

Boundary singularity problem of some nonlinear elliptic equations[☆]

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Abstract

Let L be a uniformly elliptic operator in \mathbb{R}^d . We investigate limit properties of solutions to the boundary singularity problem of non-linear equation $Lu = u^\alpha$, $1 < \alpha \leq 2$, by a probabilistic method. © 2001 Elsevier Science B.V. All rights reserved

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1. Introduction

Let D be a bounded C^2 -domain in \mathbb{R}^d with boundary ∂D , and let f be a continuous and non-negative function on ∂D . Suppose 0 belongs to ∂D . Consider the boundary singularity problem

$$\begin{cases} Lu = u^\alpha & \text{in } D, \\ u|_{\partial D \setminus \{0\}} = f, \end{cases} \quad (1.1)$$

where L is a uniformly elliptic operator in \mathbb{R}^d , $1 < \alpha \leq 2$.

The boundary value problem (1.1) has been studied recently by probabilistic and purely analytic methods.

Gmira and Véron (1991) showed that, in the case $d < 1 + 2/(\alpha - 1)$, there are three classes of solutions of (1.1): removable singularity, weak singularity and strong singularity, in which the solutions of (1.1) are in the distribution sense. Gmira and Véron's treatment about problem (1.1) is purely analytic.

Le Gall (1997) succeeded in describing all positive solutions of the equation $\Delta u = u^2$ in a smooth domain D in \mathbb{R}^2 . He established a 1–1 correspondence between all positive solutions and all pairs (Γ, ν) , where Γ is a closed subset of ∂D and ν is a Radon measure on $\partial D \setminus \Gamma$.

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Marcus and Véron (1998) investigated the equation $\Delta u = u^\alpha$, $\alpha > 1$, in the unit d -dimensional ball by analytic methods. For every positive solution u they defined the trace (Γ, ν) of u in terms of the boundary behavior of u .

Dynkin and Kuznetsov (1998) investigated the trace of positive solutions of the nonlinear equation $Lu = u^\alpha$, $1 < \alpha \leq 2$, in a bounded $C^{2,\lambda}$ domain. Their definition of the trace is different from that of Marcus and Véron (1998). The main tool they used is the (L, α) -superdiffusion.

The objective of this paper is to describe the limit behavior of classical non-negative solutions of (1.1) near singular point 0 by a probabilistic method.

2. Main result

To state the main result of this paper let us introduce some notations.

Suppose $\xi = (\xi_t, \Pi_x)$ is a diffusion with generator L . For every open set U , let τ_U denote the first exit time of ξ from U . A point $a \in \partial D$ is called regular if $\Pi_a(\tau_D = 0) = 1$. A domain D is regular if all points $a \in \partial D$ are regular. Every C^2 -domain is regular. For non-negative measurable function f on ∂D and non-negative measurable function g on D ,

$$\begin{aligned}\Pi_x f(\xi_{\tau_D}) &= \int_{\partial D} k(x, y) f(y) \sigma(dy), \\ \Pi_x \int_0^{\tau_D} g(\xi_s) ds &= \int_D G_D(x, y) g(y) dy,\end{aligned}$$

where $G_D(x, y)$ is the Green function of L in D , $k(x, y)$ is the Poisson kernel and σ is a finite measure on ∂D such that $\int_{\partial D} k(x, y) \sigma(dy) = 1$.

We denote by $\mathcal{M}(D)$ the set of all finite measures in D . To every open set $U \subset D$, there corresponds a random measure X_U such that, for every non-negative Borel function f ,

$$P_\mu \exp\langle -f, X_U \rangle = \exp\langle -u, \mu \rangle, \quad \mu \in \mathcal{M}(D), \quad (2.1)$$

where

$$u(x) + \Pi_x \int_0^{\tau_U} u^\alpha(\xi_s) ds = \Pi_x f(\xi_{\tau_U}). \quad (2.2)$$

The joint probability distribution of X_{U_1}, \dots, X_{U_n} is determined by (2.1) and the Markov property: for every positive $\mathcal{F}_{\supset U}$ -measurable Y ,

$$P_\mu \{Y | \mathcal{F}_{\subset U}\} = P_{X_U} Y, \quad (2.3)$$

where $\mathcal{F}_{\subset U}$ is the σ -algebra generated by X_{U_1} with $U_1 \subset U$ and $\mathcal{F}_{\supset U}$ is the σ -algebra generated by X_{U_2} with $U_2 \supset U$. We call $X^D = (X_U, P_\mu; U \subset D, \mu \in \mathcal{M}(D))$ the (L, α) -superdiffusion in D . Let \mathcal{R}_D denote the support of X^D .

We say that a positive solution u of $Lu = u^\alpha$ is moderate if it is dominated by an L -harmonic function. Every positive L -harmonic function on D has a unique representation

$$h(x) = \int_{\partial D} k(x, y) \nu(dy),$$

where ν is a finite measure on ∂D . The condition $d < 1 + 2/(\alpha - 1)$ implies that the empty set is the only \mathcal{R}_D -polar set, and the formula

$$u(x) + \Pi_x \int_0^{\tau_D} u^\alpha(\xi_s) ds = \int_{\partial D} k(x, y) \nu(dy)$$

established a 1–1 correspondence between moderate solutions u and finite measures ν on ∂D (see Dynkin and Kuznetsov, 1996a). We call ν the trace of the moderate solution u .

Suppose u is an arbitrary positive solution of $Lu = u^\alpha$, according to Theorem 1.2 in Dynkin and Kuznetsov (1998), for every compact subset B of ∂D , there exists the maximal solution u_B dominated by u and equal to 0 on $\partial D \setminus B$. There exists a unique (Γ, ν) satisfies: (a) Γ is a closed subset of ∂D and $O = \partial D \setminus \Gamma$ is the maximal open subset O of ∂D such that u_B is moderate for every compact $B \subset O$; (b) ν is a measure on O such that, for every compact $B \subset O$, the trace of u_B coincides with the restriction of ν to B . We call the pair (Γ, ν) the trace of u .

Dynkin and Kuznetsov (1996b) showed that, if $d < 1 + 2/(\alpha - 1)$, then, for every $\lambda > 0$, there exists a unique moderate solution of $Lu = u^\alpha$ with trace $f(z)\sigma(dz) + \lambda\delta_0$, which we denote by $u_{f,\lambda}$. Let \mathcal{W}_f^m and \mathcal{W}_f^n be the collections of all moderate solutions of (1.1) and all non-moderate solutions of (1.1), respectively.

Now we are ready to state the main result of this paper.

Theorem 2.1. Assume that D is a bounded C^2 -domain in \mathbb{R}^d , $0 \in \partial D$, $2 \leq d < 1 + 2/(\alpha - 1)$, and f is a continuous and non-negative function on ∂D .

(i)

$$u_f(x) = -\log P_{\delta_x} \exp\langle -f, X_D \rangle, \quad x \in D \quad (2.4)$$

is the unique non-negative bounded solution of (1.1).

(ii) $\mathcal{W}_f^m = \{u_{f,\lambda}, \lambda \geq 0\}$. For every $u \in \mathcal{W}_f^n$, the trace of u is $f(z)\sigma(dz) + \lambda\delta_0$ with λ given by

$$\lambda = \lim_{x \in D, x \rightarrow 0} \frac{u(x) - \Pi_x f(\xi_{\tau_D})}{k(x, 0)} = \limsup_{x \in D, x \rightarrow 0} \frac{u(x)}{k(x, 0)}, \quad (2.5)$$

i.e., $u = u_{f,\lambda}$ with λ given by (2.5).

(iii) u belongs to \mathcal{W}_f^n if and only if the trace of u is $(\{0\}, f(z)\sigma_{\partial D \setminus \{0\}}(dz))$, where $\sigma_{\partial D \setminus \{0\}}(dz)$ is the restriction of $\sigma(dz)$ to $\partial D \setminus \{0\}$. Moreover,

$$u_{f,\infty}(x) = \lim_{\lambda \rightarrow \infty} u_{f,\lambda}(x), \quad x \in D, \quad (2.6)$$

is the minimal element of \mathcal{W}_f^n , and

$$w(x) = -\log P_{\delta_x} \{\exp\langle -f, X_D \rangle, 0 \notin \mathcal{R}_D\}, \quad x \in D, \quad (2.7)$$

is the maximal element of \mathcal{W}_f^n .

Remark 2.1. If $2 \leq d < 1 + 2/(\alpha - 1)$ and $f = 0$ on $\partial D \setminus \{0\}$, Theorem 2.1 shows that $u \equiv 0$ is the unique bounded non-negative solution of (1.1), and there are two classes of positive solutions of (1.1): moderate solutions and non-moderate solutions. This coincides with the result that, in the case $2 \leq d < 1 + 2/(\alpha - 1)$, set $\{0\} \subset \partial D$ is a non-removable singularity.

As we know, it is the first time to investigate the behavior of singular solutions of (1.1) near the singular point 0 in terms of $k(x, 0)$ (see (2.5)). Le Gall (1997) proved that the non-moderate solution of (1.1) is unique for $L = \Delta, \alpha = 2$. Marcus and Véron (1998) proved this result for $L = \Delta$ and for a ball D by analytic method. But in general case, we do not know whether there is only one element in \mathcal{W}_f^n .

3. Proof of the main result

We fix a bounded C^2 -domain D in \mathbb{R}^d . Suppose $0 \in \partial D$ and f is a non-negative bounded continuous function on ∂D .

Lemma 3.1 (Dynkin 1992, Theorem 1.2). Suppose u_n is a sequence of non-negative solutions of $Lu = u^\alpha$ in D and u_n converge pointwise in D to u . Then u is a solution of $Lu = u^\alpha$ in D .

Let O be a relatively open subset of ∂D . If u_n satisfy the boundary condition $u_n = f$ on O , then the same condition holds for u .

Lemma 3.2. Let g be a locally bounded function in an open set D and $F(x) = \Pi_x \int_0^{\tau_D} g(\xi_s) ds$. If $g \in C^{0,\lambda}(D)$ and $F(x)$ is locally bounded in D , then $F \in C^{2,\lambda}(D)$ and $LF = -g$ in D .

Proof. Lemma 3.2 is a generalization of Theorem 0.3 in Dynkin (1991). It is easy to prove by the strong Markov property of ξ , and Theorems 0.2 and 0.3 in Dynkin (1991). We omit the details. \square

Lemma 3.3. Suppose Γ is a relative closed subset of ∂D . Then

$$-\log P_{\delta_x} \{ \exp \langle -f, X_D \rangle; \mathcal{R}_D \cap \Gamma = \emptyset \}$$

is the maximal non-negative solution of $Lu = u^\alpha$ with boundary condition $u = f$ on $\partial D \setminus \Gamma$.

Proof. The proof of Lemma 3.3 is similar to that of Theorem 2.1 in Dynkin (1992). We omit the details. \square

In the following of this paper, the notation C always denotes a constant which may change values from line to line.

Lemma 3.4. If $d < 1 + 2/(\alpha - 1)$, then

$$\lim_{x \in D, x \rightarrow 0} \frac{1}{k(x, 0)} \int_D G_D(x, y) k(y, 0)^\alpha dy = 0. \quad (3.1)$$

Proof. We first quote two inequalities:

$$C_1 \rho(x) \|x - z\|^{-d} \leq k(x, z) \leq C_1^{-1} \rho(x) \|x - z\|^{-d}, \quad x \in D, z \in \partial D; \quad (3.2)$$

(see Dynkin and Kuznetsov, 1996a)

$$G_D(x, y) \leq C_2 \frac{\rho(x) \rho(y)}{\|x - y\|^d}, \quad x, y \in D, \quad (3.3)$$

where $\rho(x) = d(x, \partial D)$ is the distance from x to ∂D , and C_1 and C_2 are two positive constants depending only on L and D . For $L = \Delta$, inequality (3.3) was proved in Chung and Zhao (1995), and in the general case this follows easily from the fact that quotients of Green functions are uniformly bounded (see Hueber and Siereking, 1982).

Note that

$$\frac{1}{k(x, 0)} \int_D G_D(x, y) k(y, 0)^\alpha dy = I_1 + I_2, \quad (3.4)$$

where

$$I_1 = \int_{D \cap \{y: \|x-y\| \geq \|x\|/2\}} g(x, y) dy,$$

$$I_2 = \int_{D \cap \{y: \|x-y\| < \|x\|/2\}} g(x, y) dy,$$

$$g(x, y) = \frac{G_D(x, y) k(y, 0)^\alpha}{k(x, 0)}.$$

We first prove $\lim_{x \in D, x \rightarrow 0} I_1 = 0$. Using (3.2) and (3.3), we get

$$g(x, y) \leq C \frac{\|x\|^d}{\|x - y\|^d} \|y\|^{-\alpha(d-1)+1}.$$

Then we conclude that $\lim_{x \in D, x \rightarrow 0} g(x, y) = 0$ for any $y \in D$, and $g(x, y) \leq 2^d C \|y\|^{-\alpha(d-1)+1}$ for $y \in D \cap \{y: \|x - y\| \geq \|x\|/2\}$. Note that $\int_D \|y\|^{-\alpha(d-1)+1} dy \leq C \int_0^{\text{diam}(D)} r^{d-\alpha(d-1)} dr < \infty$ ($d < 1 + 2/(\alpha - 1)$). By the dominated convergence theorem, $\lim_{x \in D, x \rightarrow 0} I_1 = 0$.

Next we prove $\lim_{x \in D, x \rightarrow 0} I_2 = 0$. By Corollaries 6.13 and 6.25 in Chung and Zhao (1995), we have

$$\frac{G_D(x, y)k(y, 0)}{k(x, 0)} \leq \begin{cases} C(\|x - y\|^{2-d} + \|y\|^{2-d}) & \text{if } d \geq 3, \\ C[(\ln \frac{1}{\|x - y\|}) \vee 1 + (\ln \frac{1}{\|y\|}) \vee 1] & \text{if } d = 2. \end{cases} \quad (3.5)$$

Note that $\|y - x\| < \|x\|/2$ implies $\|y\| \geq \|x\|/2$. In the case $d \geq 3$, we have

$$\begin{aligned} g(x, y) &\leq C(\|x - y\|^{2-d} + \|y\|^{2-d})k(y, 0)^{\alpha-1} \\ &\leq C(\|x - y\|^{2-d} + \|y\|^{2-d})\|y\|^{(\alpha-1)(1-d)} \\ &\leq C(\|x\|^{(\alpha-1)(1-d)}\|x - y\|^{2-d} + \|x\|^{2-d+(\alpha-1)(1-d)}). \end{aligned}$$

Thus

$$\begin{aligned} I_2 &\leq C \int_{\|x - y\| \leq \frac{1}{2}\|x\|} (\|x\|^{(\alpha-1)(1-d)}\|x - y\|^{2-d} + \|x\|^{2-d+(\alpha-1)(1-d)}) dy \\ &\leq C\|x\|^{2+(\alpha-1)(1-d)} \rightarrow 0 \quad (\text{since } d < 1 + 2/(\alpha - 1)). \end{aligned}$$

In the case $d = 2$, we have

$$\begin{aligned} g(x, y) &\leq C \left[\left(\ln \frac{1}{\|x - y\|} \right) \vee 1 + \left(\ln \frac{1}{\|y\|} \right) \vee 1 \right] k(y, 0)^{\alpha-1} \\ &\leq C \left[\left(\ln \frac{1}{\|x - y\|} \right) \vee 1 + \left(\ln \frac{1}{\|y\|} \right) \vee 1 \right] \|y\|^{(\alpha-1)(1-d)}. \end{aligned}$$

Thus for sufficiently small x ($\|x\| < 2/3$).

$$\begin{aligned} I_2 &\leq C \int_{\|x - y\| \leq \|x\|/2} \left(\ln \frac{1}{\|x - y\|} + \ln \frac{1}{\|y\|} \right) \|x\|^{(\alpha-1)(1-d)} dy \\ &\leq C \left(\|x\|^{(\alpha-1)(1-d)} \int_0^{\|x\|/2} -r \ln r dr + \|x\|^{(\alpha-1)(1-d)} \ln \frac{2}{\|x\|} \right) \rightarrow 0. \end{aligned}$$

Therefore, by (3.4), we obtain the desired result (3.1). \square

Lemma 3.5. Assume $d < 1 + 2/(\alpha - 1)$. For every $a \in \partial D \setminus \{0\}$, we have

$$\lim_{x \in D, x \rightarrow a} \int_D G_D(x, y)k(x, 0)^\alpha dy = 0.$$

Proof. Note that

$$\begin{aligned} \int_D G_D(x, y) k(y, 0)^\alpha dy &= \int_{D \setminus B(a, |a|/2)} G_D(x, y) k(y, 0)^\alpha dy \\ &\quad + \int_{D \cap B(a, |a|/2)} G_D(x, y) k(y, 0)^\alpha dy. \end{aligned}$$

For $x \in B(a, |a|/4)$, $y \in D \setminus B(a, |a|/2)$, by estimates (3.2) and (3.3) we have

$$G_D(x, y) k(y, 0)^\alpha \leq C \frac{\rho(x) \rho(y)^{\alpha+1}}{\|x - y\|^d \|y\|^{2d}} \leq 4^d C |a|^{-d} \|y\|^{\alpha+1-2d}.$$

Since $\int_D \|y\|^{\alpha+1-2d} dy < \infty$ ($d < 1 + 2/(\alpha - 1)$) and $\lim_{x \in D, x \rightarrow a} G_D(x, y) = 0$, it follows from the dominated convergence theorem that

$$\int_{D \setminus B(a, |a|/2)} G_D(x, y) k(y, 0)^\alpha dy \rightarrow 0 \quad \text{as } x \in D, x \rightarrow a.$$

For $x \in B(a, |a|/4)$, $y \in D \cap B(a, |a|/2)$, we similarly have the estimate

$$G_D(x, y) k(y, 0)^\alpha \leq C \frac{\rho(y)^\alpha}{\|y\|^{2d}} G_D(x, y) \leq C |a|^{-2d} G_D(x, y).$$

Since $\lim_{x \in D, x \rightarrow a} \int_D G_D(x, y) dy = 0$, we have

$$\int_{D \cap B(a, |a|/2)} G_D(x, y) k(y, 0)^\alpha dy \rightarrow 0 \quad \text{as } x \in D, x \rightarrow a.$$

Thus we complete the proof of Lemma 3.5. \square

Proof of Theorem 2.1. (i) By Theorem 1.1 in Dynkin (1991) u_f is the unique bounded solution of $Lu = u^\alpha$ with boundary condition $u|_{\partial D} = f$, and therefore a bounded solution of (1.1). Conversely suppose u is bounded solution of (1.1). By Lemma 3.2, $h(x) = u(x) + \int_D G_D(x, y) u^\alpha(y) dy$ is a bounded solution of $Lu = 0$ in D having boundary value f on $\partial D \setminus \{0\}$. From the classical theory of the regularity of solutions of elliptic equation near a boundary point, we deduce that h and hence u can be continuously extended to \bar{D} . Thus u is a bounded solution of $Lu = u^\alpha$ with boundary condition $u|_{\partial D} = f$. Hence we have $u = u_f$.

(ii) For every $\lambda \geq 0$, $u_{f,\lambda}$ satisfies

$$u_{f,\lambda}(x) + \int_D G_D(x, y) u_{f,\lambda}^\alpha(y) dy = \Pi_x f(\xi_{\tau_D}) + \lambda k(x, 0). \quad (3.6)$$

Dynkin and Kuznetsov (1996b) showed that $u_{f,\lambda}$ is a moderate solution of $Lu = u^\alpha$. From Minkowski inequality and Lemma 3.5, $\int_D G_D(x, y) u_{f,\lambda}^\alpha(y) dy$ has boundary value zero at $a \in \partial D \setminus \{0\}$. Then, by (3.6), $u_{f,\lambda}$ has boundary value f on $\partial D \setminus \{0\}$. Therefore $u_{f,\lambda}$ is a moderate solution of (1.1).

Now suppose u is a moderate solution of (1.1). There exists a finite measure ν on ∂D such that

$$u(x) + \int_D G_D(x, y) u^\alpha(y) dy = \int_{\partial D} k(x, z) \nu(dz). \quad (3.7)$$

By Fatou's Lemma and (3.2),

$$\limsup_{x \rightarrow 0} \frac{\int_{\partial D} k(x, z) \nu(dz)}{k(x, 0)} \leq \int_{\partial D} \limsup_{x \rightarrow 0} \frac{k(x, z)}{k(x, 0)} \nu(dz) = \nu(\{0\}) < \infty.$$

Then there exists a constant $C > 0$ such that $u(x) \leq Ck(x, 0)$ in a neighborhood U of 0. As u is bounded in $D \setminus U$, we can choose C big enough such that

$$u(x) \leq C(k(x, 0) + 1). \quad (3.8)$$

By Minkowski inequality,

$$\int_D G_D(x, y) u^\alpha(y) dy \leq C \left[\left(\int_D G_D(x, y) dy \right)^{1/\alpha} + \left(\int_D G_D(x, y) k(y, 0)^\alpha dy \right)^{1/\alpha} \right]^\alpha. \quad (3.9)$$

It is easy to check that $\int_D G_D(x, y) k(y, 0)^\alpha dy$ is locally bounded in D . By (3.9), $\int_D G_D(x, y) u(y)^\alpha dy$ is locally bounded in D . Then it follows from Lemma 3.2 that $h(x) = u(x) + \int_D G_D(x, y) u(y)^\alpha dy$ is a solution of $Lh = 0$. By (3.9) and Lemma 3.5, $\int_D G_D(x, y) u^\alpha(y) dy$ has boundary value zero at $a \in \partial D \setminus \{0\}$. So h is a bounded solution of $Lh = 0$ having boundary value f on $\partial D \setminus \{0\}$. From the Martin's representation theorem (see, e.g., Theorem 4.3 and its proof in Hunt and Wheeden, 1970), there exists a constant $\lambda \geq 0$ such that

$$h(x) = \Pi_x f(\xi_{\tau_D}) + \lambda k(x, 0)$$

and therefore

$$u(x) + \int_D G_D(x, y) u^\alpha(y) dy = \Pi_x f(\xi_{\tau_D}) + \lambda k(x, 0). \quad (3.10)$$

Hence $u = u_{f, \lambda}$ belongs to \mathcal{U}_f^m .

By (3.9) and Lemma 3.4,

$$\lim_{x \in D, x \rightarrow 0} \frac{1}{k(x, 0)} \int_D G_D(x, y) u^\alpha(y) dy = 0.$$

Then by (3.10), assertion (2.5) holds

(iii) It follows easily from result (ii) that u is a non-moderate solution of (1.1) if and only if the trace of u is $(\{0\}, f(z)\sigma_{\partial D \setminus \{0\}}(dz))$. The maximum principle implies that u_λ is increasing in $\lambda > 0$. Then $u_{f, \lambda} \uparrow u_{f, \infty}$. By Lemma 3.1 $u_{f, \infty}$ is a non-moderate solution of (1.1). It is obvious that

$$\lim_{x \in D, x \rightarrow 0} \frac{u_{f, \infty}(x) - \Pi_x f(\xi_{\tau_D})}{k(x, 0)} = \limsup_{x \in D, x \rightarrow 0} \frac{u_{f, \infty}(x)}{k(x, 0)} = \infty.$$

By Lemma 3.3, w is the maximal solution of (1.1), and then the maximal element of \mathcal{U}_f^m . \square

Comparing our result with that of Gmira and Véron (1991), we find that u_f is the only removable singularity of (1.1), and \mathcal{U}_f^m and \mathcal{U}_f^n are the collections of all weak singularities of (1.1) and all strong singularities of (1.1), respectively.

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