复习.

- 总体$X \sim F_\theta$, $\theta \in \Theta$.
- 样本$\vec{X} = (X_1, \cdots, X_n)$,
- 样本(值) $\vec{x} = (x_1, \cdots, x_n)$.
- 目标: 用统计量$T(X_1, \cdots, X_n)$ 作为θ (或$g(\theta)$) 的估计.
- 最大似然估计:

  $L(\theta) = \prod_{i=1}^n p_\theta(x_i)$ 在Θ 中的最大值点$\hat{\theta}$.

§7.2 矩估计

总体$X \sim F_\theta$. 目标: 给出θ 的估计值$\hat{\theta}$.

思想: 样本矩= 真实矩. 支撑: LLN.

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  令:

  $$m_k(\hat{\theta}) = a_k, \quad k = 1, \cdots r.$$

  (其中r是使得方程组有唯一解的最小整数), 唯一解$\hat{\theta}$称为θ的矩估计.

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- 习题一、11(续). N 条鱼中有80条有标记, 100条中4条有标记. 估计N.

  $n = 1$, 总体矩$m_1 = m_1(N) = 100 * \frac{80}{N}$, 样本矩$a_1 = 4$.

  求解$m_1(\hat{N}) = a_1$, 得$\hat{N} = 2000$.

例1.1(续) 飞机最大飞行速度$X \sim N(\mu, \sigma^2)$, 数据$x_1, \cdots, x_{15}$. 估计μ, σ^2.

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解得$\hat{\mu} = \bar{x}$, $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$.

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- 定义2.1. 若待估量$g(\theta) = \phi(m_1, \cdots, m_k)$(不一定是参数本身). 定义$g(\theta)$的矩估计为$\widehat{g(\theta)} := \phi(a_1, \cdots, a_k)$.

§7.3 估计的无偏性

称$T(X_1, \cdots, X_n)$ 是$g(\theta)$ 的<u>无偏估计</u>, 如果

$$E_\theta T(X_1, \cdots, X_n) = g(\theta), \forall \theta \in \Theta.$$

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- 样本方差$\widehat{\sigma^2}=\frac{1}{n}\sum_{i=1}^n(X_i-\bar{X})^2$.
  - $\frac{1}{n}\sum_{i=1}^n(x_i-\mu)^2=\frac{1}{n}\sum_{i=1}^n(x_i-\bar{x})^2+(\mu-\bar{x})^2$.
  - $\frac{1}{n}\sum_{i=1}^n(X_i-\bar{X})^2=\frac{1}{n}\sum_{i=1}^n(X_i-\mu)^2-(\mu-\bar{X})^2$.
  - $E_\theta\widehat{\sigma^2}=\mathrm{var}(X_1)-\mathrm{var}(\bar{X})=\sigma^2-\frac{1}{n}\sigma^2=\frac{n-1}{n}\sigma^2$.

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  - $E_\theta\widehat{\sigma^2}=\mathrm{var}(X_1)-\mathrm{var}(\bar{X})=\sigma^2-\frac{1}{n}\sigma^2=\frac{n-1}{n}\sigma^2$.
  - 估小了: $Y_i=X_i-\bar{X}$, $\forall i$ 不独立, 一个约束条件$\sum_i Y_i=0$.
- $S^2=\frac{1}{n-1}\sum_{i=1}^n(X_i-\bar{X})^2$ 是σ^2 的无偏估计. (定理3.1)

  有些书称S^2 为样本方差.

例1.4+2.4. $X \sim U[0,\theta]$, 数据$x_1, \cdots, x_n$. 求θ 的最大似然估计$\hat{\theta}_1$ 与矩估计$\hat{\theta}_2$.

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  - 缺点: $\hat{\theta}_1=\max_i X_i<\theta$, 故$E_\theta\hat{\theta}_1<\theta$, 估小了!

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  - 进一步计算:

$$\begin{aligned}
E_\theta\hat{\theta}_1 &= E_\theta \max_i X_i = \int_0^\theta P(\max_{1\leqslant i\leqslant n} X_i > x)dx \\
&= \int_0^\theta (1-P(\max_{1\leqslant i\leqslant n} X_i \leqslant x))dx \\
&= \theta - \int_0^\theta (\tfrac{1}{\theta}x)^n dx = (1-\tfrac{1}{n+1})\theta.
\end{aligned}$$

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    - 缺点: 有可能$2\bar{x} < \max_i x_i$, 明显不合理.
- 不能单纯追求无偏性.

§7.4 无偏估计的优良性

假设$\hat{\theta}_1$, $\hat{\theta}_2$ 都是θ 的无偏估计, 即$E_\theta\hat{\theta}_1 = E_\theta\hat{\theta}_2 = \theta$.
若$\mathrm{var}_\theta(\hat{\theta}_1) \leqslant \mathrm{var}_\theta(\hat{\theta}_2)$, 则认为$\hat{\theta}_1$ 更好.

例. $X \sim \mathrm{Exp}(\lambda)$. 样本$X_1, \cdots, X_n$. 估计$\theta = \frac{1}{\lambda} = EX$.

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    - 理由: $\min_{1\leqslant i\leqslant n} X_i \sim \mathrm{Exp}(n\lambda)$, (第七章, 例4.8)
      $\hat{\theta}_2 \sim \mathrm{Exp}(\lambda)$, $P(\hat{\theta}_2 > x) = P(X > \frac{1}{n}x)^n = e^{n\times(-\lambda\frac{1}{n}x)}$.
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  故, $\hat{\theta}_1$ 比$\hat{\theta}_2$ 好.

(一致)最小方差无偏估计

(Uniformly Minimum Variance Unbiased Estimator, UMVUE):

假设$T = T(X_1, \cdots, X_2)$ 是θ 的无偏估计, 且对于θ 的任意无偏估计$\tilde{T} = \tilde{T}(X_1, \cdots, X_n)$ 都有

$$\mathrm{var}_\theta(T) \leqslant \mathrm{var}_\theta(\tilde{T}), \quad \forall \theta \in \Theta.$$

则称$T = T(X_1, \cdots, X_2)$ 是θ 的(一致)最小方差无偏估计.

指数族分布: 指密度(或分布列)为

$$p_\theta(x) = S(\theta)h(x)\exp\{\sum_{k=1}^m C_k(\theta)T_k(x)\}.$$

指数族分布特性: 假设总体$X \sim p_\theta(x)$, $X_1, \cdots X_n$为X的样本, 则$(X_1, \cdots, X_n)$的密度(或分布列)为

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- θ 与x 分离. 例4.8, 4.9, 4.10.

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- 正态: $p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2-2\mu x+\mu^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\mu^2}{2\sigma^2}}e^{-\frac{1}{2\sigma^2}x^2+\frac{\mu}{\sigma^2}x}$

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$$\Pi_{i=1}^n p_\theta(x) = S^n(\theta)\Pi_{i=1}^n h(x_i)\exp\{\sum_{k=1}^m C_k(\theta)\sum_{i=1}^n T_k(x_i)\}$$

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- θ 与x 分离. 例4.8, 4.9, 4.10.
- 指数: $p_\lambda(x) = \lambda e^{-\lambda x}1_{\{x>0\}} = \lambda 1_{\{x>0\}}e^{-\lambda x}$.
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- 二项: $P_{n,p}(X = k) = C_n^k p^k(1-p)^{n-k} = C_n^k e^{k\log p+(n-k)\log(1-p)} = C_n^k e^{(\log p-\log(1-p))k+n\log(1-p)}$

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定理(4.2+4.3). 若

- Θ 是$\mathbb{R}^m$ 中有内点的集合.
- $(C_1, \cdots C_m) : \Theta \to \mathbb{R}^m$ 一一对应, 连续.
- $C_k(\theta)$, $1 \leqslant k \leqslant m$ 线性无关; $T_k(x)$, $1 \leqslant k \leqslant m$ 线性无关. (例如, 当$T_1(x) = x$, $T_2(x) = x^2$, $T_3(x) = x + x^2$ 时, 应化简为$(C_1(\theta) + C_3(\theta))x + (C_2(\theta) + C_3(\theta))x^2$.)
- $\hat{\theta} = \phi(T_1(\vec{X}), \cdots, T_m(\vec{X}))$ 是θ (或$g(\theta)$) 的无偏估计.

则$\hat{\theta}$ 是UMVUE.

注: $T_k(\vec{X}) = \sum_{i=1}^n T_k(X_i)$, $\quad k = 1, \cdots, m$.

例4.14. $X \sim N(\mu, \sigma^2)$.

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  $C_1(\theta) = \frac{\mu}{\sigma^2}$, $C_2(\theta) = -\frac{1}{2\sigma^2}$ 线性无关.

  $T_1(x) = x$, $T_2(x) = x^2$ 线性无关.

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- $T_1(\vec{X}) = \sum_{i=1}^n X_i = n\bar{X}$,

$$T_2(\vec{X}) = \sum_{i=1}^n X_i^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n\bar{X}^2 = (n-1)S^2 + n\bar{X}^2.$$

$\phi(T_1(\vec{X}), T_2(\vec{X})) = \varphi(\bar{X}, S^2)$,

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  (注: $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$)
- 若$\varphi(\bar{X}, S^2)$ 是无偏估计, 则是UMVUE.
  特别地, $\bar{X}$, S^2 分别是μ, σ^2 的UMVUE.