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# On States of Total Weighted Occupation Times for Superdiffusions

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**Abstract** Suppose $X$ is a superdiffusion in $\mathrm{R}^d$ with general branching mechanism $\psi$, and $Y_{\tau_D}$ denotes the total weighted occupation time of $X$ in a bounded smooth domain $D$. We discuss the conditions on $\psi$ to guarantee that $Y_{\tau_D}$ has absolutely continuous states. And for particular $\psi_(z) = z^{1+\beta}, 0 < \beta \leq 1$, we prove that, in the case $d < 2 + 2/\beta$, $Y_{\tau_D}$ is absolutely continuous with respect to the Lebesgue measure in $\overline{D}$, whereas in the case $d > 2 + 2/\beta$, it is singular. As we know the absolute continuity and singularity of $Y_{\tau_D}$ have not been discussed before.

**Keywords** Total weighted occupation time, Superdiffusion, Absolutely continuous state, Singular states
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## 1 Introduction and Main Results

Let $L$ be a uniformly elliptic differential operator in $\mathrm{R}^d$, $\xi := \{\xi_s, \Pi_x, s \geq 0, x \in \mathrm{R}^d\}$ denote the diffusion in $\mathrm{R}^d$ with generator $L$, and let

$$\psi(x, z) = a(x)z + b(x)z^2 + \int_0^\infty (e^{-uz} - 1 + uz)n(x, du), \tag{1.1}$$

where $n$ is a kernel from $\mathrm{R}^d$ to $(0, \infty)$ and $a(x), b(x)$ and $\int_0^\infty u \wedge u^2 n(x, du)$ are positive bounded Borel functions on $\mathrm{R}^d$.

For every Borel-measurable space $(E, \mathcal{B}(E))$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}(E)$ endowed with the topology of weak convergence. The expression $\langle f, \mu\rangle$ stands for the integral of $f$ with respect to $\mu$. We write $f \in \mathcal{B}(E)$ if $f$ is a $\mathcal{B}(E)$-measurable function. Writing $f \in p\mathcal{B}(E)(b\mathcal{B}(E))$ means that, in addition, $f$ is positive (bounded). We put

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$bp\mathcal{B}(E) = b\mathcal{B}(E) \cap p\mathcal{B}(E)$. If $E = \mathrm{R}^d$, we simply write $\mathcal{B}$ instead of $\mathcal{B}(E)$ and $M$ instead of $M(\mathrm{R}^d)$.

We denote by $\mathcal{T}$ the set of all exit times from open sets in $\mathrm{R}^d$. Set $\mathcal{F}_{\le r} = \sigma(\xi_s, s \le r)$; $\mathcal{F}_{>r} = \sigma(\xi_s, s > r)$ and $\mathcal{F}_\infty = \vee\{\mathcal{F}_{\le r}, r \ge 0\}$. For $\tau \in \mathcal{T}$, we put $F \in \mathcal{F}_{\ge\tau}$ if $F \in \mathcal{F}_\infty$ and if, for each $r$, $\{F, \tau > r\} \in \mathcal{F}_{>r}$.

According to Dynkin [1] there exists a Markov process $X = (X_t, P_\mu)$ in $M$ such that the following conditions are satisfied:

(a) If $f$ is a bounded continuous function, then $\langle f, X_t\rangle$ is right continuous in $t$ on $\mathrm{R}^+$;

(b) For every $\nu \in M$ and for every $f \in bp\mathcal{B}$,

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v_t, \mu\rangle, \quad \mu \in M, \tag{1.2}$$

where $v$ is the unique solution of the integral equation

$$v_t(x) + \Pi_x\Big[\int_0^t \psi(\xi_s, v_{t-s}(\xi_s))ds\Big] = \Pi_x f(\xi_t). \tag{1.3}$$

Moreover, for every $\tau \in \mathcal{T}$, there are corresponding random measures $X_\tau$ and $Y_\tau$ on $\mathrm{R}^d$ associated with the first exit time $\tau$ such that, for $f, g \in bp\mathcal{B}$,

$$P_\mu \exp\{-\langle f, X_\tau\rangle - \langle g, Y_\tau\rangle\} = \exp\langle -u, \mu\rangle, \quad \mu \in M, \tag{1.4}$$

where $u$ is the unique solution of the integral equation

$$u(x) + \Pi_x\Big[\int_0^\tau \psi(\xi_s, u(\xi_s))ds\Big] = \Pi_x\Big[f(\xi_\tau) + \int_0^\tau g(\xi_s)ds\Big]. \tag{1.5}$$

We call $X = \{X_t, X_\tau, Y_\tau; P_\mu\}$ the superdiffusion with branching mechanism $\psi$ (enhanced model). Throughout this paper $\tau_D$ denotes the first exit time of $\xi$ from an open set $D$ in $\mathrm{R}^d$, i.e., $\tau_D = \inf\{t : \xi_t \notin D\}$. And we call $Y_\tau$ the total weighted occupation time of $X$ in $D$. It is obvious that the support of $Y_{\tau_D}$ is contained in $\overline{D}$.

For bounded smooth domains $D$, the states of exit measures $X_{\tau_D}$ have been studied by some authors. When $\psi(x,z) = z^{1+\beta}, 0 < \beta \le 1$, the states of exit measures $X_\tau$ were studied by Sheu [2]. Ren [3] discussed the absolute continuity of $X_{\tau_D}$ with general branching mechanism $\psi$ and general branching rate function $A(dt)$. (In this paper, the branching rate function is $dt$). But as we know the absolute continuity of $Y_{\tau_D}$ has not been discussed before. And the studying of states of $Y_{\tau_D}$ should have the same importance as that of states of $X_{\tau_D}$. So in this paper we discuss when the states of $Y_{\tau_D}$ are absolutely continuous with respect to the Lebesgue measure in $\overline{D}$ and when they are singular.

Before giving the statements of the main results of this article, we introduce some notations. From this point on we always assume that $D$ is a bounded smooth domain in $\mathrm{R}^d$. Let $G_D(x,y)$ denote the Green function of the diffusion $\xi$ in $D$. For $f \in b\mathcal{B}(D)$ and $\nu \in M_1(D)$, define

$$G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s)ds = \int_D G_D(x,y)f(y)dy; \quad G_D\nu(x) = \int_D G_D(x,y)\nu(dy).$$

Obviously, if $\nu(dy) = f dy$, $G_D f = G_D\nu$.

We write $\mu \in M_c(D)$ if $\mu \in M(D)$ and has a compact support in $D$, and $\mu \in M_0(D)$ if $\mu \in M(D)$ and $\mu$ has finite points support. Let $M_1(D)$ denote the set of all measures $\nu$ in $M(D)$ such that $G_D\nu$ is super-harmonic in $D$. Set $N_\nu = \{x, G_D\nu(x) = \infty\}$. Then $N_\nu$ is a closed set having zero Lebesgue measure. Clearly $M_0(D) \subset M_1(D)$ and for $\nu \in M_0(D)$, $N_\nu = \operatorname{supp}\nu$. For $\nu = \sum_{i=1}^m \lambda_i \delta_{y_i}$, $y_1, \ldots, y_m \in D$, let

$$\nu_n(dy) = f_n(y)dy, \tag{1.6}$$

where

$$f_n(y) = \sum_{i=1}^m \lambda_i f_n^{y_i}(y); \quad f_n^{y_i}(y) = \begin{cases} \dfrac{1}{V(B(y_i, 1/n))}, & y \in B(y_i, 1/n), \\ 0, & y \notin B(y_i, 1/n), \end{cases} \tag{1.7}$$

$V(B(y_i, 1/n))$ being the volume of $B(y_i, 1/n)$. Clearly as $n \to \infty$, $\nu_n$ converges weakly to $\nu$.

Consider the following integral equation:

$$u(x) + \Pi_x \int_0^{\tau_D} \psi(\xi_s, u(\xi_s))dt = G_D\nu(x), \quad x \in D \backslash N_\nu, \tag{1.8}$$

where $\nu \in M_1(D)$.

**Theorem 1.1** *Assume that there exist a sequence of bounded smooth domains $D_n$ satisfying $D_n \uparrow D$ and a Borel subset $N$ of Lebesgue measure 0 such that, for all $\nu \in M_0(D)$ with finite points support contained in $D \backslash N$, $\psi$ satisfies one of the following conditions* (C1) *and* (C2):

(C1) *$\psi$ is given by* (1.1) *with $a(x) \equiv 0$ and,*

$$\limsup_{n\to\infty} \Pi_x \int_{\tau_{D_k}}^{\tau_D} (G_D\nu_n(\xi_s))^2 ds \to 0 (k \to \infty), \quad \textit{for } x \in D \backslash N_\nu, \tag{1.9}$$

*with $\nu_n$ defined by* (1.6)*;*

(C2) *$\psi$ is given by the particular form:*

$$\psi(x, z) = \gamma(x) z^{1+\beta}, \quad 0 < \beta \le 1, \tag{1.10}$$

*with $\gamma \in bp\mathcal{B}$, and*

$$\limsup_{n\to\infty} \Pi_x \int_{\tau_{D_k}}^{\tau_D} (G_D\nu_n(\xi_s))^{1+\beta} ds \to 0 (k \to \infty), \quad \textit{for } x \in D \backslash N_\nu, \tag{1.11}$$

*with $\nu_n$ defined by* (1.6).

*Then we have:*

(1) *For fixed $\mu \in M_c(D) \cap M_1(D)$, there exists a random measurable function $y_D$ defined on $\overline{D}$ such that*

$$P_\mu\{Y_{\tau D}(dy) = yD(y)dy\} = 1;$$

(2) *For each finite collection $y_1, \ldots, y_\kappa$ of points in $D \backslash N \backslash N_\mu$, the Laplace function of the random vector $[y_D(y_1), \ldots, y_D(y_m)]$ with respect to $P_\mu$ is given by*

$$P_\mu \exp\Big[ -\langle f, Y_{\tau_D}\rangle - \sum_{i=1}^{\kappa} \lambda_i y_D(y_i)\Big] = \exp\langle -u, \mu\rangle, \qquad \lambda_1, \ldots, \lambda_\kappa \ge 0, \tag{1.12}$$

*where* $\mu \in M_c(D) \cap M_1(D)$, *and* $u$ *is the unique positive solution of* (1.8) *with* $\nu(dy) = f(y)dy + \sum_{i=1}^{\kappa} \lambda_i \delta_{yi}(dy)$.

**Theorem 1.2** (1) *If* $\psi(x,z)$ *is given by the general form* (1.1) *with* $a(x) \equiv 0$, *then in the case* $d \le 3$, *condition* (1.9) *holds for all* $\nu \in M_0(D)$.

(2) *If* $\psi(x,z)$ *is given by the form* (1.10), *then in the case* $d < 2 + 2/\beta$, *condition* (1.11) *holds for all* $\nu \in M_0(D)$.

*Therefore, under one of the above two conditions, the states* $Y_{\tau_D}$ *are absolutely continuous.*

**Theorem 1.3** *Assume that* $\psi(x,z) = z^{1+\beta}, 0 < \beta \le 1$ *and* $d > 2+2/\beta$. *For every* $\mu \in M_c(D)$, $Y_{\tau_D}$ *is* $P_\mu$*-a.s. singular with respect to the Lebesgue on* $\overline{D}$.

Fundamental solutions of the integral equation (1.8) play an important role in the investigation of absolute continuity of $Y_{\tau_D}$. So, in Section 2 we first discuss fundamental solutions of (1.8). The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 3. The last section is devoted to the proof of Theorem 1.3.

Throughout this article the notation $C$ always denotes a constant which may change values from line to line.

## 2 Fundamental Solutions

For $c \in p\mathcal{B}$ put

$$H^c(r_1, r_2) = \exp\Big( - \int_{r_1}^{r_2} c(\xi_s)ds \Big), \quad 0 \le r_1 \le r_2.$$

**Theorem 2.1** (Fundamental Solutions) *Suppose* $\psi$ *is given by* (1.1), $D$ *is a bounded smooth domain. Let* $\nu \in M_0(D)$ *and let* $\nu_n$ *and* $f_n$ *be defined by* (1.6) *and* (1.7), *respectively. Assume that there exists a sequence of bounded smooth domains* $\{D_n\}$ *satisfying* $D_n \uparrow D$ *as* $n \uparrow \infty$ *such that condition* (1.9) *holds.*

(1) (Existence and Uniqueness) *There is exactly one measurable non-negative function* $U[\nu]$ *defined on* $D$ *which satisfies Eq.* (1.8).

(2) (Continuity of Regularization) *The solution* $U[\nu]$ *is continuous with respect to the operation of regulation of* $\nu$ *in the following sense:*

$$U[\nu_n](\cdot) \xrightarrow{bp} U[\nu](\cdot)(n \to \infty) \quad \text{in each compact subset } K \text{ of } D \setminus N_\nu. \tag{2.1}$$

(3) (First Derivative with Respect to Small Parameter) *If* $a(x) \equiv 0$ *in the formula of* $\psi$ *given by* (1.1), *then*

$$\lambda^{-1} U[\lambda\nu](\cdot) \xrightarrow{bp} G_D\nu(\cdot)(\lambda \to \infty) \quad \text{in each compact subset } K \text{ of } D \setminus N_\nu. \tag{2.2}$$

*Proof* For simplicity, we write $\tau$ and $\tau_n$ in parallel with $\tau_D$ and $\tau_{D_n}$ respectively.

Assume that, for each $n$, $u_n$ is a non-negative solution of the cumulate Eq. (1.8) with $\nu$ replaced by $\nu_n$. We want to show that, for any compact subset $K$ of $D\backslash N_\nu$, there exists an integer $n_0$, such that $u_n$ is uniformly bounded in $K$, and for fixed $x \in K$, $\{u_n(x); n \geq 1\}$ is a Cauchy sequence. Note that there is a constant $C$ depending only on $D$ such that

$$G_D(x,y) \leq C_\rho(x)\rho(y)\|x-z\|^{-d}, \quad x \in D, \; y \in D, \tag{2.3}$$

where $\rho(x) = d(x, \partial D)$. First of all, for a fixed compact subset $K$ of $D \setminus N_\nu$, there exists an integer $n_0$ such that, for $n \geq n_0, f_n = 0$ in a neighborhood of $K$. Thus we have the following domination:

$$0 \leq u_n(x) \leq \int_D G_D(x,y) f_n(y) \mathrm{d}y \leq C \int_D f_n(y) dy = C\nu_n(D) \leq C, \quad \text{for } x \in K, n \geq n_0. \tag{2.4}$$

Therefore, for every $k$, there exists an integer $n_k$ such that, for $n \geq n_k, f_n = 0$ and $G_D f_n$ are uniformly bounded in $D_k$. Let

$$R_k(x,z) = c_k(x)z - \psi(x,z); \quad c_k(x) = (a + \lambda_k)(x);$$

$$\lambda_k(x) = 2b(x) + \int_0^\infty u \wedge u^2 n(x,du) M_k; \quad M_k = (\sup_{x \in D_k, n \geq n_k} G_D f_n(x)) \vee 1.$$

Then

$$|R_k(x,z_1) - R_k(x,z_2)| \leq \lambda_k(x)|z_1 - z_2|, \quad x \in \mathrm{R}^d, \quad 0 \leq z_2, \quad z_1 \leq M_k. \tag{2.5}$$

(See Ren [3].) Using Lemma 2.1 in Ren [3] with $c = c_k$, $F = \int_{\tau_k}^{\tau} (f_n - \psi(\xi_s, u_n(\xi_s)))ds$, $g(x) = u_n(x)$ and $\omega(x) = f_n(x) + R_k(x, u_n(x))$, and by noticing that, for all $n \geq n_k, f_n = 0$ in $D_k$, we get

$$u_n(x) = \Pi_x H^{c_k}(0,\tau_k) \left[ \int_{\tau_k}^{\tau} (f_n(\xi_s) - \psi(\xi_s, u_n(\xi_s)))ds \right] + \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) R_k(\xi_s, u_n(\xi_s))ds.$$

For $x \in \mathrm{R}^d$, put

$$q_{m,n}(x) = \psi(x, G_D f_m(x)) + \psi(x, G_D f_n(x)). \tag{2.6}$$

Note that $u_n(x) \leq M_k$ for $n \geq n_k$, $x \in D_k$. By (2.5) we have, for sufficiently large $k$ (satisfying $x \in D_k$), and $m, n \geq n_k$,

$$\begin{aligned} |u_m - u_n|(x) \quad \leq \quad & \Pi_x H^{c_k}(0,\tau_k) \left| \Pi_{\xi_{\tau_k}} \int_0^{\tau} (f_n - f_m)(\xi_s) ds \right| \\ & + \Pi_x H^{c_k}(0,\tau_k) \Pi_{\xi_{\tau_k}} \int_0^{\tau} q_{m,n}(\xi_s) ds \\ & + \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) \left( \lambda_k |u_m - u_n| \right)(\xi_s) ds. \end{aligned}$$

Iterating the above inequality $l \geq 1$ times yields

$$\begin{aligned} |u_m - u_n|(x) \leq \quad & \Pi_x \left| \Pi_{\xi_{\tau_k}} \int_0^{\tau} (f_n - f_m)(\xi_s) ds \right| + \Pi_x \int_{\tau_k}^{\tau} q_{m,n}(\xi_s) ds \\ & + C M_k \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) \frac{(\int_0^s \lambda_k(\xi_r) dr)^l}{l!} ds. \end{aligned} \tag{2.7}$$

Notice that for fixed $k$,

$$\Pi_x \int_0^{\tau} f_n(\xi_s) ds = G_D f_n(x) \xrightarrow{bp} G_D \nu(x)(n \to \infty), \quad \text{for } x \in \overline{D}_k.$$

From the dominated convergence theorem and domination (2.4), we obtain

$$\lim_{m,n\to\infty} \Pi_x \left|\Pi_{\xi_{\tau_k}} \int_0^{\tau} (f_n - f_m)(\xi_s)ds\right| = 0, \tag{2.8}$$

and

$$\lim_{l\to\infty} \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) \frac{(\int_0^s \lambda_k(\xi_r)dr)^l}{l!} ds = 0. \tag{2.9}$$

Since $\psi(x,z) \le C(z+z^2), z \ge 0, x \in \mathrm{R}^d$, by (1.9) and the Hölder inequality, for every $x \in D$,

$$\limsup_{m,n\to\infty} \Pi_x \int_{\tau_k}^{\tau} q_{m,n}(\xi_s)ds \to 0, \text{ as } k \uparrow \infty. \tag{2.10}$$

Combining (2.7), (2.8), (2.9) and (2.10), we have

$$\limsup_{m,n\to\infty} |u_n(x) - u_m(x)| = 0, \quad x \in D \setminus N_\nu.$$

Therefore there exists a non-negative measurable function $u$ in $D \setminus N_\nu$ such that, for each compact subset $K \subset D \setminus N_\nu$, $u_n(x) \xrightarrow{bp} u(x)(n \to \infty)$ in $K$. Thus Statement (1) holds. The proof of Statements (2) and (3) is similar to that of Theorem 2.1 in Ren [3]. We omit the details.

**Remark** Checking the above proof we find that if $\psi(x,z)$ is given by the particular form (1.10) and if condition (1.9) is replaced by condition (1.11), the results of Theorem 2.1 also hold.

## 3 Proofs of Theorems 1.1 and 1.2

*Proof of Theorem* 1.1 Note that

$$\langle U(\nu), \mu\rangle \le \langle G_D\nu, \mu\rangle = \langle G_D\mu, \nu\rangle \le C|\lambda|, \tag{3.1}$$

where $|\lambda| = \max_i \lambda_i$. For fixed $y \in D \setminus N_\nu$, $U(\lambda\delta_y)/\lambda \le G_D(\cdot, y)$, $\langle G_D(\cdot,y), \mu\rangle < \infty$, and $U(\lambda\delta_y)/\lambda \to G_D(\cdot, z)$ a.s. -$\mu$, as $\lambda \to 0$. By the dominated convergence theorem, we have

$$\lim_{\lambda\to 0} \langle U(\lambda\delta_y)/\lambda, \mu\rangle = \langle G_D(\cdot, y), \mu\rangle. \tag{3.2}$$

Repeating the arguments of Theorem 2.1 in Ren and Wang [4], and using Theorem 2.1, the Remark in Seclion 2, and (3.1) and (3.2) above, we find that the results of Theorem 1.1 hold.

*Proof of Theorem* 1.2 We prove only (2); assertion (1) can be proved similarly. For $x \in D$ and any integer $k$,

$$\lim_{n\to\infty} \Pi_x \int_0^{\tau_k} (G_D\nu_n)^{1+\beta}(\xi_s)ds = \Pi_x \int_0^{\tau_k} (G_D\nu)^{1+\beta}(\xi_s)ds < \infty.$$

Consequently, condition (1.11) is satisfied if

$$\lim_{n\to\infty} \int_D G_D(x,y)(G_D\nu_n)^{1+\beta}(y)dy = \int_D G_D(x,y)(G_D\nu)^{1+\beta}(y)dy < \infty. \tag{3.3}$$

From the dominated convergence theorem,

$$\lim_{n\to\infty}\int_{D_k} G_D(x,y)(G_D\nu_n)^{1+\beta}(y)dy = \int_{D_k} G_D(x,y)(G_D\nu)^{1+\beta}(y)dy < \infty.$$

From Fatou's lemma, to prove (3.3) it is sufficient to prove

$$\sup_{n\geq 1}\int_{D\setminus D_k} G_D(x,y)(G_D\nu_n)^{1+\beta}(y)dy \to 0, \quad \text{as } k\uparrow\infty. \tag{3.4}$$

Without loss of generality, we can assume that $x$ belongs to $D_k$, $k\geq 1$. Then there exists a constant $C$ such that $G_D(x,y)\leq C$, $y\in D\setminus D_k$. Hence it is sufficient to show

$$\sup_{n\geq 1}\int_{D\setminus D_k} (G_D\nu_n)^{1+\beta}(y)dy \to 0, \quad \text{as } k\uparrow\infty. \tag{3.5}$$

Put $g_n(x) = G_D\nu_n(x)$, $\alpha_n(\lambda) = \int_{D\cap(G_D\nu_n>\lambda)} dy$. For $M>0$, we have

$$\int_{D\setminus D_k} (G_D\nu_n)^{1+\beta}(y)dy \leq M^{1+\beta}\int_{D\setminus D_k} dy + \int_{D\cap(g_n>M)} g_n^{1+\beta}(y)dy \tag{3.6}$$

and

$$\int_{D\cap(g_n>M)} g_n^{1+\beta}dy = -\int_M^\infty \lambda^{1+\beta}d\alpha_n(\lambda). \tag{3.7}$$

Then estimate

$$\begin{aligned}\lambda\alpha_n(\lambda) &\leq \int_{D\cap(g_n>\lambda)} g_n(y)dy = \int_D \nu_n(dy_1)\int_{D\cap(g_n>\lambda)} G_D(y,y_1)dy\\ &\leq C\sup_{y_1\in D}\int_{D\cap(g_n>\lambda)} G_D(y,y_1)dy.\end{aligned}$$

Choose $\alpha>1+\beta$; from Hölder's inequality, we have

$$\lambda\alpha_n(\lambda)\leq C\sup_{y_1\in D}(B(y_1))^{\frac{1}{\alpha}}(\alpha_n(\lambda))^{\frac{1}{\alpha'}},$$

where $B(y_1) = \int_D G_D(y,y_1)^\alpha dy$, $\frac{1}{\alpha}+\frac{1}{\alpha'} = 1$.

When $d\geq 3$,

$$B(y_1)\leq C\int_D \|y-y_1\|^{(2-d)\alpha}dy \leq C\int_0^{\text{diam}D} r^{(2-d)\alpha+d-1}dr,$$

where diam $D$ is the diameter of $D$. Since $d<2+2/\beta$, we can choose $\alpha>1+\beta$ such that $\int_0^{\text{diam}D} r^{(2-d)\alpha+d-1}dr<\infty$.

When $d=2$,

$$B(y_1)\leq C\int_D(\log^+\|y-y_1\|+1)^\alpha \quad\leq\quad C\left[\int_0^1(-\log r)^\alpha r dr + \int_1^{\text{diam}D} r dr\right]<\infty.$$

When $d=1$, it is obvious that $B(y_1)\leq C$.

Thus we conclude that in any dimension $d\geq 1$,

$$\alpha_n(\lambda)\leq C\alpha_n(\lambda)^{\frac{1}{\alpha'}}, \qquad \text{for all } \lambda>0, \quad n\geq 1.$$

Since $\alpha > 1+\beta$, we have, by integration by parts, that

$$-\int_M^\infty \lambda^{1+\beta} d\alpha_n(\lambda) = M^{1+\beta}\alpha_n(M) + (1+\beta)\int_M^\infty \alpha_n(\lambda)\lambda^\beta d\lambda \le CM^{1+\beta-\alpha}. \tag{3.8}$$

Therefore (3.5) follows easily from (3.6) (3.7) and (3.8).

## 4 Proof of Theorem 1.3

Let $F$ be a closed subset of $\overline{D}$. Consider the following boundary value problem:

$$\begin{cases} Lu = u^{1+\beta}, & \text{in } D \setminus F, \\ u = 0, & \text{on } \partial D \setminus F, \end{cases} \tag{4.1}$$

where $0 < \beta \le 1$.

**Lemma 4.1** *If $u$ is a solution of the boundary value problem* (4.1)*, then*

$$u(x) \le Cd(x,F)^{-2/\beta}, \quad x \in D \setminus F, \tag{4.2}$$

*where $C$ is a constant depending only on $L$, $\beta$ and $D$.*

*Proof* The proof is similar to that of Sheu [2]. We give only an outline here. Put $\omega(x) = u(x) - 1, x \in D$, and $h(x) = g(\omega(x))I_{\overline{D}\setminus F}$ for all $x \in \mathrm{R}^d$, where

$$g(r) = \begin{cases} 0, & \text{if } r < 0, \\ r^2/2, & \text{if } 0 \le r < 1, \\ r - \frac{1}{2}, & \text{if } r \ge 1. \end{cases}$$

Note that, for every $x \in D \setminus F$, we have either $w(x) = u - 1 \le 1$ or $h(x) = u(x) - 3/2$. Since $D$ is bounded, it is sufficient to show that

$$h(x) \le Cd(x,F)^{-2/\beta}, \quad x \in D \setminus F. \tag{4.3}$$

From arguments similar to those of Sheu [5], we can check that

$$Lh \ge h^{1+\beta} \quad \text{on } \mathrm{R}^d \setminus F. \tag{4.4}$$

For every $x \in D \setminus F$, using Lemma 3.1 and Theorem 0.5 in Dynkin [1] in $B(x, d(x,F))$, we obtain that (4.3) holds.

**Lemma 4.2** *Let $K$ be a compact subset of $D$. For every integer $n$, put*

$$K_n = \{x : x \in D, d(x,K) \le d(K, \partial D)/n\}. \tag{4.5}$$

*Then for every $n$, there exist two constants $C$ and $\epsilon_0$ (depending only on $L$, $\beta$, $D$, $K$ and $n$) such that if $u_\epsilon$ is a solution of* (4.1) *with $F = \overline{B}(x_0, \epsilon) \cap \overline{D}$ for some $x_0 \in \overline{D} \setminus K_n$ and $\epsilon < \epsilon_0$, then we have*

$$u_\epsilon \le C\epsilon^{d-2-2/\beta}, \quad x \in K. \tag{4.6}$$

*Proof* Choose $\epsilon_0 = \frac{1}{2n}d(K, \partial D)$. Then, for $\epsilon < \epsilon_0$, we have $K \cap B(x_0, \epsilon) = \emptyset$. Let $\tau_\epsilon = \min\{\tau_D, \tau_{B(x_0,2\epsilon)}\}$. From (1.5),

$$u_\epsilon(x) \le \Pi_x(u_\epsilon(\xi_{\tau_\epsilon})) = \Pi_x(u_\epsilon(\xi_{\tau_\epsilon}); \tau_{B(x_0,2\epsilon)} < \tau_D). \tag{4.7}$$

On $\tau_{B(x_0,2\epsilon)} < \tau_D$, we have, from Lemma 4.1, $u_\epsilon(\xi_{\tau_\epsilon}) \le Cd(\xi_{\tau_\epsilon}, F)^{-2/\beta} \le C\epsilon^{-2/\beta}$. Then from (4.7),

$$u_\epsilon(x) \le C\epsilon^{-2/\beta}\Pi_x(\tau_{B(x_0,2\epsilon)} < \infty). \tag{4.8}$$

From Theorem 3.1.6 in Port and Stone [6], $\Pi_x(\tau_{B(x_0,2\epsilon)} < \infty) = (2\epsilon/\|x - x_0\|)^{d-2} \wedge 1$. Noticing that $\|x - x_0\| \ge d(K, \partial D)/n > 2\epsilon$, we get

$$\Pi_x(\tau_{B(x_0,2\epsilon)} < \infty) \le \left[\frac{2n}{d(K, \partial D)}\right]^{d-2} \epsilon^{d-2}. \tag{4.9}$$

Our conclusion follows from (4.8) and (4.9).

**Lemma 4.3** *Suppose $F$ is a closed subset of $\overline{D}$. $v(x) = -\log P_{\delta_x}(Y_{\tau_D}(F) = 0)$ is a solution of* (4.1).

*Proof* Note that $v$ is, as $\lambda \to \infty$, the limit of functions $v_\lambda = -\log P_{\delta_x} \exp(-\lambda Y_{\tau_D}(F))$, and $v_\lambda$ satisfies

$$v_\lambda(x) + \Pi_x \int_0^{\tau_D} v_\lambda^\alpha(\xi_s)ds = \lambda\Pi_x \int_0^{\tau_D} I_F(\xi_s)ds. \tag{4.10}$$

For every $x \in D \setminus F$, choose $\epsilon$ small enough such that $B(x, \epsilon) \subset D \setminus F$. By the strong Markov property of $\xi$,

$$v_\lambda(x) + \Pi_x \int_0^{\tau_B} v_\lambda^\alpha(\xi_s)ds = \Pi_x\Big[\Pi_{\xi_{\tau_B}} \int_0^{\tau_D} (\lambda I_F + v_\lambda^\alpha)(\xi_s)ds\Big],$$

where $\tau_B$ is the first exit time of $\xi$ from $B(x, \epsilon)$. Thus, from Theorem 1.1 in Dynkin [1], $Lv_\lambda = v_\lambda^\alpha$ in $B(x, \epsilon)$, and hence in $D \setminus F$. From (4.10), it is obvious that $v_\lambda = 0$ on $\partial D \setminus F$. So, for every $\lambda > 0$, $v_\lambda$ is a solution of (4.1). Note that similar results to Theorem 1.2 in Dynkin [7] hold for the elliptic case. Therefore $v$, the limit of $v_\lambda$, is also a solution of (4.1).

*Proof of Theorem* 1.3 Fix $\mu \in M_c(D)$ and put $K = \mathrm{supp}\,(\mu)$. Let $K_n$ be defined by (4.5) and let $Y_{\tau_D}^{\overline{D}\setminus K_n}$ be the restriction of $Y_{\tau_D}$ to $\overline{D} \setminus K_n$. It is sufficient to prove that, for every $n$, $Y_{\tau_D}$ as a measure on $\overline{D} \setminus K_n$ is $P_\mu$-a.s., singular with respect to the Lebesgue measure on $\overline{D} \setminus K_n$.

Fix $n$, and set $K_n' = \mathrm{supp}(Y_{\tau_D}^{\overline{D}\setminus K_n})$. For all $m \ge 1$, let $\{B_{m,i}\}_{i\in I_m}$ be an open covering of $\overline{D} \setminus K_n$, and $\mathrm{diam}(B_{m,i}) = 2^{-m}$. By the regularity of $D$, we can assume the cardinality of $I_m$ is less than $C2^{md}$, where $C$ is a constant independent of $m$. Set

$$H_m = \sum_{i\in I_m} I_{B_{m,i}\cap K_n' \ne \emptyset}; \quad v_{m,i} = -\log P_{\delta_x}(Y_{\tau_D}(B_{m,i}) = 0).$$

Then from Lemma 4.3, $v_{m,i}$ is a solution of (4.1) with $F = \overline{B}_{m,i} \cap \overline{D}$. Thus from Lemma 4.2,

$$\begin{aligned} P_\mu H_m &= \sum_{i\in I_m} P_\mu(Y_{\tau_D} > 0) = \sum_{i\in I_m} (1 - \exp\langle -v_{m,i}, \mu\rangle) \\ &\le \sum_{i\in I_m} \langle v_{m,i}, \mu\rangle \le C(2^{-m})^{-2-2/\beta}, \end{aligned}$$

which implies, for $m$ sufficiently large,

$$P_\mu((2^{-m})^{2+2/\beta} H_n) \le C < \infty.$$

Then from Fatou's lemma and the definition of the Hausdorff dimension, $P_\mu(\dim(K_n^{'}) \le 2 + 2/\beta) = 1$. Since $\dim(\overline{D} \setminus K_n) = d > 2 + 2/\beta$, $Y_{\tau_D}$ as a measure on $\overline{D} \setminus K_n$ is $P_\mu$-a.s., singular.

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