t-t\wedge\tau_n, B_{t\wedge\tau_n})))^2\,\mathrm{d}s. \tag{57}$$

Since $h(t,x)$ is a solution to Cauchy Problem (44), $M_s = h(t-s,B_s)$ is a local martingale on $[r,t)(r<t)$ with respect to $\Pi_{r,\mu}$. Then we have

$$\Pi_{0,\mu}(h(t-t\wedge\tau_n, B_{t\wedge\tau_n})) = \langle h(t,\cdot), \mu\rangle. \tag{58}$$

and

$$\Pi_{0,\mu}\int_0^{t\wedge\tau_n} k(B_s)(\Pi_{s,B_s}(h(t-t\wedge\tau_n, B_{t\wedge\tau_n})))^2\,\mathrm{d}s$$
$$= \Pi_{0,\mu}\int_0^{t\wedge\tau_n} k(B_s)(h(t-s,B_s))^2\,\mathrm{d}s \tag{59}$$

Then (56)–(59) imply that

$$\hat{P}_{0,\mu}|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle| \geqslant |\langle \mu, h(t,\cdot)\rangle| \tag{60}$$

and

$$\begin{aligned}&\hat{P}_{0,\mu}(\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle)^2 \\ &\quad = (\langle \mu, h(t,\cdot)\rangle)^2 + 2\Pi_\mu \int_0^{t\wedge\tau_n} k(B_s)(h(t-s,B_s))^2 \,\mathrm{d}s.\end{aligned} \tag{61}$$

By (55), (60) and (61), we get

$$\begin{aligned}&\hat{P}_{0,\mu}(|\langle \hat{X}_{t\wedge\tau_n}(\cdot \cap (\tau_n(\hat{y}) \leqslant t)), \hat{h}(t-\cdot,\cdot)\rangle| > 0) \\ &\quad \geqslant \left(1 + 2\,\frac{\Pi_\mu \int_0^{t\wedge\tau_n} k(B_s)(h(t-s,B_s))^2 \,\mathrm{d}s}{(\langle h(t,\cdot),\mu\rangle)^2}\right)^{-1}.\end{aligned} \tag{62}$$

Assume for the moment that

$$\Pi_\mu \int_0^t k(B_s)(h(t-s,B_s))^2 \,\mathrm{d}s < \infty. \tag{63}$$

Letting $n \to \infty$ in (62) and noticing that $\langle h(t,\cdot),\mu\rangle \neq 0$, we get (54) holds.

It remains to verify (63). Using (45), we have, for any $\varepsilon > 0$,

$$\begin{aligned}&\Pi_\mu \int_0^t k(B_s)(h(t-s,B_s))^2 \,\mathrm{d}s \\ &\quad \leqslant c_{t,\varepsilon}\Pi_\mu \int_0^t k(B_s)\,(\exp(c_{t,\varepsilon}|B_s|^{2+\varepsilon}))^2 \,\mathrm{d}s \\ &\quad = c_{t,\varepsilon} \int \mu(\mathrm{d}x) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})P(s,x,y)\,\mathrm{d}y \\ &\quad = c_{t,\varepsilon} \int \mu(\mathrm{d}x) \int_{\mathbb{R}^d} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})g_t(x,y)\,\mathrm{d}y,\end{aligned} \tag{64}$$

where $P(s,x,y) = (2\pi s)^{-d/2}\exp(-|y-x|^2/2s)$ is the transition density of Brownian motion and

$$g_t(x,y) = \int_0^t P(s,x,y)\,\mathrm{d}s. \tag{65}$$

To prove (63), we only need to show that

$$\sup_{x\in\operatorname{supp}\mu} \int_{\mathbb{R}^d} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})g_t(x,y)\,\mathrm{d}y < \infty. \tag{66}$$

Let R_0 be a constant such that $\operatorname{supp}\mu \subset B(0,R_0)$. It is easy to see that $g_t(x,y) \leqslant c_t$, for every $x \in \operatorname{supp}\mu$ and every $y \in B^c(0,R_0+1)$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})g_t(x,y)\,\mathrm{d}y \\
&\quad\leqslant c_t \int_{\|y\|>R_0+1} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})\,\mathrm{d}y \\
&\qquad + \int_{\|y\|<R_0+1} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})g_t(x,y)\,\mathrm{d}y \\
&\quad\leqslant c_t \int_{\|y\|>R_0+1} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})\,\mathrm{d}y \\
&\qquad + c_{k,R_0}\int_{\|y\|<2R_0+1} g_t(y)\,\mathrm{d}y, \quad \forall x \in \operatorname{supp}\mu. \qquad (67)
\end{aligned}$$

It is easy to check that for fixed $t>0$,

$$\int_{\|y\|<2R_0+1} g_t(y)\,\mathrm{d}y < \infty. \qquad (68)$$

Assumption (50) implies

$$\int_{\mathbb{R}^d} k(y)\exp(2c_{t,\varepsilon}|y|^{2+\varepsilon})\,\mathrm{d}y < \infty \qquad (69)$$

for sufficiently small $\varepsilon>0$. Hence (66) follows from (67)–(69). We have now completed the proof of Theorem 1.1(1). □

Proof of Theorem 1.1(2). Recall that X_t, $\hat{X}_t$, X_τ and $\hat{X}_\tau$ are all defined on the same sample space Ω. The relations among them are given by (31) and (33). If we can prove there exists $t_0>0$ such that, for every $\mu \in M_c(\mathbb{R}^d)$, $t \in [0,t_0]$,

$$P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \text{ is bounded}\right) = 1, \qquad (70)$$

then by the Markov property, X possess the compact support property.

We first prove that

$$P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \subset \overline{B(0,n)}\right) \geqslant \hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y}) \leqslant t) = 0). \qquad (71)$$

For every closed set Γ, there exists a positive bounded continuous function f such that $\Gamma = (f=0)$. Let Q denote the set of rational numbers in $\mathbb{R}^1$. Since $\langle X_s, f\rangle$ is right

continuous in $s \in \mathbb{R}^+$ (see 1.2.A in Dynkin, 1991a),

$$\begin{aligned}\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \subset \Gamma\right) &= (\operatorname{supp} X_s \subset \Gamma,\ \forall s \in [0,t]) \\ &= (\langle X_s, f\rangle = 0,\ \forall s \in [0,t]) \\ &= (\langle X_s, f\rangle = 0,\ \forall s \in (Q \cap [0,t]) \cup \{t\}).\end{aligned}$$

Then we have

$$\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \subset \Gamma\right) = \left(\bigcup_{s\in(Q\cap[0,t])\cup\{t\}} \operatorname{supp} X_s \subset \Gamma\right). \tag{72}$$

Put

$$\tau_n(\hat{y}) = \inf\{t \geqslant 0;\ \|l(\hat{y})\| \geqslant n\}, \quad \hat{y} \in \hat{C}. \tag{73}$$

Recall the definition of $l(\cdot)$ from (29) and (30). By (72), for $\mu \in M_c(\mathbb{R}^d)$, we have

$$\begin{aligned}P_\mu\left(\bigcup_{0\leqslant s\leqslant t} \operatorname{supp} X_s \subset \overline{B(0,n)}\right) &= P_\mu\left(\bigcup_{s\in([0,t]\cap Q)\cup\{t\}} \operatorname{supp} X_s \subset \overline{B(0,n)}\right) \\ &= \hat{P}_{0,\mu}\left(\bigcup_{s\in([0,t]\cap Q)\cup\{t\}} \operatorname{supp} l(\hat{X}_s) \subset \overline{B(0,n)}\right) \\ &\geqslant \hat{P}_{0,\mu}(\hat{X}_s(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0, \\ &\quad \forall s \in ([0,t]\cap Q)\cup\{t\}).\end{aligned} \tag{74}$$

Use the special Markov property to see that

$$\begin{aligned}&\hat{P}_{0,\mu}((\hat{X}_s(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0) \cap (\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0)) \\ &\quad = \hat{P}_{0,\mu}(P_{\hat{X}_{\tau_n}(\cdot\cap(\tau_n\leqslant s))}(\hat{X}_s = 0);\ \hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0) \\ &\quad = \hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0), \quad \forall s > 0,\ \forall n \geqslant 1,\end{aligned}$$

which means that

$$\begin{aligned}&\hat{P}_{0,\mu}((\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0) \setminus (\hat{X}_s(\hat{y};\tau_n(\hat{y}) \leqslant s) = 0)) = 0, \\ &\quad \forall s > 0,\ \forall n \geqslant 1.\end{aligned} \tag{75}$$

Since for measurable sets A_n, B_n, $n \geqslant 1$ on a probability space $(\Omega, \mathscr{F}, P)$, $P(A_n \setminus B_n)=0$ implies

$$P\left(\left(\bigcap_{n=1}^{\infty} A_n\right) \setminus \left(\bigcap_{n=1}^{\infty} B_n\right)\right) = 0.$$

Therefore, (75) implies

$$\hat{P}_{0,\mu}\left(\bigcap_{s\in([0,t]\cap Q)\cup\{t\}}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y})\leqslant s)=0)\Big\backslash\bigcap_{s\in([0,t]\cap Q)\cup\{t\}}(\hat{X}_s(\hat{y};\tau_n(\hat{y})\leqslant s)=0)\right)=0.$$

Consequently we have

$$\begin{aligned}&\hat{P}_{0,\mu}(\hat{X}_s(\hat{y};\tau_n(\hat{y})\leqslant s)=0;\ \forall s\in([0,t]\cap Q)\cup\{t\})\\&\quad\geqslant\hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y})\leqslant s)=0;\ \forall s\in([0,t]\cap Q)\cup\{t\})\\&\quad\geqslant\hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y})\leqslant t)=0).\end{aligned}\tag{76}$$

Then (71) holds by (74) and (76).

Now we continue the domination of the right hand side of (71). It is obvious that

$$\hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_n(\hat{y})\leqslant t)=0)\geqslant\hat{P}_{0,\mu}(\hat{X}_{\tau_n}(\hat{y};\tau_{n-1}(\hat{y})\leqslant t)=0).\tag{77}$$

Using the special Markov property and relations (31) and (33) again, we see that the right hand side above equals

$$\hat{P}_{0,\mu}(\hat{P}_{\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))}(\hat{X}_{\tau_n}=0))=\hat{P}_{0,\mu}(P_{l[\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))]}(X_{\tau_n}=0)).\tag{78}$$

Using Theorem 1.2 in Dynkin (1991a), we see that the above equals

$$\begin{aligned}&\hat{P}_{0,\mu}(\exp\langle l[\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))],-v_n\rangle)\\&\quad\geqslant 1-\hat{P}_{0,\mu}\langle l[\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))],v_n\rangle.\end{aligned}\tag{79}$$

where $v_n(x)$ satisfies

$$\begin{cases}\frac{1}{2}\Delta v_n(x)=k(x)v_n^2(x), & x\in B(0,n),\\ v_n|_{\partial B(0,n)}=\infty.\end{cases}\tag{80}$$

Combining (71), and (77)–(79), we arrive at

$$P_\mu\left(\bigcup_{0\leqslant s\leqslant t}\operatorname{supp}X_s\subset\overline{B(0,n)}\right)\geqslant 1-\hat{P}_{0,\mu}\langle l[\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))],v_n\rangle.\tag{81}$$

Note that $l(\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}\leqslant t))$ has compact support contained in $\partial B(0,(n-1))$. So, to continue the domination in (81) we need to dominate v_n on $\partial B(0,(n-1))$. For every $x_0\in\partial B(0,(n-1))$ and $n\geqslant 1$, put

$$u_n^{x_0}(x)=c\exp(Mn^2)(1-r^2)^{-2}\tag{82}$$

with $r=\|x-x_0\|$ and a positive constant c depending only on d. By Lemma 2.1, $u_n^{x_0}(x)$ satisfies

$$\begin{cases} \frac{1}{2}\Delta u_n^{x_0}(x) \leqslant \exp(-Mn^2)(u_n^{x_0}(x))^2, & x\in B(x_0,1),\\ u_n^{x_0}|_{\partial B(x_0,1)} = \infty. \end{cases}$$

Since for $x\in B(0,n)$, we have $\exp(-Mn^2)\leqslant k(x)$, $u_n^{x_0}(x)$ satisfies

$$\begin{cases} \frac{1}{2}\Delta u_n^{x_0}(x) \leqslant k(x)(u_n^{x_0}(x))^2, & x\in B(x_0,1),\\ u_n^{x_0}|_{\partial B(x_0,1)} = \infty. \end{cases}$$

By the maximum principle,

$$v_n(x)\leqslant u_n^{x_0}(x), \quad x\in B(x_0,1).$$

In particular,

$$v_n(x_0)\leqslant u_n^{x_0}(x_0) = c\exp(Mn^2). \tag{83}$$

So, by (81) and (83), we get

$$\begin{aligned} &P_\mu\left(\bigcup_{0\leqslant s\leqslant t}\operatorname{supp} X_s\subset\overline{B(0,n)}\right)\\ &\quad\geqslant 1-c\exp(Mn^2)\hat{P}_{0,\mu}\langle l(\hat{X}_{\tau_{n-1}}(\cdot\cap(\tau_{n-1}(\hat{y})\leqslant t))),1\rangle\\ &\quad= 1-c\exp(Mn^2)\hat{P}_{0,\mu}(\hat{X}_{\tau_{n-1}}((\tau_{n-1}(\hat{y})\leqslant t))). \end{aligned} \tag{84}$$

Let R_0 be a constant such that $\operatorname{supp}\mu\subset B(0,R_0)$. Now from the expectation formula (25) with $\hat{f}(s,y^s)=I_{[0,t]}(s)$, we see that the above equals

$$\begin{aligned} &1-c\exp(Mn^2)\int \Pi_x(\tau_{n-1}\leqslant t)\mu(\mathrm{d}x)\\ &\quad\geqslant 1-c\exp(Mn^2)\int \Pi_0(\tau_{n-1-R_0}\leqslant t)\mu(\mathrm{d}x)\\ &\quad= 1-\mu(\mathbb{R}^d)c\exp(Mn^2)\Pi_0(\tau_{n-1-R_0}\leqslant t). \end{aligned} \tag{85}$$

By (84) and (85), using Lemma 2.2, we have, for every $n>0$,

$$\begin{aligned} &P_\mu\left(\bigcup_{0\leqslant s\leqslant t}\operatorname{supp} X_s\subset\overline{B(0,n)}\right)\\ &\quad\geqslant 1-\mu(\mathbb{R}^d)c\exp(Mn^2)\frac{\sqrt{t}}{n-1-R_0}\exp\left(-\frac{(n-1-R_0)^2}{2d^2t}\right). \end{aligned} \tag{86}$$

Put $t_0=1/4Md^2$. Letting $n\to\infty$, we get (70) holds for $t\in[0,t_0]$. The proof is now complete. □

3. Proof of Theorem 1.2

We first state a lemma, which is easy to check by the special Markov property of X.

Lemma 3.1. *Suppose that D_n is a sequence of bounded domains increasing to a bounded domain D, and suppose that $X=(X_t,P_\mu)$ is a super-Brownian motion with branching rate $k(x)$. Let $\mu\in M_F(\mathbb{R}^d)$ be such that $\operatorname{supp}\mu\subset\bar{D}_n$ for every $n\geqslant 1$ and let τ_{D_n} denote the first exit time of D_n. If h is a non-negative harmonic function on $D\subset\mathbb{R}^d$ and continuous on $\bar{D}$, then $\langle X_{\tau_{D_n}},h\rangle$ is a martingale under P_μ.*

Theorem 3.1. *Let $X=(X_t,P_\mu)$ denote the super-Brownian motion with branching rate $k(x)\geqslant M(\|x\|^{-2}\wedge 1)$ starting at $\mu\in M_c(\mathbb{R}^d)$, where $M>0$ is a constant. Then the global support of X is compact.*

Proof. Let τ_n be the first exit time of Brownian motion from the ball $B(0,2^n)$, and let X_{τ_n} denote stopped measure associated to τ_n. Then

$$(\text{Gsupp}\,(X)\ \text{is bounded})=\bigcup_{n=1}^{\infty}(X_{\tau_n}=0). \tag{87}$$

It suffices to show that

$$P_\mu\left(\bigcup_{n=1}^{\infty}(X_{\tau_n}=0)\right)=1. \tag{88}$$

For large n (such that $\operatorname{supp}\mu\subset\subset B(0,2^n)$), by the special Markov property

$$P_\mu(X_{\tau_n}=0)=P_\mu(P_{X_{\tau_{n-1}}}(X_{\tau_n}=0)).$$

Using Theorem 1.1 in Dynkin (1991a), we get

$$P_\mu(X_{\tau_n}=0)=P_\mu(\exp\langle X_{\tau_{n-1}},-v_n\rangle). \tag{89}$$

where $v_n(x)$ satisfies

$$\begin{cases}\frac{1}{2}\Delta v_n(x)=k(x)v_n^2(x) & \text{in } B(0,2^n),\\ v_n|_{\partial B(0,2^n)}=\infty.\end{cases} \tag{90}$$

We claim that

$$v_n\leqslant c_M \quad \text{on } \partial B(0,2^{n-1}). \tag{91}$$

By (91) and noticing that $\operatorname{supp}X_{\tau_{n-1}}\subset\partial B(0,2^{n-1})$, we obtain

$$\begin{aligned}P_\mu(X_{\tau_n}=0)&\geqslant P_\mu(\exp\langle X_{\tau_{n-1}},-c_M\rangle)\\ &=P_\mu(X_{\tau_{n-1}}=0)+P_\mu(\exp\langle X_{\tau_{n-1}},-c_M\rangle;\ X_{\tau_{n-1}}>0).\end{aligned} \tag{92}$$

Since $(X_{\tau_n}=0)\uparrow\bigcup_{n=1}^{\infty}(X_{\tau_n}=0)$, letting $n\uparrow\infty$ in the above inequality, we get

$$\lim_{n\to\infty}P_\mu(\exp\langle X_{\tau_{n-1}},-c_M\rangle;\ X_{\tau_{n-1}}>0)=0. \tag{93}$$

By Lemma 3.1, $\{\langle X_{\tau_n},1\rangle\}$ is a martingale under P_μ with mean value $\mu(\mathbb{R}^d)$. The martingale convergence theorem implies that $\lim_{n\to\infty}\langle X_{\tau_n},1\rangle$ exists and is finite a.s. P_μ. Hence, exchanging the order of the limit and expectation in (93), we get

$$P_\mu\left(\exp\Big(-\lim_{n\to\infty}\langle X_{\tau_{n-1}},c_M\rangle\Big);\ \bigcap_{n=1}^{\infty}(X_{\tau_{n-1}}>0)\right)=0, \tag{94}$$

Since $P_\mu(\exp(-\lim_{n\to\infty}\langle X_{\tau_{n-1}},c_M\rangle)>0))=1$, (94) implies $P_\mu\left(\bigcap_{n=1}^{\infty}(X_{\tau_{n-1}}>0)\right)=0$, which is equivalent to (88).

It remains to verify (91) for fixed n. For every $x_0\in\partial B(0,2^{n-1})$, put

$$u_n^{x_0}(x)=c(M2^{-2n})^{-1}2^{2(n-1)}(2^{2(n-1)}-r^2)^{-2},\quad x\in B(x_0,2^{n-1}) \tag{95}$$

with $r=\|x-x_0\|$. Using Lemma 2.1 with $R=2^{n-1}$ and $K=M2^{-2n}$, $u_n^{x_0}(x)$ satisfies

$$\begin{cases}\frac{1}{2}\Delta u_n^{x_0}(x)\leqslant M2^{-2n}(u_n^{x_0}(x))^2 & \text{in } B(x_0,2^{n-1}),\\ u_n^{x_0}|_{\partial B(x_0,2^{n-1})}=\infty.\end{cases}$$

Since $M2^{-2n}\leqslant k(x)$ for $x\in B(x_0,2^{n-1})$, $u_n^{x_0}(x)$ satisfies

$$\begin{cases}\frac{1}{2}\Delta u_n^{x_0}(x)\leqslant k(x)(u_n^{x_0}(x))^2 & \text{in } B(x_0,2^{n-1}),\\ u_n^{x_0}|_{\partial B(x_0,2^{n-1})}=\infty.\end{cases}$$

Then, by the maximum principle, $v_n(x)\leqslant u_n^{x_0}(x)$ in $B(x,2^{n-1})$. In particular,

$$v_n(x_0)\leqslant u_n^{x_0}(x_0)\leqslant c_M$$

with constant $c_M>0$. Claim (91) follows.

Theorem 3.2. *If $d=1$, the result in Theorem 3.1 holds under the following weaker condition: $k(x)\geqslant M(\|x\|^{-3}\wedge 1)$ for some constant $M>0$.*

Proof. We divided the proof into three steps.

Step 1: Let $X_{\tau_{(0,n)}}$ denote the stopped measure associated with the first exit time $\tau_{(0,n)}$ of $(0,n)$. Then for every $x>0$ and $c>0$,

$$P_{c\delta_x}\left(\bigcup_{n=1}^{\infty}(X_{\tau_{(0,n)}}(\{n\})=0)\right)=1. \tag{96}$$

Similarly, for $x<0$ and $c>0$,

$$P_{c\delta_x}\left(\bigcup_{n=1}^{\infty}(X_{\tau_{(0,-n)}}(\{-n\})=0)\right)=1. \tag{97}$$

Intuitively, (96) means that, if we stop every particle's branching and migration at the moment when they reach point 0, the set of locations ever occupied by particles is bounded. Then, using the special Markov property, we will be able to obtain that the global support of X is compact (see Steps 2 and 3). So, the main reason that for $d=1$, the global support is compact under a weaker condition on k is that Brownian motion is point recurrent in dimension one.

We only prove (96). The proof of (97) is similar. Eq. (96) is equivalent to

$$P_{c\delta_x}\left(\bigcup_{n=1}^{\infty}(X_{\tau_{(0,2^n)}}(\{2^n\})=0)\right)=1. \tag{98}$$

By the special Markov property,

$$\begin{aligned}P_{c\delta_x}(X_{\tau_{(0,2^n)}}(\{2^n\})=0)&=P_{c\delta_x}(P_{X_{(0,2^{n-1})}}(X_{\tau_{(0,2^n)}}(\{2^n\})=0))\\&=P_{c\delta_x}\exp\langle X_{\tau_{(0,2^{n-1})}},-v_n\rangle=P_{c\delta_x}\exp(-X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\})v_n(2^{n-1})),\end{aligned} \tag{99}$$

where v_n is the minimal solution of

$$\begin{cases}\frac{1}{2}\Delta v_n(x)=k(x)v_n^2(x), & x\in(0,2^n),\\ v_n(0)=0, \quad v_n(2^n)=\infty.\end{cases}$$

Put

$$u_n(x)=c\,(M2^{-3n})^{-1}2^{2(n-1)}(2^{2(n-1)}-r^2)^{-2}$$

with $r=|x-2^{n-1}|$. Use Lemma 2.1 with $R=2^{n-1}$, $K=M2^{-3n}$ and $x_0=2^{n-1}$, and the maximum principle to see that

$$v_n(x)\leqslant u_n(x), \quad x\in(0,2^n).$$

In particular,

$$v_n(2^{n-1})\leqslant u_n(2^{n-1})\leqslant c_M 2^{n-1}. \tag{100}$$

Then by (99) and (100),

$$\begin{aligned}&P_{c\delta_x}(X_{\tau_{(0,2^n)}}(\{2^n\})=0)\\&\quad\geqslant P_{c\delta x}\exp(-c_M2^{n-1}X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\}))\\&\quad=P_{c\delta x}(X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\})=0)\\&\qquad+P_{c\delta x}\exp(-c_M2^{n-1}X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\});\ X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\})>0).\end{aligned}$$

Letting $n\to\infty$ in the above inequality, we obtain

$$\lim_{n\to\infty}P_{c\delta_x}\exp(-c_M2^{n-1}X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\});\ X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\})>0)=0. \tag{101}$$

Using Lemma 3.1 with $h(x)=x$, $\{nX_{\tau_{(0,n)}}(\{n\}), n\geqslant 1\}$ is a martingale under $P_{c\delta_x}$ with mean value cx. The martingale convergence theorem implies that, a.s. $P_{c\delta_x}$,

$$\lim_{n\to\infty} 2^{n-1}X_{\tau_{(0,2^{n-1})}}(\{2^{n-1}\})<\infty. \tag{102}$$

Exchanging the order of limit and expectation in (101), and by noticing (102), we get (98).

Step 2: Let $X_{\tau_{\{0\}}}$ denote the stopped measure associated to $\tau_{\{0\}}$, the first hitting time of point 0.

(1) For $\mu\in M_F(\mathbb{R}^1)$ with $\operatorname{supp}\mu\subset(0,\infty)$, both limits $\lim_{n\to\infty}\langle X_{\tau_{(0,n)}},1\rangle$ and $\lim_{n\to\infty}X_{\tau_{(0,n)}}(\{0\})$ exist and are finite P_μ-a.s. Moreover,

$$\lim_{n\to\infty}\langle X_{\tau_{(0,n)}},1\rangle \overset{\text{a.s. }P_\mu}{=} \lim_{n\to\infty}X_{\tau_{(0,n)}}(\{0\}) \overset{\mathrm{d}}{=} X_{\tau_{\{0\}}}. \tag{103}$$

(2) For $\mu\in M_F(\mathbb{R}^1)$ with $\operatorname{supp}\mu\subset(-\infty,0)$, both limits $\lim_{n\to\infty}\langle X_{\tau_{(-n,0)}},1\rangle$ and $\lim_{n\to\infty}X_{\tau_{(-n,0)}}(\{0\})$ exist and are finite P_μ-a.s. Moreover,

$$\lim_{n\to\infty}\langle X_{\tau_{(-n,0)}},1\rangle \overset{\text{a.s. }P_\mu}{=} \lim_{n\to\infty}X_{\tau_{(-n,0)}}(\{0\}) \overset{\mathrm{d}}{=} X_{\tau_{\{0\}}}. \tag{104}$$

We are only going to prove (1). The proof of (2) is similar. By Step 1,

$$\lim_{n\to\infty}X_{\tau_{(0,n)}}(\{n\})=0, \quad \text{a.s. } P_\mu$$

Then

$$\lim_{n\to\infty}\langle X_{\tau_{(0,n)}},1\rangle=\lim_{n\to\infty}X_{\tau_{(0,n)}}(\{0\}), \quad \text{a.s.} P_\mu$$

which is the first part of (103). We want to get the Laplace functional of the above limit. We start with the Laplace function of $X_{\tau_{(0,n)}}(\{0\})$:

$$P_\mu\exp(-\lambda X_{\tau_{(0,n)}}(\{0\}))=\exp\langle\mu,-v_{\lambda,n}\rangle \tag{105}$$

with $\lambda\geqslant 0$, where $v_{\lambda,n}$ is the unique solution of integral equation:

$$v_{\lambda,n}(x)+\Pi_x\int_0^{\tau_{(0,n)}}k(B_s)v_{\lambda,n}^2(B_s)\,\mathrm{d}s=\lambda\Pi_x(\tau_{\{0\}}=\tau_{(0,n)}), \tag{106}$$

and $v_{\lambda,n}$ is also the minimal solution of

$$\begin{cases}\frac{1}{2}v''(x)=k(x)v^2(x), & x\in(0,n),\\ v(0)=\lambda, \quad v(0)=0.\end{cases} \tag{107}$$

$v_{\lambda,n}$ is non-decreasing in n by the maximum principle. Set

$$v_{\lambda,n}\uparrow v_{\lambda,\infty}.$$

Letting $n\to\infty$ in (105) and (106), we get the Laplace functional of $\lim_{n\to\infty}X_{\tau_{(0,n)}}(\{0\})$:

$$P_\mu\exp\Big(-\lambda\lim_{n\to\infty}X_{\tau_{(0,n)}}(\{0\})\Big)=\exp\langle\mu,-v_{\lambda,\infty}\rangle, \tag{108}$$

with $v_{\lambda,\infty}$ satisfying;

$$v_{\lambda,\infty}(x) + \Pi_x \int_0^{\tau_{\{0\}}} k(B_s) v^2_{\lambda,\infty}(B_s)\,\mathrm{d}s = \lambda. \tag{109}$$

The Laplace functional of $\lim_{n\to\infty} X_{\tau_{(0,n)}}(\{0\})$ is exactly that of $X_{\tau_{\{0\}}}$. This completes the proof of (1).

Step 3: Set

$$A = (\mathrm{Gsupp}\,(X) \text{ is bounded}).$$

Then $(X_{\tau_{(-n,n)}} = 0) \uparrow A$. By Lemma 1.1,

$$P_\mu(A) = \exp\langle -\mu, v\rangle \tag{110}$$

with v being the maximum non-negative solution of $\frac{1}{2}v'' = kv^2$ in $\mathbb{R}^1$. To prove $P_\mu(A)=1$ is equivalent to prove $v \equiv 0$, which is equivalent to $v(0) = 0$. To this end, for every $n > x > 0$, we use the special Markov property to get

$$P_{\delta_x}(A) = P_{\delta_x}(P_{X_{\tau_{(0,n)}}}(A)) = P_{\delta_x} \exp[-v(0)X_{\tau_{(0,n)}}(\{0\}) - v(n)X_{\tau_{(0,n)}}(\{n\})].$$

In the second equality we used (110). Letting $n \to \infty$, using the results of Steps 1 and 2, we have

$$P_{\delta_x}(A) = P_{\delta_x} \exp[-v(0)X_{\tau_{\{0\}}}(\{0\})] = \exp(-u_{v(0)}(x)). \tag{111}$$

where $u_{v(0)}$ is the unique solution of the integral equation:

$$u_{v(0)}(x) + \Pi_x \int_0^{\tau_{\{0\}}} k(B_s) u^2_{v(0)}(B_s)\,\mathrm{d}s = v(0). \tag{112}$$

By (110) we have

$$P_{\delta_x}(A) = \exp(-v(x)). \tag{113}$$

Comparing (111) and (113), we get

$$v(x) = u_{v(0)}(x), \quad x \in (0,\infty).$$

Then by (112), $v(x) \leqslant v(0)$ for all $x \in (0,\infty)$. Similarly, we have $v(x) \leqslant v(0)$ in $(-\infty, 0)$. Therefore, 0 is the maximum point of v in $(-\infty,\infty)$, which means $v''(0)=0$. Since $\frac{1}{2}v'' = kv^2$ and $k > 0$ in $\mathbb{R}^d$, we obtain $v(0) = 0$. □

Corollary 3.1. *If there exists constant $M > 0$ such that $k(x) \geqslant M\|x\|^{-3}$ for large x, and $k(x) > 0$ in $\mathbb{R}^1$, then Eq. (6) has no positive solution.*

Corollary 3.2. *Suppose $k(x) > 0$ is a radial function in $\mathbb{R}^2$. If there exists a constant $M > 0$ such that $k(r) \geqslant Mr^{-2}(\ln r)^{-3}$ for large r, then (6) has no positive symmetric solution in $\mathbb{R}^2$, where $r = \|x\|$.*

Proof. Assume for the moment that $u(x)=u(r)$ is a positive symmetric solution to (6). Put $u(r)=\bar{u}(t)$ with $t=\ln r$, then $\frac{1}{2}\bar{u}''(t)=e^{2t}k(e^t)\bar{u}^2(t)$ for $t\in\mathbb{R}^1$, which means the equation $\frac{1}{2}u''(t)=e^{2|t|}k(e^{|t|})u^2(t)$ has a positive solution in $\mathbb{R}^1$. But, by assumption, $e^{2|t|}k(e^{|t|})\geqslant M|t|^{-3}$ for large t. Corollary 3.1 implies that $\frac{1}{2}u''(t)=e^{2|t|}k(e^{|t|})u^2(t)$ has no positive solution, and we got a contradiction. □

Proof of Theorem 1.2(1). We have proved the result for $d=1$ and $\geqslant 3$ in Theorem 3.1, Theorem 3.2. We only need to prove that for $d=2$ if $k\geqslant M\|x\|^{-2}(\ln\|x\|)^{-3}$ for large $\|x\|$, then the global support of X is compact. By Corollary 3.2, it is true for radial k. But for general k, there exists a radial $\bar{k}$ and a constant c such that $k(x)\geqslant c\bar{k}(x)$, $x\in\mathbb{R}^2$. For a super-Brownian motion $\bar{X}$ with branching rate $c\bar{k}$, the global support is compact. So, the global support of X with bigger branching rate is compact. □

Proof of Theorem 1.2(2). Let τ_n denote the first exit time of $B(0,n)$. Since

$$(\text{Gsupp}\,(X)\ \text{is bounded})=\bigcup_{n=1}^{\infty}(X_{\tau_n}=0).$$

It suffices to prove

$$\lim_{n\to\infty}P_\mu(X_{\tau_n}>0)>0. \tag{114}$$

For $d=1$, (7) is equivalent to one of the following conditions:

$$\Pi_x\int_0^{\tau_{(0,\infty)}}k(B_s)h^2(B_s)\,\mathrm{d}s<\infty\quad\text{for } x>0, \tag{115}$$

$$\Pi_x\int_0^{\tau_{(-\infty,0)}}k(B_s)h^2(B_s)\,\mathrm{d}s<\infty\quad\text{for } x<0. \tag{116}$$

Suppose condition (115) holds. The other case can be proved similarly. If we can prove (114) for $\mu=c\delta_x$ for every $c>0$ and $x>0$, then by the special Markov property,

$$\begin{aligned}\lim_{n\to\infty}P_\mu(X_{\tau_{(-n,n)}}>0)&=\lim_{n\to\infty}P_\mu(P_{X_{\tau_{(-x,x)}}}(X_{\tau_{(-n,n)}}>0))\\&\geqslant\lim_{n\to\infty}P_\mu(P_{X_{\tau_{(-x,x)}}(\{x\})\delta_x}(X_{\tau_{(-n,n)}}>0))\\&=P_\mu\left(\lim_{n\to\infty}P_{X_{\tau_{(-x,x)}}(\{x\})\delta_x}(X_{\tau_{(-n,n)}}>0)\right)>0.\end{aligned}$$

In the last equality above, we used the dominated convergence theorem.

It remains to prove (114) for $\mu=c\delta_x$ with $c>0$ and $x>0$. It is obvious that

$$P_{c\delta_x}(X_{\tau_{(-n,n)}}>0)\geqslant P_{c\delta_x}(X_{\tau_{(0,n)}}(\{n\})>0).$$

So, we only need to prove

$$\limsup_{n\to\infty}P_{c\delta_x}(X_{\tau_{(0,n)}}(\{n\})>0)>0 \tag{117}$$

for $c > 0$ and $x > 0$. Recall that h is defined by (11). The expectation formula (42) and the covariance formula (43) give that

$$\begin{aligned} P_{c\delta_x}(X_{\tau_{(0,n)}}(\{n\}) > 0) &= P_{c\delta_x}(\langle X_{\tau_{(0,n)}}, h\rangle > 0) \\ &\geqslant \frac{[P_{c\delta_x}\langle X_{\tau_{(0,n)}}, h\rangle]^2}{P_{c\delta_x}\langle X_{\tau_{(0,n)}}, h\rangle^2} \\ &= \frac{[c\Pi_x h(B_{\tau_{(0,n)}})]^2}{[c\Pi_x h(B_{\tau_{(0,n)}})]^2 + 2c\Pi_x \int_0^{\tau_{(0,n)}} k(B_s)h^2(B_s)\,\mathrm{d}s}. \end{aligned}$$

Note that $h(x) = x$ is a harmonic function in $\mathbb{R}^1$, $\Pi_x h(B_{\tau_{(0,n)}}) = h(x)$. The above implies

$$P_{c\delta_x}(X_{\tau_{(0,n)}}(\{n\}) > 0) \geqslant \frac{c^2 h(x)^2}{c^2 h^2(x) + 2c\Pi_x \int_0^{\tau_{(0,n)}} k(B_s)h^2(B_s)\,\mathrm{d}s}.$$

Letting $n \to \infty$, using condition (115), we obtain (117).

For $d = 2$, condition (7) is equivalent to

$$\Pi_x \int_0^{\tau_{B^c(0,1)}} k(B_s)h^2(B_s)\,\mathrm{d}s < \infty \quad \text{for } x \in B^c(0,1). \tag{118}$$

Let $\tau_{1,n}$ be the first exit time of $B(0,n) \setminus B(0,1)$. Suppose $\operatorname{supp}\mu \subset B(0,N)$. Then for $n > N$, by the special Markov property,

$$\begin{aligned} P_\mu(X_{\tau_n} > 0) &= P_\mu[P_{X_{\tau_N}}(X_{\tau_n} > 0);\ X_{\tau_N} > 0] \\ &\geqslant P_\mu[P_{X_{\tau_N}}(X_{\tau_{1,n}}(\partial B(0,n)) > 0);\ X_{\tau_N} > 0]. \end{aligned}$$

Note that $h(x) = \ln\|x\|$ is harmonic in $B^c(0,1)$ having boundary value 0 at $\partial B(0,1)$. Using the expectation formula (42) and the covariance formula (43), the above estimate can be continued as follows:

$$\begin{aligned} &= P_\mu[P_{X_{\tau_N}}(\langle X_{\tau_{1,n}}, h\rangle > 0);\ X_{\tau_N} > 0] \\ &\geqslant P_\mu\left[\frac{(P_{X_{\tau_N}}\langle X_{\tau_{1,n}}, h\rangle)^2}{P_{X_{\tau_N}}\langle X_{\tau_{1,n}}, h\rangle^2};\ X_{\tau_N} > 0\right] \\ &= P_\mu\left[\frac{\langle X_{\tau_N}, h\rangle^2}{\langle X_{\tau_N}, h\rangle^2 + 2\left\langle X_{\tau_N}, \Pi_\cdot \int_0^{\tau_{1,n}} k(B_s)h^2(B_s)ds\right\rangle};\ X_{\tau_N} > 0\right]. \end{aligned}$$

We know that for fixed N, $P_\mu(X_{\tau_N} > 0) > 0$. By hypothesis (118) and the Fatou's Lemma, we get (114).

For $d \geqslant 3$, condition (7) is equivalent to

$$\Pi_x \int_0^{\infty} k(B_s)h^2(B_s)\,\mathrm{d}s < \infty \quad \text{in } \mathbb{R}^d. \tag{119}$$

The proof for $d \geqslant 3$ is easy. Recall that τ_n denotes the first exit time of $B(0,n)$. For large n (such that $\operatorname{supp}\mu \subset B(0,n)$),

$$P_\mu(X_{\tau_n} > 0) = P_\mu(\langle X_{\tau_n}, 1\rangle > 0) \geqslant \frac{(\mu(\mathbb{R}^d))^2}{(\mu(\mathbb{R}^d))^2 + 2\left\langle \mu, \Pi_\cdot \int_0^{\tau_n} k(B_s)ds \right\rangle}.$$

Letting $n \to \infty$, using hypothesis (119), we get (114). □

Acknowledgements

This work was completed while the author was visiting the Department of Mathematics at the University of British Columbia. She thanks Edwin Perkins for his valuable discussions and careful reading of the manuscript. The author also thanks the referee for very helpful comments on the first version of this paper and for informing her the recent preprint, Engländer and Pinsky (2003), in which a result similar to Theorem 1.1 of this paper for more general parabolic equations of the form $u_t = Lu + Vu - \gamma u^p$ is proved by using purely analytic techniques. The proof of Theorem 1.1 in this paper is probabilistic and is independent of the proof in Engländer and Pinsky (2003).

References

Cheng, K.-S., Ni, W.-N., 1992. On the structure of the conformal scalar curvature equation on $\mathbb{R}^n$. Indiana Univ. Math. J. 41, 261–278.

Dawson, D.A., Perkins, E., 1991. Historical processes. Mem. Amer. Math. Soc. 93 (454), 1–179.

Dawson, D.A., Fleischmann, K., Mueller, C., 2000. Finite time extinction of super-Brownian motions with catalysts. Ann. Probab. 28, 603–642.

Dynkin, E.B., 1991a. A probabilistic approach to one class of nonlinear differential equations. Probab. Theory Related Fields 89 (1), 89–115.

Dynkin, E.B., 1991b. Path processes and historical superprocesses. Probab. Theory Related Fields 90, 1–36.

Engländer, J., 2000. Criteria for the existence of positive solutions to the equation $\rho(x)\Delta u = u^2$ in $\mathbb{R}^d$ for all $d \geqslant 1$: a new probabilistic approach. Positivity 4, 327–337.

Engländer, J., Fleischmann, K., 2000. Extinction properties of super-Brownian motions with additional spatially dependent mass production. Stochastic Process. Appl. 88, 37–58.

Engländer, J., Pinsky, R.G., 1999. On the construction and support properties of measure-valued diffusions on $D \subset R^d$ with spatially dependent branching. Ann. Probab. 27, 684–730.

Engländer, J., Pinsky, R.G., 2003. Uniqueness/nonuniqueness for nonnegative solutions of second order parabolic equations of the form $u_t = Lu + Vu - \gamma u^p$ in R^n. J. Differ. Equations 192, 396–428.

Friedman, A., 1964. Partial Differential Equations of Parabolic Type. Prentice-Hall, Englewood Cliffs, NJ.

Perkins, E., 2002. Dawson–Watanabe superprocesses and measure-valued diffusions. Lecture Notes in Mathematics, Vol. 1781. Springer-Verlag, Heidelberg, pp. 135–192.

Ren, Y.X., 2003. The structure of nonlinear elliptic equations on unbounded domains in dimensions 1 and 2-A probabilistic approach. Comput. Math. Appl. 45, 771–782.

Sheu, Y.-C., 1995. On positive solutions of some nonlinear differential equations—A probabilistic approach. Stochastic Process. Appl. 59, 43–53.