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Interior singularity problem of some nonlinear elliptic equations

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Abstract

Let L be a uniformly elliptic operator in $\mathbf{R}^d$. We investigate positive solutions to the interior singularity problem of the nonlinear equation $Lu = u^\alpha, 1 < \alpha \leqslant 2$, by a probabilistic method.

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1. Introduction and main results

Let D be a bounded C^2-domain in $\mathbf{R}^d$ with boundary ∂D, and let f be a continuous and nonnegative function on ∂D. Suppose 0 belongs to D. Consider the interior singularity problem

$$\begin{cases} Lu = u^\alpha & \text{in } D \setminus \{0\}, \\ u|_{\partial D} = \varphi, \end{cases} \tag{1.1}$$

where L is a uniformly elliptic operator in $\mathbf{R}^d$, $1 < \alpha \leqslant 2$.

The interior value problem (1.1) has been studied by purely analytic methods. Particularly, when $L = \Delta$, Véron (1981) showed that, in the case $d < 2+2/(\alpha-1)$, there are three classes of solutions of (1.1): removable singularity, weak singularity and strong singularity. But in the case $d > 2+2/(\alpha-1)$ problem (1.1) has one unique bounded solution.

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The objective of this paper is to describe the limit behavior of classical nonnegative solutions of (1.1), near the singular point 0, by a probabilistic method and represent all nonnegative solutions of (1.1) in terms of superdiffusion.

To state the main result of this paper let us introduce some notations.

Suppose $\xi=(\xi_t,\Pi_x)$ is a diffusion with generator L. For every open set U, let τ_U denote the first exit time of ξ from U. A point $a\in\partial D$ is called regular if $\Pi_a(\tau_D=0)=1$. A domain D is regular if all points $a\in\partial D$ are regular. Every C^2-domain is regular.

For every Borel-measurable space $(E,\mathscr{B}(E))$, we denote by $M(E)$ the set of all finite measures on $\mathscr{B}(E)$ endowed with the topology of weak convergence. The expression $\langle f,\mu\rangle$ stands for the integral of f with respect to μ. We write $f\in\mathscr{B}(E)$ if f is a $\mathscr{B}(E)$—measurable function. Writing $f\in p\mathscr{B}(E)(b\mathscr{B}(E))$ means that, in addition, f is positive (bounded). We put $bp\mathscr{B}(E)=b\mathscr{B}(E)\cap p\mathscr{B}(E)$. If $E=\mathbf{R}^d$, we simply write $\mathscr{B}$ instead of $\mathscr{B}(E)$ and M instead of $M(\mathbf{R}^d)$.

We denote by $\mathscr{T}$ the set of all exit times from open sets in $\mathbf{R}^d$. Set $\mathscr{F}_{\leqslant r}=\sigma(\xi_s,s\leqslant r)$; $\mathscr{F}_{>r}=\sigma(\xi_s,s>r)$ and $\mathscr{F}_\infty=\vee\{\mathscr{F}_{\leqslant r},r\geqslant 0\}$. For $\tau\in\mathscr{T}$, we put $F\in\mathscr{F}_{\geqslant\tau}$ if $F\in\mathscr{F}_\infty$ and if, for each r, $\{F,\tau>r\}\in\mathscr{F}_{>r}$.

According to Dynkin (1991) there exists a Markov process $X=(X_t,P_\mu)$ in M such that the following conditions are satisfied:

(a) If f is a bounded continuous function, then $\langle f,X_t\rangle$ is right continuous in t on $\mathbf{R}^+$.
(b) For every $v\in M$ and for every $f\in bp\mathscr{B}$,

$$P_\mu\exp\langle -f,X_t\rangle=\exp\langle -v_t,\mu\rangle,\quad \mu\in M, \tag{1.2}$$

where v is the unique solution of the integral equation

$$v_t(x)+\Pi_x\left[\int_0^t v_{t-s}^{\alpha}(\xi_s)\,\mathrm{d}s\right]=\Pi_x f(\xi_t). \tag{1.3}$$

Moreover, for every $\tau\in\mathscr{T}$, there correspond random measures X_τ and Y_τ on $\mathbf{R}^d$ associated with the first exit time τ such that, for $f,g\in bp\mathscr{B}$,

$$P_\mu\exp\{-\langle f,X_\tau\rangle-\langle g,Y_\tau\rangle\}=\exp\langle -u,\mu\rangle,\quad \mu\in M, \tag{1.4}$$

where u is the unique solution of the integral equation

$$u(x)+\Pi_x\left[\int_0^\tau u^{\alpha}(\xi_s)\,\mathrm{d}s\right]=\Pi_x\left[f(\xi_\tau)+\int_0^\tau g(\xi_s)\,\mathrm{d}s\right]. \tag{1.5}$$

We call $X=\{X_t,X_\tau,Y_\tau;P_\mu\}$ the (L,α)-superdiffusion (enhanced model). Throughout this paper τ_D denotes the first exit time of ξ from an open set D in $\mathbf{R}^d$, i.e., $\tau_D=\inf\{t\colon\ \xi_t\notin D\}$. We call X_{τ_D} the exit measure, and Y_{τ_D} the total weighted occupation time of X in D.

Let $\mathscr{F}_{\subset U}$ denote the σ-algebra generated by $X_{\tau_{U_1}}$ with $U_1\subset U$, and let $\mathscr{F}_{\supset U}$ denote the σ-algebra generated by $X_{\tau_{U_2}}$ with $U_2\supset U$. Then we have the following special Markov property: for every positive $\mathscr{F}_{\supset U}$-measurable Y

$$P_\mu\{Y|\mathscr{F}_{\subset U}\}=P_{\tau_U}Y. \tag{1.6}$$

We call $X^D=(X_{\tau_U},P_\mu;U\subset D,\mu\in M(D))$ the (L,α)-superdiffusion in D. Let $\mathscr{R}_D$ denote the support of X^D. $\mathscr{R}_D$ is also called the range of superdiffusion X in D.

For a bounded smooth domain D, the absolute continuity of the exit measure X_{τ_D} is closely related to the following boundary singularity problem:

$$\begin{cases} Lu = u^{\alpha}, & \text{in } D, \\ u|_{\partial D\setminus\{0\}} = f, \end{cases} \tag{1.7}$$

where L is a uniformly elliptic operator in $\mathbf{R}^d$, $0\in\partial D$, and $1<\alpha\leqslant 2$. If $d<1+2/(\alpha-1)$, X_{τ_D} is absolutely continuous with respect to the surface measure $S(\mathrm{d}z)$ on ∂D (see Sheu, 1996 and Ren, 2000), and the corresponding boundary singularity problem (1.7) has three classes of solutions: removable singularity, weak singularity and strong singularity (see Gmira and Véron, 1991). But, if $d>1+2/(\alpha-1)$, X_{τ_D} is singular and problem (1.7) has one unique bounded solution. By using this relationship, Ren discussed all nonnegative solutions of problem (1.7) (see Ren, 2001).

So, we easily think that the interior singularity problem (1.1) is closely related to the absolute continuity of Y_{τ_D}. In Ren (2002), Ren discussed the absolute continuity of Y_{τ_D} with general branching mechanism.

Let $G_D(x,y)$ denote the Green function of the diffusion ξ in D. For $f\in b\mathscr{B}(D)$ and $v\in M(D)$, define

$$G_D f(x) = \Pi_x \int_0^{\tau_D} f(\xi_s)\,\mathrm{d}s = \int_D G_D(x,y)f(y)\,\mathrm{d}y, \quad G_D v(x) = \int_D G_D(x,y)v(\mathrm{d}y).$$

Obviously, if $v(\mathrm{d}y) = f\,\mathrm{d}y$, $G_D f = G_D v$.

We write $\mu\in M_c(D)$ if $\mu\in M(D)$ and has a compact support in D. Let $M_1(D)$ denotes the set of all measures v in $M(D)$ such that $G_D v$ being super-harmonic in D. Set $N_v=\{x, G_D v(x)=\infty\}$. Then N_v is a closed set having zero Lebesgue measure.

The following Proposition 1.1 is the main result of Ren (2002) for particular branching mechanism $\psi(z)=z^{\alpha}$.

Proposition 1.1. *Suppose D is a bounded C^2-domain in $\mathbf{R}^d$. If $d<2+2/(\alpha-1)$, then we have*:

(1) *for fixed $\mu\in M_c(D)\cap M_1(D)$, there exists a random measurable function y_D defined on $\bar{D}$ such that*

$$P_\mu\{Y_{\tau_D}(\mathrm{d}y) = y_D(y)\,\mathrm{d}y\} = 1,$$

(2) *for each finite collection $y_1,\ldots,y_k$ of points in $D\setminus N_\mu$, the Laplace function of the random vector $[y_D(y_1),\ldots,y_D(y_m)]$ with respect to P_μ is given by*

$$P_\mu \exp\left[-\langle f, Y_{\tau_D}\rangle - \sum_{i=1}^{k}\lambda_i y_D(y_i)\right] = \exp\langle -u,\mu\rangle, \quad \lambda_1,\ldots,\lambda_k\geqslant 0, \tag{1.8}$$

where $\mu\in M_c(D)\cap M_1(D)$, and u is the unique positive solution of the following integral equation:

$$u(x) + \Pi_x\int_0^{\tau_D} u^{\alpha}(\xi_s))\,\mathrm{d}s = G_D v(x), \quad x\in D\setminus N_v \tag{1.9}$$

with $v(\mathrm{d}y) = f(y)\,\mathrm{d}y + \sum_{i=1}^{k}\lambda_i\delta_{y_i}(\mathrm{d}y)$.

If $d>2+2/(\alpha-1)$. For every $\mu\in M_c(D)$, Y_{τ_D} is P_μ-a.s. singular with respect to the Lebesgue measure on $\bar{D}$.

In this paper, we, using the relationship between the absolute continuity of Y_{τ_D} and interior interior singularity problem (1.1), discuss all nonnegative solutions to problem (1.1).

Definition 1.1. Suppose u is a nonnegative unbounded solution of (1.1). u is called a weak singularity of (1.1) at 0 if u satisfies

$$\limsup_{x\to 0}\frac{u(x)}{G_D(x,0)}<\infty$$

u is called a strong singularity of (1.1) at 0 if u satisfies

$$\limsup_{x\to 0}\frac{u(x)}{G_D(x,0)}=\infty.$$

The following Theorem 1.1 is the main result of this paper.

Theorem 1.1. *Suppose D is a bounded C^2-domain, $0\in D$, and $2\leqslant d<2+2/(\alpha-1)$.*

(1) $u_\varphi(x)=-\log P_{\delta_x}\exp\langle-\varphi,X_{\tau_D}\rangle$ *is the unique bounded solution of* (1.1).

(2) u *is a weak singularity of* (1.1) *at* 0 *iff there exists a* $\lambda>0$ *such that*

$$u(x)=-\log P_{\delta_X}\exp\{-\langle\varphi,X_{\tau_D}\rangle-\lambda y_D(0)\}$$

and λ *is uniquely determined by the formula*

$$\lambda=\lim_{x\to 0}\frac{u(x)}{G_D(x,0)}<\infty. \tag{1.10}$$

(3) $-\log P_{\delta_x}\{\exp\langle-\varphi,X_{\tau_D}\rangle;y_D(0)=0\}$ *is the minimal strong singularity of* (1.1) *at* 0; $-\log P_{\delta_x}\{\exp\langle-\varphi,X_{\tau_D}\rangle;\mathscr{R}_D\cap\{0\}=\emptyset\}$ *is the maximal strong singularity of* (1.1) *at* 0.

2. Properties of the range $\mathscr{R}_D$

In this section, we first study some properties of the range $\mathscr{R}_D$.

Lemma 2.1 (Dynkin (1992), Theorem 1.2). *Suppose u_n is a sequence of nonnegative solutions of $Lu=u^\alpha$ in D and u_n converge pointwise in D to u. Then u is a solution of $Lu=u^\alpha$ in D.*

Let O be a relatively open subset of ∂D. If u_n satisfy the boundary condition $u_n=f$ on O, then the same condition holds for u.

Lemma 2.2. *Let $U\subset D$ be open sets. Then*

$$\{\mathscr{R}_D\subset\bar U\}\subset\{X_{\tau_D}(\bar U^c)=0\}\quad P_\mu\text{- a.s.}\quad \text{for every } \mu\in M(\mathbf{R}^d), \tag{2.1}$$

$$\{X_{\tau_U}(D)=0\}\subset\{\mathscr{R}_D\subset\bar U\}\quad P_\mu\text{- a.s.}\quad \text{for every } \mu\in M(\mathbf{R}^d) \tag{2.2}$$

and

$$\{\mathscr{R}_D\subset\bar U\}=\{X_{\tau_D}(\bar U^c)=0\}\quad P_\mu\text{- a.s}\quad \text{for every } \mu\in M(\bar U). \tag{2.3}$$

Proof. (2.1) and (2.2) follow from Lemmas 2.1 and 2.2 in Dynkin (1992). To prove (2.3), it is sufficient to prove that

$$\{X_{\tau_D}(\bar{U}^c)=0\} \subset \{X_{\tau_U}(D)=0\} \quad P_\mu\text{-a.s.} \quad \text{for } \mu\in M(\bar{U}).$$

But this inclusion follows easily from the special Markov property (1.6). In fact, for every $\mu\in M(\bar{U})$

$$P_\mu(X_{\tau_U}(D)=0; X_{\tau_D}(\bar{U}^c)>0) = P_\mu(P_{X_{\tau_U}}(X_{\tau_D}(\bar{U}^c)>0); X_{\tau_U}(D)=0)=0.$$

The last inequality follows from the fact that $X_{\tau_D}=v$, P_v-a.s. if $v(D)=0$ and $v(\bar{U}^c)=0$ if v is concentrated on $\bar{U}$. □

Proposition 2.1. *Suppose D is an open set and $\Gamma\subset D$ is a compact set*:

(1) *Put*

$$D_n=\{x\in D, d(x,\Gamma)>1/n\}, \quad \Gamma_n=\{x, d(x,\Gamma)=1/n\}.$$

Then

$$\{X_{\tau_{D_n}}(\Gamma_n)=0\}=\{X_{\tau_{D_n}}(D)=0\}\uparrow\{\mathscr{R}_D\cap\Gamma=\emptyset\} \quad P_\mu\text{-a.s.} \tag{2.4}$$

for every $\mu\in M(\mathbf{R}^d)$.

(2) *If D is regular and φ is a continuous function on ∂D, then*

$$-\log P_{\delta_x}\{\exp\langle-\varphi, X_{\tau_D}\rangle;\ \mathscr{R}_D\cap\Gamma=\emptyset\}$$

is the maximal nonnegative solution of

$$\begin{cases} Lu=u^\alpha & \text{in } D\setminus\Gamma,\\ u=\varphi & \text{on } \partial D.\end{cases} \tag{2.5}$$

Proof. (1) By Lemma 2.2, we have the following inclusions:

$$\{X_{\tau_{D_n}}(\Gamma_n)=0\}\subset\{X_{\tau_{D_n}}(D)=0\}\subset\{\mathscr{R}_D\subset\bar{D}_n\}\subset\{\mathscr{R}_{D_{n+1}}\subset\bar{D}_n\}$$
$$\subset\{X_{\tau_{D_{n+1}}}(\bar{D}_n^c)=0\}\subset\{X_{\tau_{D_{n+1}}}(\Gamma_{n+1})=0\} \quad P_\mu\text{-a.s.},\quad \mu\in M.$$

Hence, (2.4) holds.

(2) Put

$$u_n(x)=-\log P_{\delta_x}\exp(\langle-\varphi, X_{\tau_D}\rangle; X_{\tau_{D_n}}(\Gamma_n)=0).$$

By (2.4)

$$u_n(x)\downarrow -\log_{\delta_x}(\exp\langle-\varphi, X_{\tau_D}\rangle; \mathscr{R}_D\cap\Gamma=\emptyset). \tag{2.6}$$

Note that $X_{\tau_D}=\mu$, P_μ-a.s. if $\mu(D)=0$. By the special Markov property, for $x\in D_n$,

$$u_n(x)=-\log P_{\delta_x}(P_{X_{\tau_{D_n}}}\exp\langle-\varphi, X_{\tau_D}\rangle; X_{\tau_{D_n}}(\Gamma_n)=0)$$
$$=-\log P_{\delta_x}(\exp\langle-\varphi, X_{\tau_{D_n}}\rangle; X_{\tau_{D_n}}(\Gamma_n)=0).$$

Thus we have

$$-\log P_{\delta_x}(\exp\langle-\varphi I_{\partial D}-\lambda I_{\Gamma_n}, X_{\tau_{D_n}}\rangle)\uparrow -\log P_{\delta_x}(\exp\langle-\varphi, X_{\tau_{D_n}}\rangle; X_{\tau_{D_n}}(\Gamma_n)=0)=u_n.$$

By Theorem 1.1 in Dynkin (1991) and Lemma 2.1, u_n is a solution of $Lu = u^\alpha$ in D_n having boundary value φ on ∂D and boundary value $+\infty$ on Γ_n. Therefore, using Lemma 2.1 again, we obtain $-\log P_{\delta_x}(\exp\langle -\varphi, X_{\tau_D}\rangle; \mathscr{R}_D \cap \Gamma = \emptyset)$ is a solution of (2.5). The maximality follows easily from the comparison principle.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need some Lemmas. Let $\rho(x) = d(x, \partial D)$ be the distance from x to ∂D.

Lemma 3.1. *Suppose $d \geqslant 3$, D is a bounded C^2-domain in $\mathbf{R}^d$. There exists a constant $C > 0$ such that if $\|x - z\| \leqslant \rho(x) \wedge \rho(z)$, then*

$$G_D(x,z) \geqslant C\|x - z\|^{2-d}.$$

Proof. If $L = \Delta$, Lemma 3.1 is a particular case of Lemma 6.7 in Chung and Zhao (1995). For a general L, Lemma 3.1 follows immediately from the fact that in a bounded C^2-domain D, quotients of Green functions are uniformly bounded (see Hueber and Sieveking, 1982). □

Lemma 3.2. *For a bounded C^2-domain D in $\mathbf{R}^2$, there exists a constant $C > 0$ such that for all $x, y \in D$*

$$\frac{1}{C}\ln\left(1 + \frac{\rho(x)\rho(y)}{\|x - y\|^2}\right) \leqslant G_D(x, y) \leqslant C\ln\left(1 + \frac{\rho(x)\rho(y)}{\|x - y\|^2}\right).$$

Proof. If $L = \Delta$, this result is exactly Theorem 6.13 in Chung and Zhao (1995). Therefore, Lemma 3.2 also holds for general L by the same reason used in the proof of Lemma 3.1. □

Lemma 3.3. *Suppose $0 \in D, 2 \leqslant d < 2 + 2/(\alpha - 1)$ and u is a solution of (1.1). If $u(x)/G_D(x, 0)$ is bounded in a neighborhood of 0, then*

$$\lim_{x\to 0} \frac{\int_D G_D(x, y)u^\alpha(y)\,\mathrm{d}y}{G_D(x, 0)} = 0. \tag{3.1}$$

Proof. Since u is continuous in $\bar{D} \setminus \{0\}$ and $u(x)/G_D(x, 0)$ is bounded in a neighborhood of 0, there exists a constant $C > 0$ such that

$$u(x) \leqslant C(G_D(x, 0) + 1), \quad x \in D \setminus \{0\}.$$

By Minkowski inequality

$$\int_D G_D(x, y)u(y)^\alpha\,\mathrm{d}y \leqslant \left[\left(\int_D C^\alpha G_D(y, 0)^\alpha G_D(x, y)\,\mathrm{d}y\right)^{1/\alpha} + \left(\int_D C^\alpha G_D(x, y)\,\mathrm{d}y\right)^{1/\alpha}\right]^\alpha.$$

Since $\int_D G_D(x,y)\,\mathrm{d}y$ is bounded in D, to prove (3.1), we only need to prove that

$$\lim_{x\to 0}\frac{1}{G_D(x,0)}\int_D G_D(y,0)^\alpha G_D(x,y)\,\mathrm{d}y=0. \tag{3.2}$$

Note that

$$\frac{1}{G_D(x,0)}\int_D G_D(y,0)^\alpha G_D(x,y)\,\mathrm{d}y=I_1+I_2,$$

where

$$I_1=\int_{D\cap\{y:\,\|y-x\|\geqslant\|x\|/2\}} f(x,y)\,\mathrm{d}y,$$

$$I_2=\int_{D\cap\{y:\,\|y-x\|<\|x\|/2\}} f(x,y)\,\mathrm{d}y,$$

$$f(x,y)=\frac{G_D(x,y)G_D(y,0)^\alpha}{G_D(x,0)}.$$

(1) In the case $d\geqslant 3$, by Lemma 3.1, for $x\in D$ satisfying $\|x\|\leqslant\rho(x)\wedge\rho(0)$,

$$f(x,y)\leqslant C\frac{\|x\|^{d-2}}{\|x-y\|^{d-2}}\,\|y\|^{(2-d)\alpha}.$$

Thus for sufficiently small x and $y\in D\cap\{y:\|y-x\|\geqslant\|x\|/2\}$

$$f(x,y)\leqslant C\|y\|^{(2-d)\alpha}.$$

The assumption $d<2+2/(\alpha-1)$ implies that $\int_D\|y\|^{(2-d)\alpha}\,\mathrm{d}y<\infty$. It is obvious that for any $y\in D\setminus\{0\}$, $f(x,y)\to 0$ as $x\to 0$. The dominated convergence theorem implies that $I_1\to 0$ as $x\to 0$.

Now we estimate I_2. By noticing that $\|y-x\|<\|x\|/2$ implies that $\|x\|/2\leqslant\|y\|$, we have

$$\begin{aligned}I_2&\leqslant C\|x\|^{(2-d)(\alpha-1)}\int_{\|y-x\|<\|x\|/2}\|x-y\|^{2-d}\,\mathrm{d}y\\&=C\|x\|^{(2-d)(\alpha-1)}\int_0^{\|x\|/2} r\,\mathrm{d}r\\&=C\|x\|^{(2-d)(\alpha-1)}\|x\|^2\\&=C\|x\|^{2\alpha-(\alpha-1)d}\to 0\quad(\|x\|\to 0)\quad(d<2+2/(\alpha-1)).\end{aligned}$$

Therefore, we have proved Claim (3.1) in the case $d\geqslant 3$.

(2) In the case $d=2$, by Lemma 3.2

$$f(x,y)\leqslant C\frac{\ln\left(1+\dfrac{\rho(x)\rho(y)}{\|x-y\|^2}\right)}{\ln\left(1+\dfrac{\rho(x)\rho(0)}{\|x\|^2}\right)}\left[\ln\left(1+\frac{\rho(0)\rho(y)}{\|y\|^2}\right)\right]^\alpha.$$

If $\|x\| < \rho(0)/2$, then $\rho(x) = \inf_{z\in\partial D}\|x-z\| \geqslant \inf_{z\in\partial D}\|z-0\| - \|x\| = \rho(0) - \|x\| \geqslant \rho(0)/2$. Therefore there exists constant $C_1, C_2, C_3 > 0$ such that

$$f(x,y) \leqslant C\,\frac{\ln\left(1+\dfrac{C_1}{\|x-y\|^2}\right)}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)}\left[\ln\left(1+\frac{C_3}{\|y\|^2}\right)\right]^{\alpha}.$$

Thus, for $x\in D$ satisfying $\|x\| < \rho(0)/2$ and $y \in D\cap\{y : \|y-x\| \geqslant \|x\|/2\}$, we have

$$f(x,y) \leqslant C\frac{\ln\left(1+\dfrac{4C_1}{\|x\|^2}\right)}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)}\left[\ln\left(1+\frac{C_3}{\|y\|^2}\right)\right]^{\alpha}.$$

By noticing that

$$\lim_{x\to 0}\frac{\ln\left(1+\dfrac{4C_1}{\|x\|^2}\right)}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)} = 4C_1/C_2$$

there exists constant $\delta < \rho(0)/2$ and $C > 0$ such that, for $\|x\| \leqslant \delta$ and $y \in D\cap\{y : \|y-\| \geqslant \|x\|/2\}$, we have

$$f(x,y) \leqslant C\left[\ln\left(1+\frac{C_3}{\|y\|^2}\right)\right]^{\alpha}.$$

It is easy to check that $\int_D [\ln(1+C_3/\|y\|^2)]^{\alpha}\,\mathrm{d}y < \infty$. Since for fixed $y \in D\setminus\{0\}$, $f(x,y)\to 0$ as $x \to 0$, by the dominated convergence theorem, we have $I_1 \to 0$ as $x\to 0$.

In the following we estimate I_2:

$$\begin{aligned}
I_2 &\leqslant C\int_{\|y-x\|<\|x\|/2}\frac{\ln\left(1+\dfrac{C_1}{\|x-y\|^2}\right)}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)}\left[\ln\left(1+\frac{C_3}{\|y\|^2}\right)\right]^{\alpha}\mathrm{d}y \\
&\leqslant C\frac{\left[\ln\left(1+\dfrac{4C_3}{\|x\|^2}\right)\right]^{\alpha}}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)}\int_0^{\|x\|/2}\ln\left(1+\frac{C_1}{r^2}\right)r\,\mathrm{d}r \\
&\leqslant C\left[\ln\left(1+\frac{4C_3}{\|x\|^2}\right)\right]^{\alpha}\|x\|\frac{1}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)}\ln\left(1+\frac{C_1}{r_0^2}\right)r_0,
\end{aligned}$$

where r_0 is a point in $(0, \|x\|/2)$. Since

$$\lim_{x\to 0}\left[\ln\left(1+\frac{4C_3}{\|x\|^2}\right)\right]^{\alpha}\|x\| = \lim_{y\to\infty}[\ln(1+4C_3y^2)]^{\alpha}/y$$

$$= \left(\lim_{y\to\infty}\frac{\ln(1+4C_3y^2)}{y^{1/\alpha}}\right)^{\alpha} = 0,$$

$$\lim_{x\to 0}\frac{1}{\ln\left(1+\dfrac{C_2}{\|x\|^2}\right)} = 0$$

and

$$\lim_{r_0\to 0}\ln\left(1+\frac{C_1}{r_0^2}\right)r_0 = 0,$$

we conclude that $\lim_{x\to 0} I_2 = 0$. Thus we proved (3.1) in the case $d = 2$. □

Lemma 3.4. *Suppose* $2 \leqslant d < 2 + 2/(\alpha - 1)$ *and* $0 \in D$. *Then* $\int_D G_D(x,y)G_D(y,0)^{\alpha}\,\mathrm{d}y$ *is locally bounded in* $D\setminus\{0\}$, *and for every* $a \in \partial D$,

$$\lim_{x\in D, x\to a}\int_D G_D(x,y)G_D(y,0)^{\alpha}\,\mathrm{d}y = 0.$$

Proof. Let K be a compact subset of $D\setminus\{0\}$ and $K_1 = \{y : d(y,K) \leqslant \rho(0,K)/2\}$. For every $y \in K_1 \cap D$ we have $\|y\| \geqslant \rho(0,K)/2$, and for every $x \in K, y \in D\setminus K_1$ we have $\|x - y\| \geqslant \rho(0,K)/2$. Hence there exists constant C such that

$$\int_D G_D(x,y)G_D(y,0)^{\alpha}\,\mathrm{d}y \leqslant C\left(\int_D G_D(x,y)\,\mathrm{d}y + \int_D G_D(y,0)^{\alpha}\,\mathrm{d}y\right), \quad x \in K.$$

From the above inequality and the assumption $d < 2 + 2/(\alpha - 1)$, it is easy to see that $\int_D G_D(x,y) G_D(y,0)^{\alpha}\,\mathrm{d}y$ is bounded in K.

For sufficiently large n ($n > 3/\rho(0)$) and $x \in D \cap B(a, \rho(0)/3)$ we have, in the case $d \geqslant 3$,

$$\int_D G_D(x,y)G_D(y,0)^{\alpha}\,\mathrm{d}y$$

$$\leqslant C\left[\int_{\|y\|\leqslant 1/n}\|x-y\|^{2-d}\|y\|^{(2-d)\alpha}\,\mathrm{d}y + \int_{D\cap\{y:\|y\|>1/n\}} G_D(x,y)\|y\|^{(2-d)\alpha}\,\mathrm{d}y\right]$$

$$\leqslant C\left[\left(\frac{\rho(0)}{3}\right)^{2-d}\int_0^{1/n} r^{(2-d)\alpha}r^{d-1}\,\mathrm{d}r + n^{(d-2)\alpha}\int_D G_D(x,y)\,\mathrm{d}y\right]$$

and in the case $d = 2$

$$\int_D G_D(x,y)G_D(y,0)^{\alpha}\,\mathrm{d}y$$

$$\leqslant C\int_{\|y\|\leqslant 1/n}\ln\left(1+\frac{\rho(x)\rho(y)}{\|x-y\|^2}\right)\left[\ln\left(1+\frac{\rho(x)\rho(0)}{\|y\|^2}\right)\right]^{\alpha}\,\mathrm{d}y$$

$$+C\int_{D\cap\{y:\,\|y\|\geqslant 1/n\}} G_D(x,y)\left[\ln\left(1+\frac{\rho(x)\rho(0)}{\|y\|^2}\right)\right]^\alpha \mathrm{d}y$$

$$\leqslant C\int_0^{1/n}\left[\ln\left(1+\frac{C}{r^2}\right)\right]^\alpha r\,\mathrm{d}r + C\ln\left(1+Cn^2\right)\int_D G_D(x,y)\,\mathrm{d}y.$$

Since $\lim_{x\in D,x\to 0}\int_D G_D(x,y)\,\mathrm{d}y=0$, letting $x\to a$ and then $n\to\infty$ in the above two inequalities, we obtain

$$\lim_{x\in D,x\to a}\int_D G_D(x,y)G_D(y,0)^\alpha\,\mathrm{d}y=0. \qquad \square$$

Lemma 3.5. *Suppose* $0\in D$. *If* h *is a* L*-harmonic function on* $D\setminus\{0\}$ *having boundary value* 0 *on* ∂D *and* $\liminf_{x\to 0} h/G_D(x,0)\geqslant 0$, *Then there exists* $\lambda\geqslant 0$ *such that* $h(x)=\lambda G_D(x,0)$, $x\in D\setminus\{0\}$.

Proof. By the assumption, there exists a constant $C>0$ such that $h/G_D(x,0)\geqslant -C$. Let $h_1=h+CG_D(x,0)$. Then $h_1\geqslant 0$ is also a L-harmonic function on $D\setminus\{0\}$ having boundary value 0 on ∂D. Since $\liminf_{x\to 0}h_1/G_D(x,0)\geqslant C>0$, and $\liminf_{x\to 0}h_1(x)=+\infty$, if letting $h_1(0)=+\infty$, then h_1 is lower semicontinuous in D. Therefore, h_1 is a positive L-superharmonic function in D. By the Riesz decomposition theorem (Blumenthal and Getoor, 1968, p. 272), $h_1=G_D\mu+h_2$ for some measure μ on D and some L-harmonic function h_2 on D such that $G_D\mu$ is L-superharmonic on D. Since h_1 has boundary value 0, $h_2\equiv 0$. Note that $G_D\mu$ is L-harmonic in $D\setminus\{0\}$, by Theorem 6.1.4 in Port and Stone (1978) (Port and Stone's result is for $L=\Delta$, but their proof also holds for L), $\mu(D\setminus\{0\})=0$. Set $\lambda_1=\mu(\{0\})$, then $h_1(x)=\lambda_1 G_D(x,0)$ and hence $h(x)=\lambda G_D(x,0),\lambda=\lambda_1-C$. The assumption $\liminf_{x\to 0}h/G_D(x,0)\geqslant 0$ implies that $\lambda\geqslant 0$. $\square$

Lemma 3.6. *Let* g *be a locally bounded function in an open set* D *and* $F(x)=\Pi_x\int_0^{\tau_D} g(\xi_s)\,\mathrm{d}s$. *If* $g\in C^{0,\lambda}(D)$ *and* $F(x)$ *is locally bounded in* D, *then* $F\in C^{2,\lambda}(D)$ *and* $LF=-g$ *in* D.

Proof. Lemma 3.6 is a generalization of Theorem 0.3 in Dynkin (1991). It is easy to prove by the strong Markov property of ξ, and Theorem 0.2 and Theorem 0.3 in Dynkin (1991). We omit the details. $\square$

Proof of Theorem 1.1. (1) It is obvious that u_φ is a bounded solution of (1.1). Suppose u is an arbitrary bounded solution of (1.1). By Lemma 3.3, $h(x)=u(x)+\int_D G_D(x,y)u^\alpha(y)\,\mathrm{d}y$ is a bounded solution of $Lh=0$ in $D\setminus\{0\}$ having boundary value φ at ∂D. From the classical theory of the regularity of solutions of elliptic equations near a interior point, we deduce that h and hence u can be continuously extended to $\bar{D}$. By Lemma 3.1 and comparison principle we get $u=u_\varphi$.

(2) Put

$$u_{\varphi,\lambda}(x)=-\log P_{\delta_x}\exp\{-\langle\varphi,X_{\tau_D}\rangle-\lambda y_D(0)\}.$$

It follows from Proposition 1.1 and its proof that, for any constant $\lambda>0$, $u_{\varphi,\lambda}$ satisfies

$$u_{\varphi,\lambda}+\int_D G_D(x,y)u^\alpha_{\varphi,\lambda}(y)\,\mathrm{d}y=\Pi_x\varphi(\xi_{\tau_D})+\lambda G_D(x,0) \tag{3.3}$$

and

$$-\log P_{\delta_x} \exp\{-\langle \varphi, X_{\tau_D}\rangle - \lambda\langle \rho_n, Y_{\tau_D}\rangle\} \to u_{\varphi,\lambda},$$

where $\rho_n(z) = I_{(B(0,1/n)\cap D)}(z)/m(B(0,1/n)\cap D)$, m is the Lebesgue measure in $\mathbf{R}^d$. By Lemma 2.1, $u_{\varphi,\lambda}$ is a solution of $Lu = u^\alpha$. The boundary condition $u_{\varphi,\lambda}|_{\partial D} = \varphi$ follows from the inequality: $u_\varphi \leqslant u_{\varphi,\lambda} \leqslant \Pi_x\varphi(\xi_{\tau_D}) + \lambda G_D(x,0)$.

From (3.3), $u_{\varphi,\lambda}/G_D(x,0)$ is bounded near 0 and therefore by Lemma 3.3

$$\lim_{x\to 0} u_{\varphi,\lambda}/G_D(x,0) = \lambda.$$

Hence $u_{\varphi,\lambda}$ admits a weak singularity of problem (1.1).

Conversely, Suppose u is a weak singularity of problem (1.1) at 0. Then there exits a constant C such that

$$u(x) \leqslant C(G_D(x,0)+1).$$

By Minkowski inequality

$$\int_D G_D(x,y)u^\alpha(y)\,\mathrm{d}y \leqslant C^\alpha\left[\left(\int_D G_D(x,y)G_D(y,0)^\alpha\,\mathrm{d}y\right)^{1/\alpha} + \left(\int_D G_D(x,y)\,\mathrm{d}y\right)^{1/\alpha}\right]^\alpha.$$

By Lemma 3.4, $\int_D G_D(x,y)u^\alpha(y)\,\mathrm{d}y$ is locally bounded in $D\setminus 0$. Put $h(x) = u(x) + \int_D G_D(x,y)\, u^\alpha(y)\,\mathrm{d}y - \Pi_x\varphi(\xi_{\tau_D})$. Then it follows from Lemma 3.6 that h satisfies $Lh=0$ in $D\setminus\{0\}$. By Lemma 3.4, h has boundary value 0 on ∂D. Hence by Lemma 3.5 there exists $\lambda \geqslant 0$ such that

$$u(x) + \int_D G_D(x,y)u^\alpha(y)\,\mathrm{d}y = \Pi_x\varphi(\xi_{\tau_D}) + \lambda G_D(x,0).$$

By Proposition 1.1 , $u_{\varphi,\lambda}$ is the unique positive solution of the above integral equation, which means $u = u_{\varphi,\lambda}$.

(3) Put

$$u_{\varphi,\infty} = -\log P_{\delta_x}\{\exp\langle -\varphi, X_{\tau_D}\rangle; y_D(0) = 0\}.$$

Note that $u_{\varphi,\lambda} \uparrow u_{\varphi,\infty}$. By Lemma 2.1, $u_{\varphi,\infty}$ is a solution of (1.1). It is obvious that

$$\lim_{x\in D, x\to 0} \frac{u_{\varphi,\infty}}{G_D(x,0)} = +\infty.$$

Hence $u_{\varphi,\infty}$ is a strong singularity of (1.1) at 0.

Suppose u is an arbitrary strong singularity of (1.1) at 0, then for any $\lambda > 0$,

$$\limsup_{x\in D, x\to 0} (u(x) - u_{\varphi,\lambda}(x)) \geqslant 0.$$

The comparison principle implies that

$$u(x) \geqslant u_{\varphi,\lambda}(x), \quad x\in D, \ \lambda > 0.$$

Letting $\lambda\to\infty$, we get $u(x) \geqslant u_{\varphi,\infty}(x), x\in D$. Hence $u_{\varphi,\infty}$ is the minimal strong singularity of (1.1) at 0.

By Proposition 2.1, $-\log P_{\delta_x}\{\exp\langle -\varphi, X_{\tau_D}\rangle; \mathscr{R}_D\cap\{0\} = \emptyset\}$ is the maximal solution of (1.1). So it is also the maximal strong singularity of (1.1) at 0. □

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