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The Structure of Nonlinear Elliptic Equations on Unbounded Domains in Dimensions 1 and 2 —A Probabilistic Approach

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Abstract—Suppose that D is an unbounded domain in $\mathbf{R}^2$ with a compact boundary ∂D and $k(x)$ is a strictly positive Hölder continuous function on D such that

$$\int_{\|x\|\geq a} (\log(\|x\|))^{\alpha}\, k(x)\, dx < \infty,$$

for some constant $a > 0$. In this paper, we study the nonlinear elliptic equation $(1/2)\Delta u = k(x)u^{\alpha}(x)$ on D, where $\alpha \in (1, 2]$ is a constant. First, we give explicit expressions in terms of super-Brownian motions for positive solutions of the above equation with the boundary conditions: $u|_{\partial D} = 0$ and $\lim_{\|x\|\to\infty}(u(x)/\log(\|x\|)) = c$ $(0 < c \leq \infty)$. Then we give a complete classification of all positive solutions of the above equation with the boundary condition $u|_{\partial D} = 0$ when k behaves like $\|x\|^{-2}(\log(\|x\|))^{-l}$ near ∞ for some constant $l > 1 + \alpha$. In the one-dimensional case, we also have similar results. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND MAIN RESULTS

Suppose that $k(x)$ is a bounded strictly positive continuous function on $\mathbf{R}^d$ and $1 < \alpha \leq 2$ is a constant. It is well known that the following nonlinear elliptic equation

$$\frac{1}{2}\Delta u = k(x)u^{\alpha}(x), \qquad x \in \mathbf{R}^d, \tag{1.1}$$

is closely connected with super-Brownian motion. In this paper, we are going to study the equation above by using this connection. We first recall the super-Brownian motion that we are going to use.

Let $W := \{W_s, \Pi_x, s \geq 0,\ x \in \mathbf{R}^d\}$ denote a Brownian motion started at $x \in \mathbf{R}^d$. Let $\mathcal{B}$ be the Borel σ-field on $\mathbf{R}^d$, M be the collection of all finite measures on $\mathcal{B}$, and let $\mathcal{T}$ be the collection of

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exit times by the Brownian motion W from open sets in $\mathbf{R}^d$. In this paper, we use the expression $\langle f, \mu\rangle$, for the integral of f with respect to μ. According to Dynkin [1], there exists a Markov process $X = (X_t, P_\mu)$ with state space M such that the following conditions are satisfied.

(a) If f is a bounded continuous function, then the function $t \mapsto \langle f, X_t\rangle$ is right continuous on $\mathbf{R}^+$.

(b) For every $\mu \in M$ and for every bounded positive $f \in \mathcal{B}$,

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v_t, \mu\rangle, \qquad \mu \in M, \tag{1.2}$$

where v is the unique solution of the integral equation

$$v_t(x) + \Pi_x\left[\int_0^t k(W_s)\, v_{t-s}^\alpha(W_s)\, ds\right] = \Pi_x f(W_t). \tag{1.3}$$

Moreover, for every $\tau \in \mathcal{T}$, there corresponds a random measure X_τ on $\mathbf{R}^d$ such that, for every bounded positive $f \in \mathcal{B}$,

$$P_\mu \exp\{-\langle f, X_\tau\rangle\} = \exp\langle -u, \mu\rangle, \qquad \mu \in M, \tag{1.4}$$

where u is the unique solution of the integral equation

$$u(x) + \Pi_x\left[\int_0^\tau k(W_s) u^\alpha(W_s)\, ds\right] = \Pi_x f(W_\tau) \tag{1.5}$$

($f(W_\tau) = 0$ if $\tau = \infty$). We call $X = \{X_t, X_\tau, P_\mu\}$ the super-Brownian motion with branching mechanism $\psi(x, z) = k(x) z^\alpha$.

By using the super-Brownian motion above, Sheu [2] studied the structure of the set of all positive solutions of the nonlinear elliptic equation (1.1) in Dimension 3 under the condition

$$\lim_{r\to\infty} \sup_{x\in\mathbf{R}^d} \int_{\|y\|>r} \|x-y\|^{2-d}\, k(y)\, dy = 0. \tag{1.6}$$

In this paper, we discuss similar problems in Dimensions 1 and 2.

Suppose that $k(x)$ is a bounded strictly positive Hölder continuous function on $\mathbf{R}^2$, and D is an unbounded domain in $\mathbf{R}^2$ with a compact nonempty boundary ∂D which consists of finitely many Jordan curves. For simplicity, we suppose $D = (0, \infty)$ is the half straight line in one dimension. We consider the structure of solutions to the problem

$$\begin{aligned} \frac{1}{2}\Delta u &= k(x)\, u^\alpha(x), && \text{in } D,\\ u &> 0, && \text{in } D,\\ u &= 0, && \text{on } \partial D. \end{aligned} \tag{1.7}$$

Problem (1.7) with a more general nonlinear term has been studied by Ufuktepe and Zhao [3]. A similar equation in Dimension 1 has been studied by Zhao [4]. By using probabilistic potential theory and fixed-point theory, they proved that, under certain conditions on k, (1.7) has solutions. But they did not provide probabilistic expressions for their solutions. The main goal of this paper is to give probabilistic expressions in terms of super-Brownian motion for all solutions to (1.7).

For any domain U, we use τ_U to denote the first exit time of W from U. Let $G_D(x, y)$ be the Green function of D. For any Borel function f in D, the Green operator is defined as

$$G_D f(x) = \Pi_x\left[\int_0^{\tau_D} f(W_t)\, dt\right] = \int_D G_D(x, y)\, f(y)\, dy.$$

For $x \in D$, put

$$h(x) = \pi \lim_{y \to \infty} G_D(x, y). \tag{1.8}$$

Then h is a harmonic function in D such that

$$\lim_{D \ni x \to z} h(x) = 0, \qquad \text{for any } z \in \partial D, \tag{1.9}$$

and

$$\lim_{\|x\| \to \infty} \frac{h(x)}{\log(\|x\|)} = 1. \tag{1.10}$$

(See [3, Proposition 2.1].)

The following two theorems are the main results of this paper.

THEOREM 1. *Suppose k satisfies*

$$\int_{\|x\| \geq a} k(x) (\log(\|x\|))^\alpha \, dx < \infty, \tag{1.11}$$

for some constant $a > 0$.

(1) *Let $h > 0$ be given by (1.8). For every $\mu \in M$ with compact support in D, $Z = \lim_{n \to \infty} \langle h, X_{\tau_{D \cap B(0,n)}} \rangle$ exists P_μ-a.s. and for every $c > 0$,*

$$u_c(x) := -\log P_{\delta_x} \exp\{-cZ\} \tag{1.12}$$

is the unique solution of (1.7) satisfying the condition

$$\lim_{n \to \infty} \frac{u(x)}{\log(\|x\|)} = c. \tag{1.13}$$

(2) *If $u(x)$ is a solution of (1.7) and satisfies $\limsup_{x \to \infty} (u(x)/\log(\|x\|)) < \infty$, then $u = u_c$ for some $c > 0$.*

(3)

$$J(x) := -\log P_{\delta_x}, \qquad (Z = 0) \tag{1.14}$$

is the smallest solution to (1.7) satisfying the condition

$$\lim_{x \to \infty} \frac{u(x)}{\log(\|x\|)} = \infty, \tag{1.15}$$

and

$$I(x) := -\log P_{\delta_x}, \qquad (X_{\tau_{D \cap B(0,n)}} = 0 \text{ for } n \text{ sufficiently large}) \tag{1.16}$$

is the largest solution to problem (1.7) satisfying condition (1.15).

If $k(x) \sim \|x\|^{-2} \log(\|x\|)^{-l}$ near ∞ for some constant $l > 1 + \alpha$, the following result shows that $I = J$ is the unique solution to (1.7) satisfying condition (1.15). (Here $f \sim g$ near ∞ means there exist two positive constants C_1, C_2 such that $C_1 f(x) \geq g(\|x\|) \geq C_2 f(x)$ for x sufficiently large.) But we do not know if $I = J$ for a general function k.

THEOREM 2. *If $k(x) \sim \|x\|^{-2} \log(\|x\|)^{-l}$ near ∞ for some constant $l > 1 + \alpha$, then (1.7) has only one solution satisfying condition (1.15). Moreover, we have*

$$I(x) = J(x) \sim (\log(\|x\|))^q, \tag{1.17}$$

where $q = (l - 2)/(\alpha - 1)$.

We also have similar results in Dimension 1. For simplicity, we assume that $D = (0, \infty)$ in this case. So, in Dimension 1, we are dealing with the following problem:

$$\begin{aligned} \frac{1}{2} u''(x) &= k(x) u^\alpha(x), && \text{in } (0, \infty), \\ u &> 0, && \text{in } (0, \infty), \\ u(0) &= 0. \end{aligned} \tag{1.18}$$

The analogue of Theorem 1 in this case is as follows.

THEOREM 3. *Suppose k satisfies*

$$\int_{\|x\|\geq a} k(x)\,|x|^{\alpha}\,dx < \infty,$$

for some constant $a > 0$.

(1) *Let h be the function $h(x) = x$. For every $\mu \in M$ with compact support in $(0, \infty)$, $Z = \lim_{n\to\infty}\langle h, X_{\tau_{D\cap B(0,n)}}\rangle$ exists P_μ-a.s. and for every $c > 0$,*

$$u_c(x) := -\log P_{\delta_x} \exp\{-cZ\}$$

is the unique solution of (1.18) satisfying the condition

$$\lim_{n\to\infty} \frac{u(x)}{|x|} = c.$$

(2) *If $u(x)$ is a solution of (1.18) and satisfies $\limsup_{x\to\infty}(u(x)/|x|) < \infty$, then $u = u_c$ for some $c > 0$.*

(3)

$$J(x) := -\log P_{\delta_x}, \qquad (Z = 0)$$

is the smallest solution to (1.18) satisfying the condition

$$\lim_{x\to\infty} \frac{u(x)}{|x|} = \infty, \tag{1.19}$$

and

$$I(x) := -\log P_{\delta_x}, \qquad \left(X_{\tau_{D\cap B(0,n)}} = 0,\ \text{for } n \text{ sufficiently large}\right)$$

is the largest solution to problem (1.18) satisfying condition (1.19).

Here is the analogue of Theorem 2 in the one-dimensional case.

THEOREM 4. *If $k(x) \sim |x|^{-l}$ near ∞ for some constant $l > 1 + \alpha$, then (1.18) has only one solution satisfying condition (1.19). Moreover, we have*

$$I(x) = J(x) \sim |x|^q, \tag{1.20}$$

where $q = (l-2)/(\alpha - 1)$.

We are only going to prove Theorems 1 and 2 in this paper, the proof of Theorems 3 and 4 are similar to the proof of Theorems 1 and 2, respectively. We omit the details.

2. PROOF OF THEOREM 1

In this section, we are going to give the proof of Theorem 1. In order to do that, we need some preparations first. The following result is a particular case of Theorem 0.5 in [1].

LEMMA 2.1. MAXIMUM PRINCIPLE.

Suppose U is a bounded domain and $\psi : U \times \mathbf{R}^+ \to \mathbf{R}^+$ satisfy the condition

$$\psi(x,u) \geq \psi(x,v), \qquad \text{for every } u \geq v \in \mathbf{R}^+ \text{ and every } x \in U.$$

If $u, v \geq 0$ belong to $C^2(U)$ and satisfy the conditions:

$$\frac{1}{2}\Delta u(x) - \psi(x,u(x)) \geq \frac{1}{2}\Delta v(x) - \psi(x,v(x)), \qquad \text{in } U,$$

and

$$\limsup_{x\to a, x\in U} [u(x) - v(x)] \leq 0, \qquad \text{for all } a \in \partial U,$$

then $u(x) \leq v(x)$ in U.

The following results will be used repeatedly in this paper.

LEMMA 2.2. *Suppose that U is a bounded regular domain and that u is a solution of $(1/2)\Delta u = ku^\alpha$ on U. If $U_1 \subset U$ is a bounded regular domain such that $\bar{U}_1 \subset U$, then*

$$u(x) = -\log \exp \left\langle -u, X_{\tau_{U_1}} \right\rangle, \qquad x \in U_1. \tag{2.1}$$

PROOF. It follows from Theorem 1.1 in [1] that $-\log \exp\langle -u, X_{\tau_{U_1}}\rangle$ is the unique bounded solution of $(1/2)\Delta u = ku^\alpha$ on U_1 with the boundary value u on ∂U_1. By our assumption, u is also a bounded solution of $(1/2)\Delta u = ku^\alpha$ on U_1 with boundary value u on ∂U_1. Therefore, (2.1) is true.

LEMMA 2.3. *Suppose that $B(0,r)$ is a disk such that $\partial D \subset B(0,r)$ and that $\varphi \geq 0$ is a bounded continuous function on ∂D. If $\{u_n\}$ is a sequence of positive solutions of $(1/2)\Delta u = ku^\alpha$ in $D \cap B(0,r)$ and if $u = \lim_{n\to\infty} u_n$ in $D \cap B(0,r)$, then u is also a solution of $(1/2)\Delta u = ku^\alpha$ in $D \cap B(0,r)$. Furthermore, if, for each n, u_n satisfies the boundary condition $u_n = \varphi$ on ∂D, then the same condition holds for u.*

PROOF. We first prove that u satisfies $(1/2)\Delta u = ku^\alpha$ on $D \cap B(0,r)$. Let $U \subset D \cap B(0,r)$ be an arbitrary smooth domain such that $\bar{U} \subset D \cap B(0,r)$. It follows from Lemma 2.2 that

$$u_n(x) = -\log P_{\delta_x} \exp \langle u_n, X_{\tau_U} \rangle, \qquad x \in U. \tag{2.2}$$

Let U_1 be a smooth domain such that $\bar{U} \subset U_1 \subset \bar{U}_1 \subset D \cap B(0,r)$. Since $\inf_{x\in U_1} k(x) > 0$, we can use Theorem 2.1 in [1] to conclude that there exists a nonnegative solution v of $(1/2)\Delta v = kv^\alpha$ in U_1 with the boundary value ∞ on ∂U_1. By the maximum principle, $u_n(x) \leq v(x)$ for $x \in \bar{U}$. Thus, $\{u_n\}$ is uniformly bounded in $\bar{U}$. Applying the bounded convergence theorem, we get that $\lim_{n\to\infty}\langle u_n, X_{\tau_U}\rangle = \langle u, X_{\tau_U}\rangle$, P_{δ_x}-a.s. for $x \in U$. Upon letting $n \to \infty$ in (2.2), we get that

$$u(x) = -\log P_{\delta_x} \exp \langle -u, X_U \rangle, \qquad x \in U.$$

Using Theorem 1.1 in [1] again, we see that u is a solution of $(1/2)\Delta u = ku^\alpha$ on U. Since the smooth domain $U \subset \bar{U} \subset D \cap B(0,r)$ is arbitrary, u is a solution in $D \cap B(0,r)$.

Next, we prove that u has boundary value $\varphi(z)$ at $z \in \partial D$. Choose $0 < r_0 < r$ such that $\partial D \subset B(0, r_0)$. It follows from Lemma 2.2 that for any $x \in D \cap B(0,r_0)$,

$$\begin{aligned} u_n(x) &= -\log P_{\delta_x} \exp \left\langle -u_n, X_{\tau_{D\cap B(0,r_0)}} \right\rangle \\ &= -\log P_{\delta_x} \exp \left(-\int_{\partial D} \varphi(z) X_{\tau_{D\cap B(0,r_0)}}(dz) - \int_{S(0,r_0)} u_n(z) X_{\tau_{D\cap B(0,r_0)}}(dz) \right), \end{aligned}$$

where $S(0,r_0)$ is the circle of radius r_0 centered at 0. From the proof of the first part, we know that $\{u_n\}$ is uniformly bounded on $S(0,r_0)$. Letting $n \to \infty$ and applying the bounded convergence theorem, we obtain that

$$u(x) = -\log P_{\delta_x} \exp \left\langle -\bar{\varphi}, X_{\tau_{D\cap B(0,r_0)}} \right\rangle,$$

where

$$\bar{\varphi}(z) = \begin{cases} \varphi(z), & z \in \partial D, \\ u(z), & x \in S(0, r_0). \end{cases}$$

Now, applying Theorem 1.1 in [1], we get that u has boundary value $\varphi(z)$ at $z \in \partial D$.

The following result is a modified version of Theorems 4.6.6 and 4.6.7 in [5].

LEMMA 2.4. *Suppose that ρ is a positive bounded integrable function on D. Then we have*

(1) $\lim_{D\ni x\to a} G_D\rho(x) = 0$, *for every $a \in \partial D$;*
(2) $G_D\rho \in C^{0,\lambda}(D)$;
(3) *if $\rho \in C^{0,\lambda}(D)$, then $G_D\rho \in C^{2,\lambda}(D)$ and $(1/2)\Delta G_D\rho = -\rho$ in D.*

Pick a fixed point $a \in \mathbf{R}^2 \setminus \bar{D}$ and a number $r > 0$ such that $D \supset B_r^* = \mathbf{R}^2 \setminus \overline{B(a,r)}$. Using the explicit formula for $G_{B_r^*}(\cdot,\cdot)$, one can easily prove the following result.

LEMMA 2.5. *The family of functions* $\{G_D(x,\cdot)k(\cdot)h^{\alpha-1}(\cdot) : x \in D\}$ *is uniformly integrable over* D, *where* h *is given by (1.8).*

LEMMA 2.6. *If* $f > 0$ *is a harmonic function having boundary value* 0 *on* ∂D, *then* f *is a constant multiple of the function* h *defined in (1.8).*

PROOF. Pick a fixed point $a \in \mathbf{R}^2 \setminus \bar{D}$ and a number $r > 0$ such that $D \supset B_r^* = \mathbf{R}^2 \setminus \overline{B(a,r)}$. The Kelvin transform of f relative to the circle $S(a,r)$ is $f^*(x^*) = f(x)$, where $x = a + (r^2/\|x^* - a\|^2)(x^* - a)$. $f^*(x^*)$ is a positive harmonic function on $D^* \setminus \{a\}$. By defining $f^*(a) = \liminf_{x^* \to a} f^*(x^*)$, we get a function f^* which is superharmonic on D^* with 0 boundary value on ∂D^*. By the Riesz decomposition theorem and Theorem 6.1.4 in [5], there exists a constant $c > 0$ such that $f^*(x^*) = cG_{D^*}(x^*, a)$, for every $x^* \in D^*$, and hence, $f(x) = ch(x)$ for every $x \in D$.

PROPOSITION 2.1. *Let* $\{D_n\}$ *be a sequence of bounded domains such that* $D_n \uparrow D$ *and* h *be defined by (1.8).*

(1) *There exists a random variable* Z *such that for every* $\mu \in M$ *with compact support in* D, $Z = \lim_{n\to\infty}\langle h, X_{\tau_{D_n}}\rangle < \infty$, P_μ-a.s.

(2) *If* u *is a solution to (1.7), then there exists a random variable* Z_u *such that for every* $\mu \in M$ *with compact support in* D, $Z_u = \lim_{n\to\infty}\langle u, X_{\tau_{D_n}}\rangle < \infty$, P_μ-a.s.

PROOF. Result (2) is proved in Section 5.5 in Dynkin [1]. Here we only give the outline of the proof of Result (1).

It follows from (1.4),(1.5) and the special Markov property (see [6, Section 2.1.A]) that $\exp\langle -h, X_{\tau_{D_n}}\rangle$ is a bounded submartingale. Thus, for every $\mu \in M$ with compact support in D, $Z = \lim_{n\to\infty}\langle h, X_{\tau_{D_n}}\rangle$ exists P_μ-a.s. It follows from (1.4),(1.5) that $P_\mu\langle h, X_{\tau_{D_n}}\rangle = \langle h, \mu\rangle < \infty$ for any $\mu \in M$ with compact support in D. Using Fatou's lemma, we get that $P_\mu Z \le \liminf_{n\to\infty}\langle h, X_{\tau_{D_n}}\rangle < \infty$, which implies that $Z < \infty$, P_μ-a.s., for all $\mu \in M$ with compact support in D.

We can further prove the limits do not depend on μ and the choice of D_n. For details, please see the proof of Theorem 2.2(1) in [7].

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1.

(1) Set

$$u_{c,n}(x) =: -\log P_{\delta_x} \exp\left\langle -ch, X_{\tau_{D\cap B(0,n)}}\right\rangle. \tag{2.3}$$

It follows from (1.4),(1.5) that $u_{c,n}$ satisfies the equation

$$u_{c,n}(x) + \Pi_x \int_0^{\tau_{D\cap B(0,n)}} k(W_s) u_{c,n}^\alpha(W_s)\, ds = ch(x), \qquad x \in D \cap B(0,n). \tag{2.4}$$

We get from Proposition 2.1 that $u_c(x) = \lim_{n\to\infty} u_{c,n}(x)$, for all $x \in D$. It follows from (1.10),(1.11) that $k(y)h^\alpha(y)$ is integrable in D, and therefore, $\int_D G_D(x,y)k(y)h^\alpha(y)\,dy < \infty$. Note that, for fixed $c>0$, each $u_{c,n}$ is dominated by ch. Thus, $\Pi_x \int_0^{\tau_{D\cap B(0,n)}} k(W_s)u_{c,n}^\alpha(W_s)\,ds \le c^\alpha \Pi_x \int_0^{\tau_D} k(W_s)h^\alpha(W_s)\,ds = c^\alpha \int_D G_D(x,y)k(y)h^\alpha(y)\,dy < \infty$. Letting $n \to \infty$ in (2.4) and applying dominated convergence, we get that

$$u_c(x) + G_D(ku_c^\alpha)(x) = ch(x), \qquad x \in D. \tag{2.5}$$

By Lemma 2.4(2), we know that $G_D(ku_c^\alpha) \in C^{0,\lambda}(D)$; thus, by (2.5), we get that $u_c \in C^{0,\lambda}(D)$. From Lemma 2.4(3), we know that $G_D(ku_c^\alpha) \in C^{2,\lambda}(D)$ and that $(1/2)\Delta G_D(ku_c^\alpha) = -ku_c^\alpha$, and therefore, $(1/2)\Delta u_c^\alpha = ku_c^\alpha$ in D. From Lemma 2.4(1), we know that $G_D(ku_c^\alpha)$ has the boundary value 0 on ∂D, and thus, from (2.5), we get that u_c has the boundary value 0 on ∂D. Now we check that $\lim_{\|x\|\to\infty}(u_c(x)/h(x)) = c$. It is enough to prove that

$$\lim_{\|x\|\to\infty} \frac{G_D(ku_c^\alpha)(x)}{h(x)} = 0. \tag{2.6}$$

It follows from (2.5) that $u_c(x) \le ch(x)$, $x \in D$. So we only need to prove

$$\lim_{\|x\|\to\infty} \frac{G_D(kh^\alpha)(x)}{h(x)} \left(= \int_D \frac{G_D(x,y)\,k(y)\,h^\alpha(y)}{h(x)}\,\mathrm{d}y \right) = 0. \tag{2.7}$$

By Theorem 2.2 in [3], we know that there exists a constant $C > 0$ depending on D only such that

$$\frac{G_D(x,y)\,G_D(y,z)}{G_D(x,z)} \le C\left(G_D(x,y) + G_D(y,z) + 1\right), \qquad x,y,z \in D.$$

This equality is called the (3-G) inequality for Green functions on D. Using the (3-G) inequality, we get

$$\begin{aligned} \frac{G_D(x,y)\,k(y)h^\alpha(y)}{h(x)} &= \lim_{z\to\infty} \frac{G_D(x,y)\,G_D(y,z)\,k(y)\,h^{\alpha-1}(y)}{G_D(x,z)} \\ &\le \lim_{z\to\infty} C\left(G_D(x,y) + G_D(y,z) + 1\right) k(y)\,h^{\alpha-1}(y) \\ &\le C\left(G_D(x,y) + h(y) + 1\right) k(y)\,h^{\alpha-1}(y). \end{aligned}$$

It follows from Lemma 2.5 that the family of functions $\{G_D(x,\cdot)k(\cdot)h^{\alpha-1}(\cdot) : x \in d\}$ is uniformly integrable over D. Using the fact that $\int_D (h(y)+1)k(y)h^{\alpha-1}(y)\,dy < \infty$, we get that the family $\{G_D(x,\cdot)k(\cdot)h^\alpha(\cdot)/h(x) : x \in D\}$ is uniformly integrable over D. Since $G_D(x,y) \to 0$ as $x \to \infty$ for fixed $y \in D$, we have $G_D(x,y)k(y)h^\alpha(y)/h(x) \to 0$ as $x \to \infty$. Thus, (2.7) holds.

Suppose that $u(x)$ is a solution of (1.7) satisfying $\lim_{x\to\infty}(u(x)/h(x)) = c$. Then $G_D(ku^\alpha) < \infty$ on D, and $\lim_{x\to\infty}(G_D(ku^\alpha)(x)/h(x)) = 0$. Let $\bar{h}(x) = u(x) + G_D(ku^\alpha)(x)$. It follows from Lemma 2.4 that $(1/2)\Delta G_D(ku^\alpha)(x) = -k(x)u^\alpha(x)$, $x \in D$, and that $G_D(ku^\alpha)$ has the boundary value 0 on ∂D. Thus, $\bar{h}$ is a positive harmonic function on D having the boundary value 0 on ∂D and satisfies $\lim_{x\to\infty}(\bar{h}(x)/h(x)) = c$. Now Lemma 2.6 implies that $\bar{h}(x) = ch(x)$, which means that

$$u(x) + G_D(ku^\alpha)(x) = ch(x). \tag{2.8}$$

It follows from Lemma 2.2 that, for n large enough,

$$u(x) = -\log P_{\delta_x} \exp\left\langle -u, X_{\tau_{D\cap B(0,n)}} \right\rangle. \tag{2.9}$$

Since $u(x) \le ch(x)$, $x \in D$, we have $u(x) \le -\log P_{\delta_x} \exp\langle -ch, X_{\tau_{D\cap B(0,n)}}\rangle$. Letting $n \to \infty$, we get

$$u(x) \le -\log P_{\delta_x} \exp(-cZ). \tag{2.10}$$

However, from (2.8) and (2.9), we know that, for n large enough,

$$\begin{aligned} u(x) &= -\log P_{\delta_x} \exp\left\langle (-ch + G_D(ku^\alpha)), X_{\tau_{D\cap B(0,n)}} \right\rangle \\ &= -\log P_{\delta_x} \exp\left\langle -chI_{\partial B(0,n)}(\cdot)\left(1 - \frac{G_D(ku^\alpha)}{ch}\right), X_{\tau_{D\cap B(0,n)}} \right\rangle. \end{aligned}$$

Since $\lim_{\|x\|\to\infty}(G_D(ku^\alpha)(x)/h(x)) = 0$, we know that, for every $\epsilon > 0$, there exists an integer N such that $G_D(ku^\alpha)(x)/ch(x) \le \epsilon$ for $n > N$ and $x \in \partial B(0,n)$, and hence,

$$u(x) \ge -\log P_{\delta_x} \exp\left\langle -c(1-\epsilon)\,h, X_{\tau_{D\cap B(0,n)}} \right\rangle, \qquad \text{for } n > N.$$

Letting $\epsilon \to 0$, we get

$$u(x) \ge -\log P_{\delta_x} \exp(-cZ). \tag{2.11}$$

Combining (2.10) and (2.11), we get $u(x) = -\log P_{\delta_x} \exp(-cZ)$.

(2) If u is a solution of (1.7) satisfying $\limsup_{x\to\infty}(u(x)/h(x)) < \infty$, using the same method as above, we can prove that (2.8) holds for some constant $c > 0$, and thus, $u(x) = -\log P_{\delta_x}\exp(-cZ)$.

(3) Since $J(x) = \lim_{c\to\infty} u_c(x)$, it follows from Lemma 2.2 that $J(x)$ is a solution to (1.7). It is obvious that $\lim_{x\to\infty}(J(x)/h(x)) = \infty$. So by the maximum principle, J is the smallest solution to (1.7) with $\lim_{x\to\infty}(u(x)/h(x)) = \infty$.

For large n, put

$$I_n(x) = -\log P_{\delta_x}\left(X_{\tau_{D\cap B(0,n)}}(\partial B(0,n)) = 0\right), \qquad x \in D\cap B(0,n).$$

Note that $-\log P_{\delta_x}\exp(-\lambda X_{\tau_{D\cap B(0,n)}}(\partial B(0,n))) \uparrow I_n(x)$ as $\lambda \uparrow \infty$. Lemma 2.3 implies that I_n is a solution of $(1/2)\Delta u = ku^\alpha$ on $D\cap B(0,n)$ with the boundary value 0 on ∂D and the boundary value ∞ on $\partial B(0,n)$. Since $I(x) = \lim_{n\to\infty} I_n(x)$, We know from Lemma 2.3 that I is a solution to (1.7). The maximum principle implies that I is the largest solution to (1.7). ∎

3. PROOF OF THEOREM 2

Throughout this section, C is a positive constant whose value may change from line to line.

As an important step in proving Theorem 2, we first consider the special case where $k(x) = C\|x\|^{-2}(\log(\|x\|))^{-l}$ near ∞ for some constant $l > 1+\alpha$. To that end, we consider positive radial solutions of the equation

$$\frac{1}{2}\Delta u = Cr^{-2}(\log(r))^{-l}u^\alpha, \qquad \text{in } B^c(0,R), \tag{3.1}$$

where $r = \|x\|$ and $l > 1+\alpha$.

PROPOSITION 3.1. *Suppose that $u(r)$ is a radial solution to (3.1) and satisfies*

$$\lim_{r\to\infty}\frac{u(r)}{\log(r)} = \infty,$$

then $u(r) \sim (\log(r))^q$ near infinity, where $q = (l-2)/(\alpha-1) > 1$.

Before we give the proof of Proposition 3.1, we give three lemmas.

LEMMA 3.1. *Suppose that $l > 1+\alpha$ and $R > 0$ is a large constant. If u is a positive solution of $(1/2)u''(x) = x^{-l}u^\alpha(x)$ on the interval (R,∞) and if u satisfies $\lim_{x\to\infty}(u(x)/x) = \infty$, then $\liminf_{x\to\infty}(x^q/u(x)) < \infty$, where $q = (l-2)/(\alpha-1) > 1$.*

PROOF. Suppose the result were false and so $\lim_{x\to\infty}(x^q/u(x)) = \infty$. Set $\omega(x) = x^{-q}u(x)$, $x \in (R,\infty)$. Then $\omega \to 0$ as $x\to\infty$ and ω satisfies

$$w_{xx} + \frac{2q}{x}\omega_x + q(q-1)\frac{\omega}{x^2} - C\frac{\omega^\alpha}{x^2} = 0, \qquad x \in (R,\infty). \tag{3.2}$$

Setting $s = \log x$, (3.2) becomes

$$\omega_{ss} + (2q-1)\omega_s + \omega\left[q(q-1) - C\omega^{\alpha-1}\right] = 0, \qquad s \in (\log R, \infty).$$

Using analytic method (see the arguments in Step 4 in the proof of Theorem 4.3 in [8]), we can prove that $\omega(s) \le C\exp(-\epsilon s)$ for some constants $C, \epsilon > 0$ near ∞, thus, we have $u(x) \le Cx^{q-\epsilon}$. Therefore, for large x,

$$\frac{u(x)}{x} \le Cx^{q-1-\epsilon}. \tag{3.3}$$

Choose $\epsilon \neq (q-1)/\alpha^n$, $n = 1, 2, \ldots$. Substituting estimate (3.3) into the integral representation of u, we obtain, for $x \geq R$,

$$\begin{aligned} u(x) &= u'(R)\,x + C\int_R^x \mathrm{d}t \int_R^t s^{-l}u^{\alpha}(s)\,\mathrm{d}s + u(R) \\ &= u'(R)\,x + C\int_R^x (x-s)\,s^{-l}u^{\alpha}(s)\,\mathrm{d}s + u(R) \\ &\leq u'(R)\,x + Cx\int_R^x s^{-l+(q-\epsilon)\alpha}\,\mathrm{d}s + u(R) \\ &= u'(R)\,x + Cx\int_R^x s^{q-2-\epsilon\alpha}\,\mathrm{d}s + u(R). \end{aligned}$$

Thus, we have

$$\frac{u(x)}{x} \leq u'(R) + C\int_R^x s^{q-2-\epsilon\alpha}\,\mathrm{d}s.$$

Since $\epsilon \neq (q-1)/\alpha$, we have $q-2-\epsilon\alpha \neq -1$. If $q-2-\epsilon\alpha < -1$, then $u(x)/x \leq C$ for some constant $C > 0$, which contradicts the assumption on u. If $q-2-\epsilon\alpha > -1$, then there exists a constant $C > 0$ such that

$$\frac{u(x)}{x} \leq u'(R) + \frac{C}{q-1-\epsilon\alpha}x^{q-1-\epsilon\alpha} \leq Cx^{q-1-\epsilon\alpha}, \tag{3.4}$$

for x large, which improves (3.3). Let $K = \min\{n; n \text{ satisfies } q-1 < \alpha^n\epsilon\}$. Iterating (3.4), after K steps, we conclude that there exists a constant $C > 0$ such that

$$\frac{u(x)}{x} \leq C,$$

for x large, which contradicts the assumption on u.

Lemma 3.2. *Suppose that $l > 1 + \alpha$ and $R > 0$ is a large constant. If u is a positive solution of $(1/2)u''(x) = x^{-l}u^{\alpha}(x)$ on the interval (R, ∞) and if u satisfies $\lim_{x\to\infty}(u(x)/x) = \infty$, then $\liminf_{x\to\infty}(u(x)/x^q) < \infty$, where $q = (l-2)/(\alpha-1) > 1$.*

Proof. Suppose that the result were false and so $\lim_{x\to\infty}(u(x)/x^q) = \infty$. Set $v(x) = x^{-q}u(x)$, $x \in (R, \infty)$. Then $v \to \infty$ as $x \to \infty$, and v satisfies

$$v_{xx} + \frac{2q}{x}v_x + q(q-1)\frac{v}{x^2} - C\frac{v^{\alpha}}{x^2} = 0. \tag{3.5}$$

Rewrite (3.5) as

$$\left(x^{2q}v_x\right)_x = x^{2q-2}\left[v^{\alpha} - Cq(q-1)\,v\right], \qquad x \in (R, \infty). \tag{3.6}$$

Since $v(x) \to \infty$ at ∞ and R is large, we have

$$\left(x^{2q}v_x\right)_x \geq \frac{1}{2}x^{2q-2}v^{\alpha}, \qquad x \in (R, \infty), \tag{3.7}$$

and the inequality $v_x(R) \geq 0$ holds for large R. Integrating (3.7) first from R to t and then from R to x, we arrive at

$$v(x) \geq v(R) + \frac{1}{2(2q-1)}\int_R^x \frac{1}{s}\left[1 - \left(\frac{s}{x}\right)^{2q-1}\right]v^{\alpha}(s)\,\mathrm{d}s.$$

Now the same argument in the proof of Theorem 2.1 in [9] leads to a contradiction. Since the argument is purely analytic and is not the main point of this paper, we omit the details.

LEMMA 3.3. *Suppose that $l > 1 + \alpha$ and $R > 0$ is a large constant. If u is a positive solution of $(1/2)u''(x) = x^{-l}u^{\alpha}(x)$ on the interval (R, ∞) and if u satisfies $\lim_{x\to\infty}(u(x)/x) = \infty$, then $u(x) \sim x^q$ near infinity, where $q = (l-2)/(\alpha-1) > 1$.*

PROOF. Put $u_0(x) = [q(q-1)/2C]^{1/(\alpha-1)}x^q$, $x > 0$. It is easy to check that u_0 is a solution of $(1/2)u'' = Cku^{\alpha}$ in (R, ∞) and satisfies $\lim_{x\to\infty}(u_0(x)/x) = \infty$.

First, we prove that there exists a constant $M_1 > 0$ such that $u(x) \geq M_1 x^q$, on (R, ∞). It follows from Lemma 3.1 that $\liminf_{x\to\infty}(u_0(x)/u(x)) < \infty$. Thus, there exists a sequence $y_n \uparrow \infty$ such that $\lim_{n\to\infty}(u_0(y_n)/u(y_n)) < \infty$. Hence, there exists a constant $M > 1$ such that $u_0(R) \leq Mu(R)$ and $u_0(y_n) \leq Mu(y_n)$ for every integer n. Using Lemma 2.2 and Hölder's inequality, we get that, for large n,

$$\begin{aligned} u_0(x) &= -\log P_{\delta_x} \exp\left(-\langle u_0, X_{\tau(R,y_n)}\rangle\right) \\ &\leq -\log P_{\delta_x} \exp\left(-M\langle u, X_{\tau(R,y_n)}\rangle\right) \\ &= -\log P_{\delta_x}\left[\exp\left(-\langle u, X_{\tau(R,y_n)}\rangle\right)\right]^M \\ &\leq -\log\left[P_{\delta_x}\exp\left(-\langle u, X_{\tau(R,y_n)}\rangle\right)\right]^M = Mu(x), \qquad x \in (R, y_n). \end{aligned}$$

Consequently, we have $u_0 \leq Mu$ on (R, ∞). Therefore, $u \geq M_1 x^q$ on (R, ∞) for $M_1 = (1/M)[q(q-1)/2C]^{1/(\alpha-1)}$.

Next, we prove that there exists a constant $M_2 > 0$ such that $u(x) \leq M_2 x^q$ on (R, ∞). It follows from Lemma 3.2 that $\liminf_{x\to\infty}(u(x)/u_0(x)) < \infty$. Thus, there exists a sequence $y_n \uparrow \infty$ such that $\lim_{n\to\infty}(u(y_n)/u_0(y_n)) < \infty$. Hence, there exists a constant $M' > 1$ such that $u(R) \leq M'u_0(R)$ and $u(y_n) \leq M'u_0(y_n)$ for every integer n. Using Lemma 2.2 and Hölder's inequality again, we get that, for large n,

$$\begin{aligned} u_0(x) &= -\log P_{\delta_x} \exp\left(-\langle u_0, X_{\tau(R,y_n)}\rangle\right) \\ &\geq -\log P_{\delta_x} \exp\left(-\frac{1}{M'}\langle u, X_{\tau(R,y_n)}\rangle\right) \\ &= -\log P_{\delta_x}\left[\exp\left(-\langle u, X_{\tau(R,y_n)}\rangle\right)\right]^{1/M'} \\ &\geq -\log\left[P_{\delta_x}\exp\left(-\langle u, X_{\tau(R,y_n)}\rangle\right)\right]^{1/M'} = \frac{1}{M'}u(x), \qquad x \in (R, y_n). \end{aligned}$$

Consequently, we have $u_0 \geq (1/M')u$ on (R, ∞). Therefore, $u \leq M_2 x^q$ on (R, ∞) for $M_2 = M'[q(q-1)/2C]^{1/(\alpha-1)}$.

PROOF OF PROPOSITION 3.1. Since u is radial, we can define a function u on (R, ∞) by setting $u(r) = u(\|x\|)$ for any x satisfying $\|x\| = r$. The function u satisfies

$$u''(r) + \frac{1}{r}u'(r) = Cr^{-2}(\log r)^{-l}u^{\alpha}, \qquad r > R. \tag{3.8}$$

Putting $u(r) = v(t)$ and $t = \log r$, the equation above becomes

$$v''(t) = Ct^{-l}v^{\alpha}(t), \qquad t > \log(R). \tag{3.9}$$

From the definition of v, we can easily check that

$$\lim_{t\to\infty}\frac{v(t)}{t} = \infty. \tag{3.10}$$

Applying Lemma 3.3, we get that $v(t) \sim t^q$ near infinity, which implies that $u(r) \sim (\log r)^q$ near infinity.

PROOF OF THEOREM 2. By assumption, there exist two constants $C_1, C_2 > 0$ such that

$$C_1 \|x\|^{-2} (\log (\|x\|))^{-l} \le k(x) \le C_2 \|x\|^{-2} (\log (\|x\|))^{-l}, \tag{3.11}$$

when $\|x\|$ is large. Suppose that k_s, $s = 1, 2$, are bounded strictly positive Hölder continuous functions on $\mathbf{R}^2$ such that $k_1(x) \le k(x) \le k_2(x)$ for $x \in \mathbf{R}^2$ and $k_s = C_s\|x\|^{-2}\log(\|x\|)^{-l}$ near ∞. Suppose that $D_s = B^c(a, r_s)$, $s = 1, 2$ satisfy $\bar{D}_1 \subset D$ and $\bar{D} \subset D_2$. Let I_s, J_s denote the largest and smallest solution of (1.7) satisfying $\lim_{\|x\|\to\infty}(u(x)/\log(\|x\|)) = \infty$ with k replaced by k_s and D by D_s, respectively. The maximum principle implies that when $\|x\|$ is large enough, we have

$$I_2 (x) \le I (x) \le I_1 (x)$$

and

$$J_2 (x) \le J (x) \le J_1 (x) .$$

Thus, we have

$$\frac{I (x)}{J (x)} \le \frac{I_1 (x)}{J_2 (x)},$$

when $\|x\|$ is large enough. Since I_s and J_s are radial, Proposition 3.1 implies that I_s, $J_s \sim (\log(\|x\|))^q$, $s = 1, 2$ near infinity. Hence, there exists a constant $M > 0$ such that

$$\frac{I (x)}{J (x)} \le M, \tag{3.12}$$

when $\|x\|$ is large enough.

It follows from Proposition 2.1(2) that both

$$Z_I := \lim_{n\to\infty} \langle I, X_{\tau_{D\cap B(0,n)}} \rangle = \lim_{n\to\infty} \int_{S(0,n)} I (z) X_{\tau_{D\cap B(0,n)}} (\mathrm{d}z)$$

and

$$Z_J := \lim_{n\to\infty} \langle J, X_{\tau_{D\cap B(0,n)}} \rangle = \lim_{n\to\infty} \int_{S(0,n)} J(z) X_{\tau_{D\cap B(0,n)}} (\mathrm{d}z)$$

exist P_{δ_x}-a.s., and

$$I (x) = -\log P_{\delta_x} \exp (-Z_I), \qquad J (x) = -\log P_{\delta_x} \exp (-Z_J) .$$

Note that

$$Z_I = \lim_{n\to\infty} \int_{S(0,n)} \frac{I (z)}{h (z)} h (z) X_{\tau_{D\cap B(0,n)}} (\mathrm{d}z), \tag{3.13}$$

$$Z_J = \lim_{n\to\infty} \int_{S(0,n)} \frac{J (z)}{h (z)} h (z) X_{\tau_{D\cap B(0,n)}} (\mathrm{d}z), \tag{3.14}$$

and

$$Z = \lim_{n\to\infty} \int_{S(0,n)} h (z) X_{\tau_{D\cap B(0,n)}} (\mathrm{d}z) . \tag{3.15}$$

Since $\lim_{\|z\|\to\infty}(I(z)/h(z)) = \lim_{\|z\|\to\infty}(J(z)/h(z)) = \infty$, we get from (3.13)–(3.15) that $Z_I = Z_J = \infty$ on $(Z > 0)$. Thus, we have

$$-\log P_{\delta_x} (Z = 0) = J (x) = -\log P_{\delta_x} \exp(-Z_J) = -\log P_{\delta_x} [\exp (-Z_J) ; Z = 0] .$$

Hence, $(Z = 0)$ implies $(Z_J = 0)$, P_{δ_x}-a.s. From (3.12), we know that $I(x) \le MJ(x)$ when $\|x\|$ is sufficiently large, which implies that $Z_I = 0$ on $(Z = 0)$, P_{δ_x}-a.s. Therefore,

$$I (x) = -\log P_{\delta_x} \exp (-Z_I) = -\log P_{\delta_x} [\exp (-Z_I) ; Z = 0] = -\log P_{\delta_x} [Z = 0] = J (x) .$$

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