第九章、第十章

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  §9.2 一元线性回归(续)

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- x = 路程(可设定), y = 耗油量.

  x = 父亲身高(不可设定, 只可测量), y = 儿子身高.

  关心f, 不关心自变量如何变化. 将x 视为参数.

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  x_i 是参数, y_i 是随机变量或其取值(可观测).
- e_i 是随机变量, 但取值未知(不可观测), 因为f 未知.

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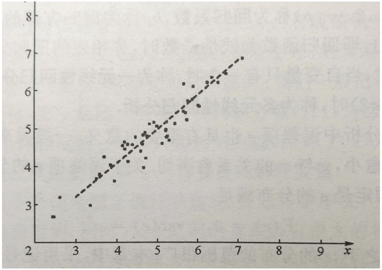
- 例1.1. $y = f(x) + e$, 一元. 数据: (x_i, y_i), $i = 1, \cdots, n = 50$. 观察散点图, 确认f 是否线性.



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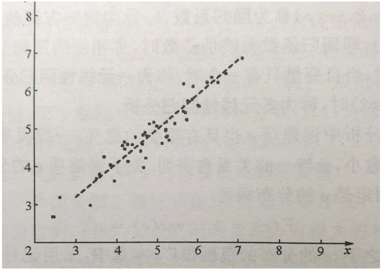

- 建立回归模型: $y_i = a + bx_i + e_i$, $i = 1, \cdots, n$.

例1.3. $x=$ 某小区人口数, $y=$ 冬季用煤量, $z=$ 室温.

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- 预测. 回归关系: $y=a+bx+e$.

  数据(x_i, y_i), $i=1,\cdots,n$.

  小区人口为x, 冬季应储备多少煤?

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  数据(x_i,y_i), $i=1,\cdots,n$.
  小区人口为x, 冬季应储备多少煤?
  自变量→ 因变量.
- 控制. 回归关系: $z=c+dy+\varepsilon$.
  数据(y_i,z_i), $i=1,\cdots,n$.
  为控制室温为18度, 冬季应储备多少煤?
  因变量→ 自变量.

§9.2 一元线性回归

$y = a + bx + e$, $e \sim N(0, \sigma^2)$, σ^2 未知.

数据: (x_i, y_i), $i = 1, \cdots, n$.

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数据: (x_i, y_i), $i = 1, \cdots, n$.

- 最小二乘拟合系数 指: 使得$Q(a,b) = \sum_{i=1}^{n}[y_i - (a + bx_i)]^2$ 达到最小的a, b. 记为$\hat{a}$, $\hat{b}$.

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- 回归模型: $y_i = a + bx_i + e_i$, $i = 1, \cdots, n$.
  $p_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - (a + bx_i))^2}$,
  视x_i 为已知参数, a, b 为未知待估参数, σ^2 为讨厌参数.
- 似然函数: $L(a, b, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}[y_i - (a + bx_i)]^2}$.

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- 最大似然估计: $L(a, b, \sigma^2)$ 的最大值点.
  a, b 的最大似然估计使得$Q(a,b)$ 达到最小, 即为$\hat{a}$, $\hat{b}$.

定理2.1. $\hat{a} = \bar{y} - \hat{b}\bar{x}, \quad \hat{b} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\ell_{xy}}{\ell_{xx}}.$

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- 样本: $P(\xi = x_i, \eta = y_i) = \frac{1}{n}$, $i = 1, \cdots, n$.
  则$E[\eta - (a + b\xi)]^2 = \frac{1}{n}\sum_{i=1}^n [y_i - (a + bx_i)]^2 = \frac{1}{n}Q(a, b)$.

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- 回顾第八次课, 第三章定理5.3.
  $a=E\eta-bE\xi \to E(\tilde{\eta}-b\tilde{\xi})^2$, $\tilde{\eta}=\eta-E\eta$, $\tilde{\xi}=\xi-E\xi$.

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- $E\xi = \bar{x}, \ E\eta = \bar{y}. \ \mathrm{var}(\xi) = \frac{1}{n}\ell_{xx}, \ \mathrm{var}(\eta) = \frac{1}{n}\ell_{yy},$
  $\mathrm{cov}(\xi, \eta) = \frac{1}{n}\ell_{xy}, \quad \rho_{\xi,\eta} = \frac{\ell_{xy}}{\sqrt{\ell_{xx}\ell_{yy}}} =: r$ 样本相关系数(2.15).

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- $E\xi = \bar{x}, E\eta = \bar{y}$. $\mathrm{var}(\xi) = \frac{1}{n}\ell_{xx}, \mathrm{var}(\eta) = \frac{1}{n}\ell_{yy}$,
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- $Q(\hat{a}, \hat{b}) = nE(\tilde{\eta} - \hat{b}\tilde{\xi})^2 = n\mathrm{var}(\eta)(1 - \rho_{\xi,\eta}^2) = \ell_{yy}(1 - r^2)$.

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- $E\xi=\bar{x}, E\eta=\bar{y}.\ \mathrm{var}(\xi)=\frac{1}{n}\ell_{xx}, \mathrm{var}(\eta)=\frac{1}{n}\ell_{yy},$
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- $Q(\hat{a},\hat{b})=nE(\tilde{\eta}-\hat{b}\tilde{\xi})^2=n\mathrm{var}(\eta)(1-\rho_{\xi,\eta}^2)=\ell_{yy}(1-r^2).$
- $Q(\hat{a},\hat{b})=Q$ 残差平方和, $r^2=1-Q/\ell_{yy}$ (2.20).
  正交分解$\stackrel{b=0}{\longrightarrow}\ell_{yy}=Q+U$ (引理2.1), U: 回归平方和.

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  - 正交分解: $\tilde{\eta} = (\tilde{\eta} - \hat{b}\tilde{\xi}) \oplus \hat{b}\tilde{\xi}$, $\eta \to y_i$, $\xi \to x_i$.
    $E\tilde{\eta}^2 = E(\tilde{\eta} - \hat{b}\tilde{\xi})^2 + E(\hat{b}\tilde{\xi})^2$.

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    $E\tilde{\eta}^2 = E(\tilde{\eta} - \hat{b}\tilde{\xi})^2 + E(\hat{b}\tilde{\xi})^2$.
  - $\tilde{\eta} = \eta - E\eta \to y_i - \bar{y} \to \ell_{yy}$.
    $\hat{b}\tilde{\xi} \to \hat{b}(x_i - \bar{x}) = \hat{f}(x_i) - \hat{f}(\bar{x}) = \hat{y}_i - \bar{y} \to U$,
    $\tilde{\eta} - \hat{b}\tilde{\xi} \to y_i - \bar{y} - (\hat{y}_i - \bar{y}) = y_i - \hat{y}_i \to Q$.

无偏估计: $E\hat{b} = b,\ E\hat{a} = a$

$\hat{a} = \bar{y} - \hat{b}\bar{x},\ \hat{b} = \frac{\ell_{xy}}{\ell_{xx}} = \frac{1}{\ell_{xx}} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}).$

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- 定理2.2. 假设$x_i,\ i = 1, \cdots, n$ 不全相等.

  (关心f, 即, 当x 变化时, y 如何跟着变化).

  那么, $\hat{a},\ \hat{b}$ 是最优线性无偏估计.

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- 其他统计量通过基本统计量计算得到:

$$\hat{b} = l_{xy}/l_{xx},$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x},$$

$$U = \hat{b}l_{xy},$$

$$Q = l_{yy} - U,$$

$$r^2 = U/l_{yy} = 1 - Q/l_{yy}.$$

残差平方和. $Q = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$. 则$\frac{1}{\sigma^2}Q \sim \chi^2(n-2)$.

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  $\ldots = (\hat{b} - b)(x_i - \bar{x}) - (e_i - \bar{e})$.
  $(\ldots)^2 = (\hat{b} - b)^2(x_i - \bar{x})^2 + (e_i - \bar{e})^2 - 2(\hat{b} - b)\underline{(x_i - \bar{x})(e_i - \bar{e})}$.

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- 若$b \neq 0$, $\frac{1}{\sigma^2} U = (\frac{1}{\sigma}\sqrt{\ell_{xx}} b + Z)^2$.
- 若$Z \sim N(0,1)$, 则$P(|Z + c| \leqslant l) \leqslant P(|Z| \leqslant l)$.
  $U_b := (\sqrt{\ell_{xx}} b + Z_2)^2$, $\quad U_0 := {Z_2}^2$
  $P(U_b \leqslant l^2) \leqslant P(U_0 \leqslant l^2)$ vs $U_b \leqslant l^2 \Rightarrow U_0 \leqslant l^2$, $\forall l$.
  在某种意义下, $U_0 \leqslant U_b$.