Finite time extinction of super-Brownian motions with deterministic catalyst

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Abstract

In this paper we consider a super-Brownian motion X with branching mechanism $k(x)z^\alpha$, where $k(x) > 0$ is a bounded Hölder continuous function on $\mathbf{R}^d$ and $\inf_{x\in\mathbf{R}^d} k(x) = 0$. we prove that if $k(x) \geq \|x\|^{-l}(0 \leq l < \infty)$ for sufficiently large x, then X has compact support property, and for dimension $d = 1$, if $k(x) \geq \exp(-l\|x\|)(0 \leq l < \infty)$ for sufficiently large x, then X also has compact support property. The maximal order of $k(x)$ for finite time extinction is different for $d = 1$, $d = 2$ and $d \geq 3$: it is $O(\|x\|^{-(\alpha+1)})$ in one dimension, $O(\|x\|^{-2}(\log\|x\|)^{-(\alpha+1)})$ in two dimension, and $O(\|x\|^2)$ in higher dimensions. These growth orders also turn out to be the maximum order for the nonexistence of a positive solution for $\frac{1}{2}\Delta u = k(x)u^\alpha$.

Keywords super-Brownian motion, compact support property, finite time extinction, positive solutions to nonlinear pde's

AMS 2000 subject classifications Primary 60J80; secondary 60J45

1 Introduction and Main Results

Suppose $X = \{X_t, P_\mu\}$ is a super-Brownian motion with branching mechanism $k(x)z^\alpha(1 < \alpha \leq 2)$ with k being a nonnegative, bounded, Hölder continuous function on $\mathbf{R}^d$. It is well known that if $\inf_{x\in\mathbf{R}^d} k(x) > 0$, then X has the compact support property and becomes extinct. In [1], Dawson, Fleischmann and Mueller investigated the finite time extinction of super-Brownian motions with catalysts in one dimension. They pointed out that if $k(x) = 0$ in some interval (a, b), then the superprocess survives in finite time, and they gave an abstract sufficient criterion for finite time extinction based on the idea of good and bad paths(see Theorem 10 in [1]). Using this abstract criterion they showed that if $k(x) = \|x\|^q \wedge 1(q > 0)$, the super-Brownian motion X dies in finite time. In this article, we investigate the compact support property and finite time extinction of super-Brownian motion X in the case $k(x) \sim \|x\|^{-l}$ as $x \to \infty$ for some constant $l \geq 0$, by using the connections between superdiffusions and partial differential equations. (Writing $f \sim g$ as $x \to \infty$ means there exist two positive constants C_1, C_2 such that $C_1 f(x) \geq g(x) \geq C_2 f(x)$ for x sufficiently large.)

Let $W := \{W_s, \Pi_x, s \geq 0, x \in \mathbf{R}^d\}$ denote the Brownian motion in $\mathbf{R}^d$. For every Borel-measurable space $(E, \mathcal{B}(E))$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}(E)$ endowed with the topology of weak convergence. The expression $< f, \mu >$ stands for the integral of f with respect to μ. We write $f \in \mathcal{B}(E)$ if f is a $\mathcal{B}(E)-$measurable function. Writing $f \in p\mathcal{B}(E)(b\mathcal{B}(E))$ means that, in addition, f is positive (bounded). We put $bp\mathcal{B}(E) = b\mathcal{B}(E) \cap p\mathcal{B}(E)$. If $E = \mathbf{R}^d$, we simply write $\mathcal{B}$ instead of $\mathcal{B}(E)$ and M instead of $M(\mathbf{R}^d)$. For an open set D, we write $\mu \in M_c(D)$ if $\mu \in M(D)$ and has a compact support in D.

We denote by $\mathcal{T}$ the set of all exit times from open sets in $\mathbf{R}^d$. Set $\mathcal{F}_{\leq r} = \sigma(W_s, s \leq r)$; $\mathcal{F}_{>r} = \sigma(W_s, s > r)$ and $\mathcal{F}_\infty = \vee\{\mathcal{F}_{\leq r}, r \geq 0\}$. For $\tau \in \mathcal{T}$, we put $F \in \mathcal{F}_{\geq\tau}$ if $F \in \mathcal{F}_\infty$ and if, for each r, $\{F, \tau > r\} \in \mathcal{F}_{>r}$.

According to Dynkin[4], for every non-negative bounded Borel function $k(x)$ in $\mathbf{R}^d$ and $1 < \alpha \leq 2$, there exists a Markov process $X = (X_t, P_\mu)$ in M such that the following conditions are satisfied.

(a) If f is a bounded continuous function, then $< f, X_t >$ is right continuous in t on $\mathbf{R}^+$.

(b) For every $\mu \in M$ and for every $f \in bp\mathcal{B}$,

$$P_\mu \exp < -f, X_t >= \exp < -v_t, \mu >, \quad \mu \in M \tag{1.2}$$

where v is the unique solution of the integral equation

$$v_t(x) + \Pi_x \left[\int_0^t k(W_s) v_{t-s}^\alpha (W_s) \mathrm{d}s \right] = \Pi_x f(W_t). \tag{1.3}$$

Moreover, for every $\tau \in \mathcal{T}$, there corresponds a random measure X_τ on $\mathbf{R}^d$ associated with the first exit time τ such that, for $f \in bp\mathcal{B}$,

$$P_\mu \exp\{- < f, X_\tau >\} = \exp < -u, \mu >, \quad , \mu \in M \tag{1.4}$$

where u is the unique solution of the integral equation

$$u(x) + \Pi_x \left[\int_0^\tau k(W_s) u^\alpha (W_s) \mathrm{d}s \right] = \Pi_x f(W_\tau) \tag{1.5}$$

($f(W_\tau) = 0$ if $\tau = \infty$). We call $X = \{X_t, X_\tau, P_\mu\}$ the super-Brownian motion with branching mechanism $\psi(x, z) = k(x)z^\alpha$ (enhanced model).

The class of all k times continuously differentiable functions in a domain D is denoted by $C^k(D)$. For constant $\lambda > 0$, we put $f \in C^{0,\lambda}(D)$ if f is locally Hölder continuous on D with exponent $\lambda > 0$ that is if for every compact set $K \subset D$, there exists a constant C_K depending only on K such that $|f(x) - f(y)| \le C_K \|x - y\|^\lambda$ for all $x, y \in K$. Put $f \in C^{k,\lambda}(D)$ if $f \in C^k(D)$ and if the derivations of order $\le k$ belong to $C^{0,\lambda}(D)$. Throughout this paper, we suppose k is a bounded function in $\mathbf{R}^d$, $k \in C^\lambda(\mathbf{R}^d)$, $\lambda \in (0, 1]$ and $k > 0$.

In the sequel we will frequently use the notation $A \subset\subset B$, which means that A is bounded and $\overline{A} \subset B$. For $0 \ne \mu \in M_c(\mathbf{R}^d)$. we say that the measure-valued process corresponding to P_μ possesses the compact support property if

$$P_\mu \left(\cup_{0 \le s \le t} \mathrm{supp} X_s \subset\subset \mathbf{R}^d \right) = 1, \quad \text{for all } t \ge 0.$$

In the first part of this paper, we will investigate the compact support property for super-Brownian motions. This problem have been discussed by Engländer and Pinsky (see [5] and [6]) for superdiffusions. According to Engländer and Pinsky's results in [5], if $\inf_{x \in \mathbf{R}^d} k(x) > 0$, then the super-Brownian motion with branching mechanism $k(x)z^\alpha (1 < \alpha \le 2)$ possesses the compact support property. By checking their proofs and making some modification, we get the following Theorems 1.1 and 1.2.

Theorem 1.1 *Suppose there exist constants $M > 0, l \ge 0$ such that $k(x) \ge M \left(\|x\|^{-l} \wedge 1 \right)$ for $x \in \mathbf{R}^d$.*

(1) The Cauchy-problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - k(x)u^\alpha, \ \ on \ \mathbf{R}^d \times (0,\infty), \\ \lim_{t\to 0} u(x,t) = 0. \end{cases} \quad (C)$$

has no nonnegative nonzero solution.

(2) The super-Brownian motion with branching mechanism $k(x)z^\alpha (1 < \alpha \le 2)$ *possesses the compact support property.*

For dimension $d = 1$, we have a stronger result.

Theorem 1.2 *Suppose* $d = 1$ *and there exists constants* $M, l > 0$ *such that* $k(x) \ge M\exp(-l\|x\|)$ *for* $x \in \mathbf{R}^1$. *Then results (1) and (2) in Theorem 1.1 hold.*

Remark In the case $d = 1$, Engländer and Pinsky [5] pointed out that if $k(x) = \exp(-(x^2 + 1)^2)$, then the process does not possess the compact support property.

A path $X(\cdot)$ of the super-Brownian motion survives if $X(t) \neq 0$ for all $t \ge 0$ and becomes extinct if $X(t) = 0$ for all sufficiently large t. The super-Brownian motion corresponding to P_μ becomes extinct (survives) if $P_\mu(X(\cdot)$ survives$) = 0 (> 0)$.

By using the connections between superdiffusions and partial differential equations, Sheu[9] studied the structure of the set of all positive solutions for nonlinear elliptic equation:

$$\frac{1}{2}\Delta u = k(x)u^\alpha(x), \quad x \in \mathbf{R}^d \tag{1.6}$$

for dimension $d \ge 3$ under the condition:

$$\lim_{r\to\infty} \sup_{x\in\mathbf{R}^d} \int_{\|y\|>r} g(x,y)k(y)dy = 0, \tag{1.7}$$

where $g(x,y)$ is the Green function of the operator $\frac{1}{2}\Delta$ on $\mathbf{R}^d$. In section 3, we give the structures of all positive solutions of (1.6) on an unbounded domain D for dimension $d \le 2$ in terms of superdiffusions (see Theorems 1.3 and 3.1). Then, in section 4, by using the connections between superdiffusions and structures of positive solutions of nonlinear partial differential equations, we obtain the finite time extinction results of super-Brownian motions (see Theorem 1.4).

Theorem 1.3 *Consider the elliptic differential equation*

$$\frac{1}{2}\Delta u = k(x)u^\alpha \ \ in \ \mathbf{R}^d \tag{E}$$

(1) *Let* $d = 1$. *If* $\int_a^\infty k(y)y^\alpha dy < \infty$(*or* $\int_{-\infty}^{-a} k(y)y^\alpha < \infty$) *for some constant* $a > 0$, *then* E *has a nonnegative solution satisfying* $\lim_{x\to\infty} \frac{u(x)}{x} = \infty$ (*or* $\lim_{x\to-\infty} \frac{u(x)}{-x} = \infty$). *If* $k(x) \ge c/\|x\|^{1+\alpha}$ *for* $\|x\| >> 1$ *and some* $c > 0$, *then* E *has no nonnegative nonzero solution.*

(2) *Let* $d = 2$. *If* $k(x) \le \overline{k}(\|x\|)$ *for* x *sufficiently large, and* $\int_a^\infty s\overline{k}(s)(\log(1+s))^\alpha ds < \infty$ *for some constant* $a > 0$, *then* E *has a nonnegative solution satisfying* $\lim_{\|x\|\to\infty} \frac{u(x)}{\ln(\|x\|)} = \infty$. *If* $k(x) \ge c/\|x\|^2(\log\|x\|)^{\alpha+1}$ *for* $\|x\| >> 1$ *and some* $c > 0$,

then E *has no nonnegative nonzero solution.*

(3) *Let* $d \ge 3$. *If* $k(x) \le \overline{k}(\|x\|)$ *for* x *sufficiently large, and* $\int_a^\infty s\overline{k}(s) < \infty$ *for some* $a > 0$, *then* E *has nonnegative nonzero solution. If* $k(x) \ge c/\|x\|^2$ *for* $\|x\| >> 1$ *and some* $c > 0$, *then* E *has no nonnegative nonzero solution.*

Theorem 1.4 *Let* X *be a super-Brownian motion with branching mechanism* $k(x)z^\alpha$.

(1) *If*

$$k(x) \le \begin{cases} c/\left[\|x\|^{1+\alpha+\epsilon}\right], & d = 1; \\ c/\left[\|x\|^2(\log\|x\|)^{1+\alpha+\epsilon}\right], & d = 2; \\ c/\|x\|^{2+\epsilon}, & d \ge 3. \end{cases} \tag{1.8}$$

for sufficiently large $\|x\|$ *and some* $\epsilon, c > 0$, *then* X *survives.*

(2) *If*

$$k(x) \ge \begin{cases} c/\left[\|x\|^{1+\alpha}\right], & d = 1; \\ c/\left[\|x\|^2(\log\|x\|)^{1+\alpha}\right], & d = 2; \\ c/\|x\|^{2}, & d \ge 3. \end{cases} \tag{1.9}$$

for sufficiently large $\|x\|$ *and some* $c > 0$, *then* X *dies in finite time.*

2 Compact Support Property (Proof of Theorem 1.1 and Theorem 1.2)

Proof of Theorem 1.1 By Theorem 3.4 in [5], statements (1) and (2) are equivalent. So, we only need to prove statement (1).

Let $u \geq 0$ be a solution to (C). Define

$$u_R(x,t) = (\lambda + \gamma r^m)(R^2 - r^2)^{-2/(\alpha-1)} e^{Kt}, \quad \|x\| < R, t \geq 0, \tag{2.1}$$

where $r = \|x\|$ and m, γ, K and λ are to be fixed later. We prove that

$$\frac{1}{2}\Delta u_R - k(x)u_R^\alpha - (u_R)_t \leq 0, \text{ for sufficiently large } R > 0, \tag{2.2}$$

if γ, K, λ are sufficiently large. Since $\lim_{r\to R} u_R(x,t) = \infty$, for x, t fixed, and since $u_R(x,0) > 0$, it follows from the parabolic maximum principle that $u(x,t) \leq u_R(x,t)$ for $\|x\| < R, t > 0$. Since $\lim_{R\to\infty} u_R(x,t) = 0$, we conclude that $u \equiv 0$. It remains to prove (2.2). Put

$$f = (\lambda + \gamma r^m), \quad g(r) = (R^2 - r^2)^{-2/(\alpha-1)}$$

Then

$$f' = m\gamma r^{m-1}, \quad f'' = m(m-1)\gamma r^{m-2},$$
$$g' = \frac{4r}{(\alpha-1)}(R^2 - r^2)^{-(\alpha+1)/(\alpha-1)},$$
$$g'' = \frac{4}{\alpha-1}(R^2 - r^2)^{-(\alpha+1)/(\alpha-1)} + \frac{8(\alpha+1)}{(\alpha-1)^2} r^2 (R^2 - r^2)^{-2\alpha/(\alpha+1)}.$$

Note that

$$\begin{aligned}
& e^{-Kt}\left[\tfrac{1}{2}\Delta u_R - k(x)u_R^\alpha - (u_R)_t\right] \\
= \;& e^{-Kt}\left[\tfrac{1}{2}u_R''(r,t) + \tfrac{d-1}{2r}u_R'(r,t) - k(x)u_R^\alpha(r,t) - (u_R)_t(r,t)\right] \\
= \;& \tfrac{1}{2}fg'' + (f' + \tfrac{d-1}{2r}f)g' + (\tfrac{1}{2}f'' + \tfrac{d-1}{2r}f' - Kf)g - k(x)f^\alpha g^\alpha e^{(\alpha-1)Kt} \\
\leq \;& \tfrac{4(\alpha+1)}{(\alpha-1)^2} r^2 (\lambda + \gamma r^m)(R^2 - r^2)^{-2\alpha/(\alpha-1)} + \tfrac{1}{\alpha-1}\left[4m\gamma r^m + 2d(\lambda + \gamma r^m)\right](R^2 - r^2)^{-(\alpha+1)/(\alpha-1)} \\
& + \left[\tfrac{1}{2}m(m-1)\gamma r^{m-2} + \tfrac{1}{2}(d-1)m\gamma r^{m-2} - K(\lambda + \gamma r^m)\right](R^2 - r^2)^{-2/(\alpha-1)} \\
& - k(x)(\lambda + \gamma r^m)^\alpha (R^2 - r^2)^{-2\alpha/(\alpha-1)} \equiv \textstyle\sum .
\end{aligned}$$

To show that $\sum \leq 0$, it is enough to prove that

$$\begin{aligned}
& \tfrac{4(\alpha+1)}{(\alpha-1)^2} r^2 (\lambda + \gamma r^m) + \tfrac{1}{\alpha-1}\left[4m\gamma r^m + 2d(\lambda + \gamma r^m)\right](R^2 - r^2) \\
& + \left[\tfrac{1}{2}m(m-1)\gamma r^{m-2} + \tfrac{1}{2}(d-1)m\gamma r^{m-2} - K(\lambda + \gamma r^m)\right](R^2 - r^2)^2 \\
& - M(r^{-l} \wedge 1)(\lambda + \gamma r^m)^\alpha \equiv I + II + III - IV \leq 0.
\end{aligned}$$

First, let $r > 1$. We consider separately the cases $R^2 \leq 2r^2$ and $R^2 > 2r^2$. When $R^2 \leq 2r^2$,

$$\begin{aligned} III &\leq \gamma \left[\tfrac{1}{2}m(m-1) + \tfrac{1}{2}(d-1)m - Kr^2)\right] r^{m-2}(R^2-r^2)^2 \\ &\leq \gamma \left[\tfrac{1}{2}m(m+d-2) - K\right] r^{m-2}(R^2-r^2)^2. \end{aligned}$$

Let $K > \frac{1}{2}m(m+d-2)$. Then $III \leq 0$.

$$\begin{aligned} I + II &\leq 2r^2 \left[\tfrac{2(\alpha+1)+d(\alpha-1)}{(\alpha-1)^2}(\lambda+\gamma r^m) + \tfrac{2m}{\alpha-1}\gamma r^m\right] \\ &\leq 2r^2 \left[\tfrac{2(\alpha+1)+d(\alpha-1)}{(\alpha-1)^2}(\lambda+\gamma r^m) + \tfrac{2m}{\alpha-1}(\lambda+\gamma r^m)\right] \\ &\leq 2 \left[\tfrac{2(\alpha+1)+(d+2m)(\alpha-1)}{(\alpha-1)^2}\right] r^2(\lambda+\gamma r^m). \end{aligned}$$

Thus,

$$\begin{aligned} I + II - IV &\leq \left[\tfrac{4(\alpha+1)+2(d+2m)(\alpha-1)}{(\alpha-1)^2}\right] r^2(\lambda+\gamma r^m) - Mr^{-l}(\lambda+\gamma r^m)^\alpha \\ &\leq (\lambda+\gamma r^m)r^2 \left[\tfrac{4(\alpha+1)+2(d+2m)(\alpha-1)}{(\alpha-1)^2} - M\gamma^{\alpha-1}r^{-l-2+(\alpha-1)m}\right] \end{aligned}$$

If we choose $m \geq \frac{l+2}{\alpha-1}$ and $\gamma \geq \left[\frac{4(\alpha+1)+2(d+2m)(\alpha-1)}{M(\alpha-1)^2}\right]^{1/(\alpha-1)}$, then $I + II - IV \leq 0$.

Consider now the case $R^2 \geq 2r^2$. Then

$$\begin{aligned} II &\leq \tfrac{1}{\alpha-1}\left[4m\gamma + 2d(\lambda+\gamma)\right] r^m(R^2-r^2) \\ &\leq \tfrac{1}{\alpha-1}\left[4m\gamma + 2d(\lambda+\gamma)\right] r^{m-2}(R^2-r^2)^2 \end{aligned}$$

$$III \leq \left[\frac{1}{2}m(m+d-2) - K\right]\gamma r^{m-2}(R^2-r^2)^2.$$

Let $K \geq \left[\frac{4m+2d}{\alpha-1} + \frac{1}{2}m(m+d-2) + \frac{2d\lambda}{(\alpha-1)\gamma}\right]$. Then

$$II + III \leq \left[\frac{4m+2d}{\alpha-1}\gamma + \frac{1}{2}m(m+d-2)\gamma + \frac{2d\lambda}{(\alpha-1)} - K\gamma\right] r^{m-2}(R^2-r^2)^2 \leq 0.$$

Note that

$$\begin{aligned} I - IV &= (\lambda+\gamma r^m)\left[\tfrac{4(\alpha+1)}{(\alpha-1)^2}r^2 - Mr^{-l}\left((\lambda+\gamma r^m)^{\alpha-1}\right]\right. \\ &\leq (\lambda+\gamma r^m)r^2\left[\tfrac{4(\alpha+1)}{(\alpha-1)^2} - M\gamma r^{-l-2+m(\alpha-1)}\right]. \end{aligned}$$

Thus for $m \geq \frac{l+2}{\alpha-1}$ and $\gamma \geq \frac{4(\alpha+1)}{(\alpha-1)^2 M}$, we have $I - IV \leq 0$.

Finally, consider the case $r \leq 1$. For λ sufficiently large, we have

$$I - IV \leq \frac{4(\alpha+1)}{(\alpha-1)^2}(\lambda+\gamma) - M\lambda^\alpha \leq 0.$$

Since, for $\lambda > 1$,

$$II + III \leq \frac{1}{\alpha-1}\left[4m\gamma + 2d(\lambda+\gamma)\right]R^2 + \left[\frac{1}{2}m(m+d-2)\gamma - K\lambda\right](R^2-r^2)^2.$$

Let $K > \frac{m(m+d-2)\gamma}{2\lambda}$. Then for sufficiently large R,

$$II + III \leq \frac{1}{\alpha-1}\left[4m\gamma + 2d(\lambda+\gamma)\right]R^2 + \left[\frac{1}{2}m(m+d-2)\gamma - K\lambda\right](R^2-1)^2 \leq 0.$$

In light of the above calculations, for R sufficiently large, the inequality $\sum \leq 0$ for all $0 < r < R$ will be satisfied if m, γ, λ and K are chosen as follows. First, choose $m = \frac{l+2}{\alpha-1}$ and $\gamma = \left[\frac{4(\alpha+1)+2(d+2m)(\alpha-1)}{M(\alpha-1)^2}\right]^{1/(\alpha-1)}$. Then choose $\lambda > 1$ so large that $I - IV \leq 0$ for $r \leq 1$. Finally, let $K > \left[\frac{4m+2d}{\alpha-1} + \frac{1}{2}m(m+d-2) + \frac{2d\lambda}{(\alpha-1)\gamma}\right] \vee \left[\frac{m(m+d-2)\gamma}{2\lambda}\right]$.

The proof of Theorem 1.2 is similar to that of Theorem 1.1. To make it easier for the reader to follow, we give its detailed proof here.

Proof of Theorem 1.2 We only need to prove statement (2). Let $u \geq 0$ be a solution to (C). Define

$$u_R(x,t) = \left[\lambda + \gamma\exp(mr)\right](R^2-r^2)^{-2/(\alpha-1)}e^{Kt}, \quad \|x\| < R, t \geq 0, \tag{2.3}$$

where $r = \|x\|$ and γ, K and λ are to be fixed later. we prove that

$$\frac{1}{2}\Delta u_R - k(x)u_R^\alpha - (u_R)_t \leq 0, \text{ for sufficiently large } R > 0, \tag{2.4}$$

if m, γ, K, λ are sufficiently large. Since $\lim_{r\to R} u_R(x,t) = \infty$, for x, t fixed, and since $u_R(x,0) > 0$, it follows from the parabolic maximum principle that $u(x,t) \leq u_R(x,t)$ for $\|x\| < R, t > 0$. Since $\lim_{R\to\infty} u_R(x,t) = 0$, we conclude that $u \equiv 0$. It remains to prove (2.4). Put

$$f(x) = \left[\lambda + \gamma\exp(mr)\right], \quad g(x) = (R^2-r^2)^{-2/(\alpha-1)}.$$

Then

$$f' = m\gamma\exp(mr) \leq mf, \quad f'' = m^2\gamma\exp(mr) \leq m^2 f$$

$$g' = \frac{4r}{(\alpha-1)}(R^2-r^2)^{-(\alpha+1)/(\alpha-1)} = \frac{4r}{\alpha-1}(R^2-r^2)g^\alpha,$$

$$
\begin{aligned}
g'' &= \tfrac{4}{\alpha-1}(R^2-r^2)^{-(\alpha+1)/(\alpha-1)} + \tfrac{8(\alpha+1)}{(\alpha-1)^2} r^2 (R^2-r^2)^{-2\alpha/(\alpha+1)} \\
&= \tfrac{4}{\alpha-1}(R^2-r^2) g^{\alpha} + \tfrac{8(\alpha+1)}{(\alpha-1)^2} r^2 g^{\alpha}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& e^{-Kt}\left[\tfrac{1}{2}\Delta u_R - k(x)u_R^{\alpha} - (u_R)_t\right] \\
\leq\ & \tfrac{1}{2} f g'' + f'g' + (\tfrac{1}{2}f'' - Kf)g - M\exp(-lr) f^{\alpha} g^{\alpha} \\
\leq\ & f g^{\alpha}\left[\tfrac{4(\alpha+1)}{(\alpha-1)^2}r^2 + \left(\tfrac{2}{\alpha-1}+4mr\right)(R^2-r^2) + (\tfrac{1}{2}m^2 - K)(R^2-r^2)^2 - M\exp(-lr) f^{\alpha-1}\right] \equiv \textstyle\sum.
\end{aligned}
$$

To show that $\sum \leq 0$, it is enough to prove that

$$
\begin{aligned}
& \tfrac{4(\alpha+1)}{(\alpha-1)^2}r^2 + \left(\tfrac{2}{\alpha-1}+4mr\right)(R^2-r^2) + (\tfrac{1}{2}m^2 - K)(R^2-r^2)^2 - M\exp(-lr) f^{\alpha-1} \\
\equiv\ & I + II + III - IV \leq 0.
\end{aligned}
$$

First, consider $r > 1$. We consider separately the cases $R^2 \leq 2r^2$ and $R^2 > 2r^2$. When $R^2 \leq 2r^2$, Let $K \geq \frac{1}{2}m^2$. Then $III \leq 0$.

$$
I + II - IV \leq \tfrac{4(\alpha+1)}{(\alpha-1)^2}r^2 + \tfrac{2r^2}{\alpha-1} + 4mr^3 - M\exp(-lr)\gamma^{\alpha-1}\exp(m(\alpha-1)r)
$$

If we choose $m > l/(\alpha-1)$, and let $\gamma \geq \sup_{r\geq 1}\left[\frac{4(\alpha+1)r^2 + 2(\alpha-1)r^2 + 4m(\alpha-1)^2 r^3}{M(\alpha-1)^2\exp(m(\alpha-1)-l)r}\right]^{1/(\alpha-1)}$, then $I + II - IV \leq 0$.

Consider now the case $R^2 \geq 2r^2$. Let $K \geq \frac{2}{\alpha-1} + 4m + \frac{1}{2}m^2$. Then

$$
II + III \leq \left[\frac{2}{\alpha-1} + 4m + \frac{1}{2}m^2 - K\right](R^2-r^2)^2 \leq 0.
$$

Note that, for $m > \frac{l}{\alpha-1}$ and $\gamma \geq \sup_{r\geq 1}\left[\frac{4(\alpha+1)r^2}{M(\alpha-1)^2\exp(m(\alpha-1)-l)r}\right]^{1/(\alpha-1)}$,

$$
I - IV = \frac{4(\alpha+1)}{(\alpha-1)^2}r^2 - M\exp(-lr)\gamma^{\alpha-1}\exp(m(\alpha-1)r) \leq 0.
$$

Finally, consider the case $r \leq 1$. For λ sufficiently large, we have

$$
I - IV \leq \frac{4(\alpha+1)}{(\alpha-1)^2} - M\exp(-l)\lambda^{\alpha-1} \leq 0.
$$

Since

$$II + III \le \left[\frac{2}{\alpha-1} + 4m\right] R^2 + (\frac{1}{2}m^2 - K)(R^2 - r^2)^2,$$

if let $K > \frac{1}{2}m^2$, then for sufficiently large R,

$$II + III \le \left[\frac{2}{\alpha-1} + 4m\right] R^2 + (\frac{1}{2}m^2 - K)(R^2 - 1)^2 \le 0.$$

In light of the above calculations, for R sufficiently large, the inequality $\sum \le 0$ for all $0 < r < R$ will be satisfied if m, γ, λ and K are chosen as follows. First, choose $m > \frac{l}{\alpha-1}$, $K \ge \frac{2}{\alpha-1} + 4m + \frac{1}{2}m^2$ and $\gamma = \sup_{r\ge1}\left[\frac{4(\alpha+1)r^2+2(\alpha-1)r^2+4m(\alpha-1)^2r^3}{M(\alpha-1)^2\exp(m(\alpha-1)-l)r}\right]^{1/(\alpha-1)}$. Then choose $\lambda > 1$ so large that $I - IV \le 0$ for $r \le 1$.

3 Probabilistic Solutions of Nonlinear Differential Equations on Unbounded Domains (Proof of Theorem 1.3)

In this section, we use D to denote the set (a, ∞) or $(-\infty, a)$ with a being a constant for $d = 1$, an unbounded domain in $\mathbf{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves for $d = 2$, and $\mathbf{R}^d$ for $d \ge 3$. Now we first study probabilistic solutions of

$$\begin{cases} \frac{1}{2}\Delta u = k(x)u^{\alpha}(x), & \text{on } D; \\ u > 0, & \text{on } D; \\ u = 0 & \text{on } \partial D (\text{in the case } d \le 2). \end{cases} \tag{3.1}$$

Zhao discussed Problem (3.1) for $d \ge 3$ and $d = 1$, respectively in 1993 and 1994 (see [11] and [12]). In 1998, Ufuktepe and Zhao[10] discussed problem (3.1) for $d = 2$. They proved existence theorems for problem (1.3) under certain conditions on k for all dimensions. The main tools used by them are probabilistic potential theory and fixed-point theory. Their proofs also hold for a more general nonlinear term. In 1995, Sheu [9], by using the connections between super-Brownian motions and nonlinear differential equations, discussed the structure of all solutions of problem (3.1) for $d \ge 3$. In this section we describe the structures of all positive solutions of problem (3.1) for $d \le 2$. To state our results, let us give some notations.

Let $G_D(x, y)$ denote the Green function for D. If $d = 1$, then $G_{(a,\infty)}(x, y) = 2(|x - a| \wedge |y - a|)$

for $x, y \in D$. For a regular domain D, the Green operator is defined as

$$G_D f(x) = \Pi_x \left[\int_0^{\tau_D} f(W_t)dt \right] = \int_D G_D(x,y)f(y)dy,$$

where, f is a Borel function on D.For $x \in D$, put

$$h(x) = \begin{cases} |x-a|, & d=1; \\ \pi \lim_{y\to\infty} G_D(x,y), & d=2; \\ 1, & d \geq 3. \end{cases} \tag{3.2}$$

For dimension $d = 2$, by Proposition 2.1 in Ufuktepe and Zhao[10], $h(x)$ is a harmonic function on D satisfying

$$\lim_{D \ni x \to z} h(x) = 0 \quad \text{for any } z \in \partial D, \tag{3.3}$$

and

$$\lim_{\|x\|\to\infty} \frac{h(x)}{\log \|x\|} = 1. \tag{3.4}$$

Put

$$g(x) = \begin{cases} \|x\|, & d=1; \\ \log(\|x\|), & d=2; \\ 1, & d=3 \end{cases} \tag{3.5}$$

for large $\|x\| > 1$.

Theorem 3.1 *Let D, h and g be defined as above. Suppose $d \leq 2$ and k satisfies*

$$\int_{\|y\|>a} k(y)g(y)^{\alpha} dy < \infty \tag{3.6}$$

(1) *For every $\mu \in M_c((0,\infty))$, $Z = \lim_{n\to\infty} < h, X_{\tau_{D\cap B(0,n)}} >$ exists P_μ-a.s. and for every $c > 0$,*

$$u_c(x) := -\log P_{\delta_x} \exp\{-cZ\} \tag{3.7}$$

is the unique solution of (3.1) *with condition*

$$\lim_{x\in D, \|x\|\to\infty} \frac{u(x)}{g(x)} = c. \tag{3.8}$$

(2) *If* $u(x)$ *is a solution of* (3.1) *and satisfies* $\limsup_{x\in D,\|x\|\to\infty}\frac{u(x)}{g(x)}<\infty$, *then* $u=u_c$ *for some* $c>0$.

(3)

$$J(x):=-\log P_{\delta_x}(Z=0) \tag{3.9}$$

is the smallest solution to problem (3.1) *with condition*

$$\lim_{x\in D,\|x\|\to\infty}\frac{u(x)}{g(x)}=\infty, \tag{3.10}$$

and

$$I(x):=-\log P_{\delta_x}(X_{\tau_{D\cap B(0,n)}}=0 \text{ for } n \text{ sufficiently large}) \tag{3.11}$$

is the largest solution to problem (3.1) *with condition (3.10).*

Sheu[9] discussed Problem (3.1) for $d\geq 3$. We state Sheu's result as follows:

Theorem 3.2 *Suppose* $d\geq 3$, $k(x)\leq \overline{k}(\|x\|)$ *for* x *sufficiently large, and* $\int_a^\infty s\overline{k}(s)<\infty$ *for some* $a>0$. *The result of Theorem 3.1 holds for* $D=\mathbf{R}^d$ *with* $d\geq 3$.

To prove Theorem 3.1, we quote some lemmas, in which the first two lemmas are refer to Sheu[9] and Wang[8].

Lemma 3.1 *Suppose* $k(x)$ *satisfies (3.6). For every sequence of open sets* $\{D_n\}$ *satisfying* $D_n\uparrow D$, *there exists a random variable* Z *such that* $Z=\lim_{n\to\infty}<h,X_{\tau_{D_n}}><\infty$, P_μ*-a.s. for every* $\mu\in M_c(D)$. *Moreover, the limit does not depend on* μ *and the choice of* D_n.

Lemma 3.2 *Suppose* $d\leq 2$. *Let* $B(0,r)=\{x\in\mathbf{R}^2,\|x\|<r\}$ *be a ball such that* $D^c\subset B(0,r)$ *for* $d=2$. *If* u_n *is a sequence of solutions of differential equation* $\frac{1}{2}\Delta u=ku^\alpha$ *on* $D\cap B(0,r)$ *and* $u=\lim_{n\to\infty}u_n$ *on* $D\cap B(0,r)$, *then* u *is also a solution of* $\frac{1}{2}\Delta u=ku^\alpha$ *on* $D\cap B(0,r)$. *Moreover, if* u_n *has boundary value* u_n *at* ∂D, *and* $\lim_{n\to\infty}u_n(z)$ *exists for every* $z\in\partial D$, *then* u *has boundary value* $\lim_{n\to\infty}u_n(z)$ *at* z.

Lemma 3.3 *If* g *is a positive bounded integrable function on* D. *Then*

(1) $G_Dg\in C^{0,\lambda}(D)$;

(2) if $g\in C^{0,\lambda}(D)$, *then* $G_Dg\in C^{2,\lambda}(D)$ *and* $\frac{1}{2}\Delta G_Dg=-g$ *in* D.

Lemma 3.3 is Theorem 4.6.6 in Port and Stone[7] with some modifications.

Now let us give two results for dimension $d=2$.

Pick a fixed point $a\in\mathbf{R}^2\setminus\overline{D}$ and $r>0$ (small enough) such that $D\supset B_r^*=\mathbf{R}^2\setminus\overline{B(a,r)}$. Let $x^*=a+r^2\frac{(x-a)}{\|x-a\|^2}$ be the Kelvin inversion from $D\cup\{\infty\}$ to D^*, where $D^*=\{x^*\in B(a,r):x\in D\cup\{\infty\}\}$. Using the explicit formula of $G_{B_r^*}(\cdot,\cdot)$, we can prove:

Lemma 3.4 Suppose $d = 2$. For positive bounded function k, the family of functions $\{G_D(x,\cdot)k(\cdot)h^{\alpha-1}(\cdot)\}$ with parameter $x \in D$ is uniformly integrable over D.

Lemma 3.5 *Suppose $d = 2$. If $f > 0$ is a harmonic function having boundary value 0 on ∂D, then $f = ch$ on D for some constant $c > 0$ with h given by (3.2).*

Proof The Kelvin transformation of f relative to $S_r(a)$ is $f^*(x^*) = f(x)$, where $x = a + \frac{r^2}{\|x^*-a\|^2}(x^*-a)$. $f^*(x^*)$ is a positive harmonic function on $D^*\backslash\{a\}$. Letting $f^*(a) = \liminf_{x^*\to a} f^*(x^*)$, f^* is a super-harmonic function on D^* having boundary value 0 on ∂D^*. By the Riesz decomposition theorem and Theorem 6.1.4 in Port and Stone[7], there exists a constant $c > 0$ such that $f^*(x^*) = cG_{D^*}(x^*, a)$ for $x^* \in D^*$ and hence $f(x) = ch(x)$ for $x \in D$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1 We only give a proof for $d = 2$. The proof for $d = 1$ is similar.

(1) By (1.4) and (1.5),

$$u_{c,n}(x) = -\log P_{\delta_x} \exp < -ch, X_{\tau_{D\cap B(0,n)}} > \tag{3.13}$$

satisfies the following integral equation:

$$u_{c,n}(x) + \Pi_x \int_0^{\tau_{D\cap B(0,n)}} k(W_s)u^\alpha_{c,n}(W_s)\mathrm{d}s = ch(x), \quad x \in D \cap B(0,n). \tag{3.14}$$

By Lemma 3.1, $u_c(x) = \lim_{n\to\infty} u_{c,n}(x), x \in D$. By (3.4) and (3.6), $k(y)h^\alpha(y)$ is integrable in D. Thus $\int_D G_D(x,y)k(y)h^\alpha(y)dy < \infty$. Letting $n \to \infty$ in (3.14), by dominated convergence, we have

$$u_c(x) + G_D(ku^\alpha_c) = ch(x), \quad x \in D. \tag{3.15}$$

By Lemma 3.3(1), $G_D(ku^\alpha_c) \in C^{0,\lambda}(D)$. Then by (3.15), $u_c \in C^{0,\lambda}(D)$. Using Lemma 3.3(2), $G_D(ku^\alpha_c) \in C^{2,\lambda}(D)$ and $\frac{1}{2}\Delta G_D(ku^\alpha_c) = -ku^\alpha_c$, and therefore $\frac{1}{2}\Delta u^\alpha_c = ku^\alpha_c$ in D. Now we check that $\lim_{\|x\|\to\infty} \frac{u_c(x)}{h(x)} = c$. It is enough to prove that

$$\lim_{\|x\|\to\infty} \frac{G_D(ku^\alpha_c)}{h(x)} = 0. \tag{3.16}$$

Note that

$$\frac{G_D(ku^\alpha_c)}{h(x)} \le c^\alpha \int_D \frac{G_D(x,y)k(y)h^\alpha(y)}{h(x)}. \tag{3.17}$$

By the (3-G) inequality for Green functions on D (see Theorem 2.2 in Ufuktepe and Zhao[10]), there exists a constant $C > 0$ such that

$$\frac{G_D(x,y)k(y)h^\alpha(y)}{h(x)} = \pi \lim_{z\to\infty} \frac{G_D(x,y)G_D(y,z)k(y)h^{\alpha-1}(y)}{G_D(x,z)}$$

$$\le \lim_{z\to\infty} C(G_D(x,y)+G_D(y,z)+1)k(y)h^{\alpha-1}(y)$$

$$\le C(G_D(x,y)+h(y)+1)k(y)h^{\alpha-1}(y).$$

Since $\int_D (h(y)+1)k(y)h^{\alpha-1}(y)dy < \infty$ and the family of functions $G_D(x,\cdot)k(\cdot)h^{\alpha-1}(\cdot)$ with parameter $x \in D$ is uniformly integrable over D by Lemma 3.4, the family of functions $\frac{G_D(x,\cdot)k(\cdot)h^\alpha(\cdot)}{h(x)}$ is uniformly integrable over D. It is obvious that $\frac{G_D(x,y)k(y)h^\alpha(y)}{h(x)} \to 0$ as $x \to \infty$. Thus (3.16) holds by (3.17).

Suppose $u(x)$ is a solution of (3.1) satisfying $\lim_{x\to\infty} \frac{u(x)}{h(x)} = c$. Then $\int_D G_D(x,y)k(y)u^\alpha(y)\mathrm{d}y < \infty$, and $\lim_{x\to\infty} \frac{\int_D G_D(x,y)k(y)u^\alpha(y)\mathrm{d}y}{h(x)} = 0$. Let $\overline{h}(x) = u(x) + \int_D G_D(x,y)k(y)u^\alpha(y)\mathrm{d}y$. Then $\overline{h}$ is a positive harmonic function on D having boundary value 0 at ∂D and satisfies $\lim_{x\to\infty} \frac{\overline{h}(x)}{h(x)} = c$. Hence, by Lemma 3.5, $\overline{h}(x) = ch(x)$, which means that

$$u(x) + \int_D G_D(x,y)k(y)u^\alpha(y)\mathrm{d}y = ch(x). \tag{3.18}$$

By the maximum principle,

$$u(x) = -\log P_{\delta_x} \exp < -u, X_{\tau_{D\cap B(0,n)}} > . \tag{3.19}$$

On one hand,

$$u(x) \le -\log P_{\delta_x} \exp < -ch, X_{\tau_{D\cap B(0,n)}} > . \tag{3.20}$$

On the other hand, by (3.19), we have, for n large enough (such that $D^c \subset B(0,n)$),

$$\begin{aligned} u(x) &= -\log P_{\delta_x} \exp < (-ch + G_D k u^\alpha), X_{\tau_{D\cap B(0,n)}} > \\ &= -\log P_{\delta_x} \exp < -chI_{\partial B(0,n)} \left(1 - \frac{G_D(ku^\alpha)}{ch}\right), X_{\tau_{D\cap B(0,n)}} > . \end{aligned}$$

Since $\lim_{\|x\|\to\infty} \frac{G_D(ku^\alpha)(x)}{h(x)} = 0$, we have, for every $\epsilon > 0$, there exists an integer N such that $\frac{G_D(ku^\alpha)(x)}{ch(x)} \le \epsilon$ for $n > N$ and $x \in \partial B(0,n)$, and hence

$$u(x) \ge -\log P_{\delta_x} \exp < -c(1+\epsilon)h, X_{\tau_{D\cap B(0,n)}} > .$$

Letting $\epsilon \to 0$, we get

$$u(x) \ge -\log P_{\delta_x} \exp(-cZ). \tag{3.21}$$

Combining (3.20) and (3.21), we get $u(x) = -\log P_{\delta_x} \exp(-cZ)$.

(2) If u is a solution of (3.1) satisfying $\limsup_{x\to\infty} \frac{u(x)}{h(x)} < \infty$, using the same method as above, we can prove that (3.18) holds for some constant $c > 0$, and then $u(x) = -\log P_{\delta_x} \exp(-cZ)$.

(3) Since $J(x) = \lim_{c\to\infty} u_c(x)$, it follows from Lemma 2.2 that $J(x)$ is a solution to problem (3.1). It is obvious that $\lim_{x\to\infty} \frac{J(x)}{h(x)} = \infty$. By the maximum principle, J is the minimal solution to problem (3.1) with $\lim_{x\to\infty} \frac{u(x)}{h(x)} = \infty$.

For large n (such that $D^c \subset B(0,n)$), put

$$I_n(x) = -\log P_{\delta_x}(X_{\tau_{D\cap B(0,n)}}(\partial B(0,n)) = 0), \quad x \in D \cap B(0,n).$$

Note that $-\log P_{\delta_x} \exp(-\lambda X_{\tau_{D\cap B(0,n)}}(\partial B(0,n)) \uparrow I_n(x)$ as $n \uparrow \infty$. Using Lemma 3.2, we have I_n is a solution of $u'' = ku^\alpha$ on $D \cap B(0,n)$ having boundary value 0 at ∂D and boundary value ∞ at $\partial B(0,n)$. Since $I(x) = \lim_{n\to\infty} I_n(x)$, by Lemma 3.2, I is a solution to problem (3.1). The maximum principle implies that I is the largest solution to problem (3.1). □

Theorem 3.3 *If $d = 1$ and $\int_a^\infty k(y)y^\alpha dy < \infty$(or $\int_{-\infty}^{-a} k(y)y^\alpha dy < \infty$) for some constant $a > 0$, then there is a non-negative solution of equation* (E) *in $\mathbf{R}^1$ satisfying*

$$\lim_{x\to\infty} \frac{u(x)}{x} = \infty \quad (\ or \lim_{x\to-\infty} \frac{u(x)}{-x} = \infty). \tag{3.22}$$

Proof Suppose $\int_a^\infty k(y)y^\alpha dy < \infty$. (The other case can be proved similarly.) By Theorem 3.2, $J(x) = -\log P_{\delta_x}(\lim_{n\to\infty} nX_{\tau_{(a,n)}}(n) = 0)$ is a solution of $\frac{1}{2}u'' = k(x)u^\alpha$ in (a,∞) satisfying $J(a) = 0$ and $\lim_{x\to\infty} \frac{J(x)}{x} = \infty$. Let $u_n(x) = -\log P_{\delta_x}(X_{\tau_{(-n,n)}} = 0)$. Then u_n is a nonnegative solution of

$$\begin{cases} \frac{1}{2}u'' = ku^\alpha, & \text{in } (-n,n) \\ u(-n) = u(n) = \infty \end{cases}.$$

By the elliptic maximum principle, for $n > a$,

$$u_n(x) \geq J(x), \quad x \in (a,n)$$

Letting $n \to \infty$, $u := \lim_{n\to\infty} u_n(x) \geq J(x)$ in (a,∞). Therefore, u is a nonnegative solution of $\frac{1}{2}u'' = ku^\alpha$ in $\mathbf{R}^1$ satisfying $\lim_{x\to\infty} \frac{u}{x} = \infty$.

Theorem 3.4 *If $d = 2$, $k(x) \leq \overline{k}(\|x\|)$ for x sufficiently large, and $\int_a^\infty s\overline{k}(s)(\log(1+s))^\alpha ds < \infty$ for some constant $a > 0$, then there is a positive solution of equation* (E) *in $\mathbf{R}^2$ satisfying*

$$\lim_{\|x\|\to\infty} \frac{u(x)}{\ln(\|x\|)} = \infty \quad . \tag{3.23}$$

Proof Let $u_n(x) = -\log P_{\delta_x}(X_{\tau_{B(0,n)}} = 0)$. Then u_n is a nonnegative solution of

$$\begin{cases} \frac{1}{2}\Delta u = ku^\alpha, & \text{in } B(0,n) \\ u|_{\partial B(0,n)} = \infty \end{cases}.$$

u_n is decreasing in n. Put $u(x) = \lim_{n\to\infty} u_n(x), x \in \mathbf{R}^2$. u is a solution of $\frac{1}{2}\Delta u = ku^\alpha$ in $\mathbf{R}^2$. For every fixed $x_0 \in \mathbf{R}^d$, there exists a ball $B(x_1, r_0)$ such that $x \in \overline{B}^c(x_1, r_0)$. By Theorem 3.1(3), there is a solution J of $\frac{1}{2}\Delta u = k(x)u^\alpha$ in $B^c(x_1, r_0)$ satisfying $J = 0$ on $\partial B(x_1, r_0)$ and

$$\lim_{\|x\|\to\infty} \frac{J(x)}{\ln(\|x\|)} = \infty. \tag{3.24}$$

By the elliptic maximum principle, for n sufficiently large (such that $B(x_1, r_0) \subset B(0, n)$ and $x_0 \in B(0, n)$),

$$u_n(x) \geq J(x), \quad x \in B(0,n) \setminus B(x_1, r_0) \tag{3.25}$$

Letting $n \to \infty$, $u \geq J(x)$ in $B^c(x_0, r_0)$. Particularly, $u \geq J > 0$ in a neighborhood of x_0. Since x_0 is an arbitrary point of $\mathbf{R}^d$, u is a positive solution of $\frac{1}{2}\Delta u = ku^\alpha$ in $\mathbf{R}^2$. By (3.24) and (3.25), u satisfies $\lim_{\|x\|\to\infty} \frac{u(x)}{\ln(\|x\|)} = \infty$.

The following Theorem 3.5 gives the nonexistence result. For details see Cheng and Lin [2].

Theorem 3.5 *If k satisfies* (1.9)*, then E does not possess any positive solution in $\mathbf{R}^d$.*

Proof of Theorem 1.3 The results of Theorem 1.3 follow from Theorems 3.1 - 3.5.

4 Finite Time Extinction (Proof of Theorem 1.4)

To prove Theorem 1.4, we need the following Theorem 4.1, which is proved in [5].

Theorem 4.1 (1) *There exists a nonnegative function $\omega(x)$, which solves the equation $\frac{1}{2}\Delta u = ku^\alpha$ on $\mathbf{R}^d$, and for which*

$$P_\mu(X(\cdot) \text{ is extinct}) = \exp\left(-\int_{\mathbf{R}^d} \omega(x)\mu(dx)\right), \mu \in M_c(\mathbf{R}^d). \tag{4.1}$$

Moreover, $\omega(x) = \lim_{t\to\infty} \lim_{n\to\infty} u_n(x,t)$*, where* $u_n(t,x)$ *is the minimal nonnegative solution to*

$$\begin{cases} u_t = \frac{1}{2}\Delta u - ku^\alpha & \text{in } \mathbf{R}^d \times (0,\infty) \\ u(\cdot,0) \equiv n. \end{cases} \tag{4.2}$$

Furthermore, ω *is either identically zero or positive everywhere in* $\mathbf{R}^d$.

(2) *Let* C *denote the event that the range of the process is compactly embedded in* $\mathbf{R}^d$*, that is*

$$C \equiv \left\{\cup_{0\le s\le\infty} \mathrm{supp} X_s \subset\subset \mathbf{R}^d\right\}.$$

There exists a maximal nonnegative solution to $\frac{1}{2}\Delta u = ku^\alpha$ *on* $\mathbf{R}^d$, ω_{max}*, and*

$$P_\mu(C) = \exp\left(-\int_{\mathbf{R}^d} \omega_{max}\mu(dx)\right), \quad \forall \mu \in M_c(\mathbf{R}^d). \tag{4.3}$$

also

$$P_\mu(C \cap \{X(\cdot) \text{ is survival }\}) = 0, \quad \forall \mu \in M_c(\mathbf{R}^d). \tag{4.4}$$

(3) *If the compact support property holds, then* $\omega \equiv \omega_{max}$.

Proof of Theorem 1.4 (1) Suppose $\overline{k}(x) = \overline{k}(\|x\|)$ is a strictly positive Hölder continuous function in $\mathbf{R}^d$ satisfying

$$\overline{k}(\|x\|) = \begin{cases} c/\left[\|x\|^{1+\alpha+\epsilon}\right], & d = 1; \\ c/\left[\|x\|^2(\log\|x\|)^{1+\alpha+\epsilon}\right], & d = 2; \\ c/\|x\|^{2+\epsilon}, & d \ge 3 \end{cases} \tag{4.5}$$

for $\|x\|$ sufficiently large, and

$$k(y) \le \overline{k}(x) = \overline{k}(\|x\|), \quad \text{for all } x \in \mathbf{R}^d. \tag{4.6}$$

Let $\overline{X}$ be the super-Brownian motion corresponding to the differential equation $\frac{1}{2}\Delta u = \overline{k}u^\alpha$. By Theorem 1.1, $\overline{X}$ has the compact support property. Then, by Theorem 4.1(2), $\omega^{\overline{X}}(x) = \omega^{\overline{X}}_{max}(x)$ in $\mathbf{R}^d$. By Theorem 1.3,

$$\lim_{\|x\|\to\infty} \frac{\omega^{\overline{X}}(x)}{g(x)} = \lim_{\|x\|\to\infty} \frac{\omega^{\overline{X}}_{max}(x)}{g(x)} = \infty \tag{4.7},$$

where $g(x)$ is defined by (3.5). By Theorem 4.1(1),

$$\omega^X(x) = \lim_{t\to\infty} \lim_{n\to\infty} u^X_n(t,x) \tag{4.8}$$

and

$$\omega^{\overline{X}}(x) = \lim_{t\to\infty} \lim_{n\to\infty} u_n^{\overline{X}}(t,x), \tag{4.9}$$

where $u_n^X(x,t)$ is the minimal nonnegative solution to (3.12), and $u_n^{\overline{X}}(x,t)$ is the minimal nonnegative solution to (4.5) with k replaced by $\overline{k}$. By the parabolic maximum principle and the proof of Lemma A1 in [5],

$$u_n^X(x,t) \geq u_n^{\overline{X}}(x,t). \tag{4.10}$$

Hence, by (4.8), (4.9) and (4.10),

$$\omega^X \geq \omega^{\overline{X}} \quad \text{in } \mathbf{R}^d \qquad (4.11).$$

Therefore, by (4.7) and (4.11),

$$\lim_{\|x\|\to\infty} \frac{\omega^X}{g(x)} = \infty, \tag{4.12}$$

and X survives.

(2) This result is obvious by Theorem 3.5 and Theorem 4.1.

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