Lattice reptiles with 2 pieces

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Basic definitions

Definition

A tiling of a set on $\mathbb{C}$ is a cover of its subsets which have no interior point in common.

Definition

For a compact metric space (X, d), a continuous mapping $f : X \to X$ is expanding if there exist constants $\lambda > 1, \eta > 0$ and $n \geq 0$ such that for all $x, y \in X$

$$d(x, y) \leq 2\eta \Rightarrow d(f^n(x), f^n(y)) \geq \lambda d(x, y) \tag{1}$$

Core concept

Definition

A lattice reptile with k pieces in $\mathbb{C}$ is a compact set $A \subset \mathbb{C}$ with non-empty interior such that:

(i) There is a lattice $\Lambda \subset \mathbb{C}$ (i.e. $\Lambda \cong \{m + n\mathbf{i} \,|\, m, n \in \mathbb{Z}\}$) such that $\{z + A \,|\, z \in \Lambda\}$ is a tiling of $\mathbb{C}$.

(ii) There exists an expanding linear map $f\colon \Lambda \to \Lambda$ and vectors $z_i \in \Lambda$ for $i = 1, 2, \ldots, k$, such that $\{(z_1 + A), (z_2 + A), \ldots, (z_k + A)\}$ is a tiling of $f(A)$. z_i are called residues.

Here $f(\Lambda)$ is a subgroup of Λ. This can be derived from the linearity of f.
An example is a square. It can be divided into 4 smaller squares, so it is a lattice reptile with 4 pieces.

Propositions which restrict α

Now we discuss $f(z) = \alpha z$(or $f(z) = \alpha\bar{z}$). There are some restrictions on α, which can help us classify lattice reptiles with k pieces.

Proposition

For a linear dilation map $f(z) = \alpha z$ (or $f(z) = \alpha\bar{z}$), if it is expanding, then $|\alpha| > 1$.

This is obvious because when $|\alpha| \leq 1$,

$$|f(z_1) - f(z_2)| = \alpha|z_1 - z_2| \leq |z_1 - z_2|.$$

Propositions which restrict α

Proposition

Given a lattice reptile $A \subset \mathbb{C}$ with k pieces and expanding linear map $f(z) = \alpha z$ or $f(z) = \alpha \bar{z}$, we have $|\alpha|^2 = k$.

This can be derived by considering $Area\, f(A)$. Because $f(A)$ consists of k pieces of A.

Lemma

For a linear expanding map (on $\mathbb{C}$) $f(z) = \alpha z$ (or $f(z) = \alpha \bar{z}$), $\alpha \in \mathbb{C}$, and has integer real and imaginary part. Denote $|\alpha|^2$ by $K \in \mathbb{N}$ ($K \geq 2$) and $2\Re(\alpha) \in \mathbb{Z}$ by m. Then $|m| \leq K$.

We suppose that α has integer real and imaginary part without loss of generation because we can choose a basis of $\mathbb{C}$ arbitrarily. This lemma can be derived by Mean inequality.

The equivalent relation

To classify the lattice reptiles, we need to define equivalent relation between them.

Definition

Let A_1, A_2 be two lattice reptiles in $\mathbb{C}$ with expanding mappings f_1, f_2 and residues z_{i1}, z_{i2} for $i = 1, 2, ..., k$. An affine bijection $\phi\colon f_1(A_1) \to f_2(A_2)$ is called an equivalence if $\phi(z_{i1} + A_1) = z_{\sigma(i)2} + A_2$ for some permutation σ on $\{1, 2, ..., k\}$ and for all i. If such ϕ exists, A_1 and A_2 are said to be equivalent.

Classifying lattice reptiles with 2 pieces

Now we classify lattice reptiles with 2 pieces. By Lemma 6, in the case $K = 2$, $m = 0, \pm 1, \pm 2$, so $\Re(\alpha) = 0, \pm\frac{1}{2}, \pm 1$ respectively. First we consider the case $f(z) = \alpha z$.

(i) When $m = 0$, we have $\alpha = \pm \mathbf{i}\sqrt{2}$, thus f is a rotation by $\frac{\pi}{2}$ (both clockwise and counterclockwise, the difference between them is a rotation, so they are equivalent) composed with a homothety with factor $\sqrt{2}$.

(ii) When $m = \pm 1$, we can just discuss $m = 1$ because these two cases are actually equivalent(the difference between them is a reflection of Imaginary axis, so they are equivalent). In this case we have $\alpha = \frac{1 \pm \mathbf{i}\sqrt{7}}{2} = \sqrt{2}(\frac{\sqrt{2}}{4} \pm \mathbf{i}\frac{\sqrt{14}}{4})$. So f is a rotation by $\arccos\frac{\sqrt{2}}{4}$ (both clockwise and counterclockwise, they are equivalent) composed with a homothety with factor $\sqrt{2}$.

Classifying lattice reptiles with 2 pieces

(iii) When $m = \pm 2$, we can just discuss $m = 2$ because these two cases are actually equivalent. This time $\alpha = 1 \pm \mathbf{i} = \sqrt{2}(\frac{\sqrt{2}}{2} \pm \mathbf{i}\frac{\sqrt{2}}{2})$. So f is a rotation by $\frac{\pi}{4}$ (both clockwise and counterclockwise, they are equivalent) composed with a homothety with factor $\sqrt{2}$.

Now we have to deal with the case $f(z) = \alpha\bar{z}$, so f is a rotation composed with a homothety composed with a reflection of Real axis. They are equivalent to the cases above respectively.

Existence of invariant set

Now it is time to construct the lattice reptiles. A theorem by Hutchinson tells us its existence.

Theorem (Hutchinson)

Let Λ be a lattice in $\mathbb{C}$, $f(z) = \alpha z$ (or $f(z) = \alpha \bar{z}$) is a linear expanding map such that $f(\Lambda) \subset \Lambda$. Let $\{z_1, z_2, ..., z_k\}$ be the right coset representatives of $f(\Lambda) \subset \Lambda$, so $\Lambda = (z_1 + f(\Lambda)) \cup (z_2 + f(\Lambda)) \cup ... \cup (z_k + f(\Lambda))$. Then there exist a unique compact subset of $\mathbb{C}$ such that $\{z_1 + A, z_2 + A, ..., z_k + A\}$ is a tiling of $f(A)$. Moreover, A is a lattice reptile with expanding linear map f.

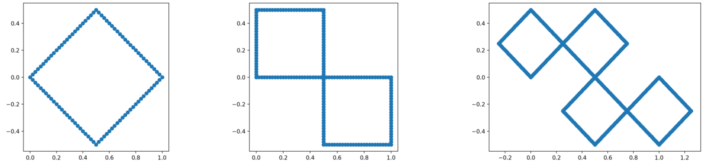
The construction by squares

We consider the case $\alpha = 1 - \mathbf{i}$.

Definition

Denote the square with vertices $0, \frac{1}{2} + \frac{1}{2}\mathbf{i}, 1, \frac{1}{2} - \frac{1}{2}\mathbf{i}$ by ψ_0. Define $g_1(z) = (\frac{1}{2} + \frac{1}{2}\mathbf{i})z$, $g_2(z) = (\frac{1}{2} + \frac{1}{2}\mathbf{i})(z - \mathbf{i})$, and $\psi_n = g_1(\psi_{n-1}) \cup g_2(\psi_{n-1})$. Denote the limit set by ψ, which means that in Hausdorff metric, $\psi_n \to \psi$ when $n \to \infty$.

We hope to prove that ψ is a lattice reptile with 2 pieces with respect to f.



(a) ψ_0 (b) ψ_1 (c) ψ_2

Definition of Hausdorff metric

About Hausdorff metric, we give its definition as a reminder.

Definition

Let (S, ρ) be a metric space. Denote the collection of all non-empty compact subset of S by $\mathbb{H}(S)$. For $A \subset S$, define

$$N_r(A) = \{y \in S \,|\, \exists\, x \in A,\ \rho(x, y) < r\}. \tag{2}$$

For $A, B \subset S$, define

$$D(A, B) = \inf_{r>0}\{A \subset N_r(B) \text{ and } B \subset N_r(A)\}. \tag{3}$$

The function D is called Hausdorff function on S. It can be proved that D is a metric on $\mathbb{H}(S)$. See (Gerald Edgar, 2008, Theorem 2.5.1).

If (S, ρ) is a complete metric space, then $(\mathbb{H}(S), D)$ is a complete metric space, as proven in (Gerald Edgar, 2008, Theorem 2.5.3).

The convergence of Hausdorff metric

The convergence of Hausdorff metric has the following property.

Proposition

Let A_n be a sequence of nonempty compact subset of S and let A be a nonempty compact subset of S. If A_n converges to A in Hausdorff metric, then

$$A = \{x \,|\, \text{There is a sequence } (x_n) \text{ with } x_n \in A_n \text{ and } x_n \to x\}. \tag{4}$$

Denote $\{x \,|\, \text{There is a sequence } (x_n) \text{ with } x_n \in A_n \text{ and } x_n \to x\}$ by B. For any $x \in A$, we can construct a sequence (x_n) which is convergent to x by the definition of Hausdorff metric. This proves $A \subset B$. On the other hand, we can prove that for $x \in B$, $\text{dist}(x, A) := \inf_{y \in A} \rho(x, y) = 0$. By the compactness of A, A is closed. Then $x \in A$, we derive $B \subset A$.

Preparations

Now we make some preparations for the proof of ψ being a lattice reptile with 2 pieces.

Lemma

The following propositions are valid for $n \in \mathbb{N}$.

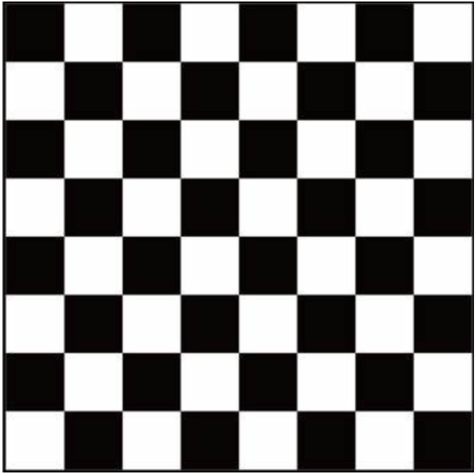
(i) *Every square in* ψ_n *intersects with at least one square in* ψ_n, *but only on vertices.*

(ii) ψ_n *intersects with* $-\mathbf{i} + \psi_n$, *but only on vertices.*

(iii) ψ_n *intersects with* $1 + \psi_n$, *but only on vertices.*

(iv) *Denote* $\Gamma_n = \operatorname{span}\left(\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)^n, (-\mathbf{i})\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)^n\right)$, $S_n = \left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)^n \psi_0$, *and* $P_n = \bigcup\{z + S_n \mid z \in \Gamma_n\}$. *Then* $\{z + \psi_n \mid z \in \operatorname{span}(1, \mathbf{i})\}$ *is a tiling of* P_n.

Actually P_n is like a chess board, and ψ_n is like the combination of some black squares.



This lemma can be proved by induction, if we suppose they are valid for $n-1$, then (i) is valid for n. Notice that

$$\begin{aligned} P_n &= g_1(P_{n-1}) \\ &= g_1((\mathrm{span}(1, -2\mathbf{i}) + (\psi_{n-1} \cup (-\mathbf{i} + \psi_{n-1})))) \\ &= \mathrm{span}\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}, 1 - \mathbf{i}\right) + \psi_n. \end{aligned}$$

By computation we have
$\left(-\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right) + \psi_n\right) \cap (-\mathbf{i} + \psi_n) = -\mathbf{i} + g_1(\psi_{n-1})$ and
$\left(-\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right) + (1 - \mathbf{i}) + \psi_n\right) \cap (-\mathbf{i} + \psi_n) = -\mathbf{i} + g_2(\psi_{n-1})$. So (ii) is proven. The proof of (iii) is very similar with (ii).

For (iv) we have
$\left(\left(\frac{1}{2}+\frac{1}{2}\mathbf{i}\right)+\psi_n\right)\cap(1+\psi_n)=\left(\frac{1}{2}+\frac{1}{2}\mathbf{i}\right)+g_2(\psi_{n-1})$ and
$\left(\left(\frac{1}{2}+\frac{1}{2}\mathbf{i}\right)+\psi_n\right)\cap(\mathbf{i}+\psi_n)=\left(\frac{1}{2}+\frac{1}{2}\mathbf{i}\right)+g_1(\psi_{n-1})$. So
$\left(\left(\frac{1}{2}+\frac{1}{2}\mathbf{i}\right)+\psi_n\right)\subset(1+\psi_n)\cup(\mathbf{i}+\psi_n)$ and
$((-1+\mathbf{i})+\psi_n)\in\{z+\psi_n\,|\,z\in\operatorname{span}(1,\mathbf{i})\}$, which means that
$\{z+\psi_n\,|\,z\in\operatorname{span}(1,\mathbf{i})\}$ contains P_n. By (ii) and (iii) they have no interior points in common.

Preparations

By the lemma, the vertices of ψ_n can be divided into 2 types.

Definition

Let $A_n =$
$\{z \,|\, z$ is a vertex of ψ_n and belongs to only one square in $\psi_n\}$
and
$B_n = \{z \,|\, z$ is a vertex of ψ_n and belongs to two squares in $\psi_n\}$.

Preparations

We have the following proposition.

Proposition

The following properties are valid for $n \in \mathbb{N}$.

(i) *In* $\{z + \psi_n \mid z \in \operatorname{span}(1, \mathbf{i})\}$, ψ_n *intersects with and only with* $\pm 1 + \psi_n$, $\pm \mathbf{i} + \psi_n$, *and* $\pm(1 - \mathbf{i}) + \psi_n$.

(ii) $A_n = \bigcup_{z \in \{\pm 1, \pm \mathbf{i}, \pm(1-\mathbf{i})\}} ((z + \psi_n) \cap \psi_n) = (g_1(A_{n-1}) \cup g_2(A_{n-1})) \setminus (g_1(A_{n-1}) \cap g_2(A_{n-1}))$.

(iii) $A_n \subset A_{n+1}$ *and* $B_n \subset B_{n+1}$.

(iv) $\partial \psi = \overline{\bigcup_{n=1}^{+\infty} A_n}$.

Take $\mathbf{i} + \psi_{n+1}$ as an example. Notice that $\psi_{n+1} = g_1(\psi_n) \cup g_2(\psi_n)$. We derive that

$$\begin{aligned}(\mathbf{i} + \psi_{n+1}) \cap \psi_{n+1} &= (\mathbf{i} + (g_1(\psi_n) \cup g_2(\psi_n))) \cap (g_1(\psi_n) \cup g_2(\psi_n)) \\ &= ((\mathbf{i} + g_1(\psi_n)) \cap g_1(\psi_n)) \cup ((\mathbf{i} + g_1(\psi_n)) \cap g_2(\psi_n)) \\ &\cup ((\mathbf{i} + g_2(\psi_n)) \cap g_1(\psi_n)) \cup ((\mathbf{i} + g_2(\psi_n)) \cap g_2(\psi_n)).\end{aligned}$$

However, by (i) for n we know
$(\mathbf{i} + g_1(\psi_n)) \cap g_2(\psi_n) = g_1((1+\mathbf{i}) + \psi_n) \cap g_1(-\mathbf{i} + \psi_n) = \emptyset$,
$(\mathbf{i} + g_2(\psi_n)) \cap g_2(\psi_n) = (g_1(1 + \psi_n)) \cap g_1(-\mathbf{i} + \psi_n) = \emptyset$, and
$(\mathbf{i} + g_1(\psi_n)) \cap g_1(\psi_n) = g_1((1+\mathbf{i}) + \psi_n) \cap g_1(\psi_n) = \emptyset$. So

$$\begin{aligned}(\mathbf{i} + \psi_{n+1}) \cap \psi_{n+1} &= (\mathbf{i} + g_2(\psi_n)) \cap g_1(\psi_n) \\ &= \left(\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right) + g_1(\psi_n)\right) \cap g_1(\psi_n).\end{aligned} \tag{5}$$

Notice that

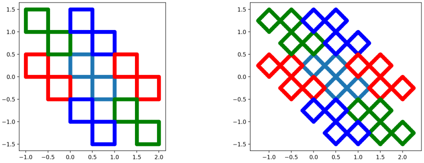
$$g_1(A_n) = \bigcup_{z \in \{\pm 1, \pm \mathbf{i}, \pm(1-\mathbf{i})\}} (g_1(z+\psi_n) \cap g_1(\psi_n))$$
$$= \bigcup_{w \in \{\pm(\frac{1}{2}+\frac{1}{2}\mathbf{i}), \pm(\frac{1}{2}-\frac{1}{2}\mathbf{i}), \pm 1\}} ((w + g_1(\psi_n)) \cap g_1(\psi_n)).$$

Similarly we have

$$g_2(A_n) = \bigcup_{z \in \{\pm 1, \pm \mathbf{i}, \pm(1-\mathbf{i})\}} (g_2(z+\psi_n) \cap g_2(\psi_n))$$
$$= \bigcup_{w \in \{0, 1, \mathbf{i}, 1-\mathbf{i}, -\frac{1}{2}-\frac{1}{2}\mathbf{i}, \frac{3}{2}-\frac{1}{2}\mathbf{i}\}} ((w + g_1(\psi_n)) \cap (\left(\frac{1}{2} - \frac{1}{2}\mathbf{i}\right) + g_1(\psi_n)))$$

Then $(\mathbf{i} + \psi_{n+1}) \cap \psi_{n+1} \subset A_{n+1} = (g_1(A_{n-1}) \cup g_2(A_{n-1})) \setminus (g_1(A_{n-1}) \cap g_2(A_{n-1}))$.

(i) and (ii) divided A_n into 6 subsets with no intersections.



(d) ψ_1 (e) ψ_2

Actually we can prove that $(z + \psi_n) \cap \psi_n \subset (z + \psi_{n+1}) \cap \psi_{n+1}$, which derives (iii).

We aim to prove (iv), which gives a construction of $\partial\psi$. For $z \in \partial\psi$ and $\varepsilon > 0$, there are two squares with side length small enough such that they are in $B(z, \varepsilon)$ and one of them belongs to ψ_n, and the other one does not belong to ψ_n. This means that there is some point in A_n which belongs to $B(z, \varepsilon)$. This proves $\partial\psi \subset \overline{\bigcup_{n=1}^{+\infty} A_n}$. On the other hand, for $z \in \bigcup_{n=1}^{+\infty} A_n$, there is $N \in \mathbb{N}$ such that for every $n > N$, $z \in A_n$. For an arbitrary $\varepsilon > 0$, there are two squares with side length small enough such that they are in $B(z, \varepsilon)$ and one of them has a vertex in $B_n \subset \mathrm{int}(\psi)$, and the other one has a vertex in $w + B_n \subset \mathbb{C} \backslash \psi$. So $z \in \partial\psi$.

Noticing that $\partial\psi$ is closed, we have $\overline{\bigcup_{n=1}^{+\infty} A_n} \subset \partial\psi$.

ψ is a lattice reptile with 2 pieces

Now we can prove the following theorem.

Theorem

ψ satisfies the following properties, thus it is a lattice reptile of $\mathbb{C}$:

(i) *There is a lattice $\Lambda = \mathrm{span}(1, \mathbf{i})$ such that $\{z + \psi \mid z \in \Lambda\}$ is a tiling of $\mathbb{C}$.*

(ii) *The expanding linear map $f(z) = (1 - \mathbf{i})z$ satisfies that $\{\psi, -\mathbf{i} + \psi\}$ is a tiling of $f(\psi)$.*

For (i), because $P_n \to \mathbb{C}$ when $n \to \infty$, ψ_n tiles P_n, we have ψ tiles $\mathbb{C}$. For (ii), noticing that $\psi_n, -\mathbf{i} + \psi_n$ is a tiling of $f(\psi_{n+1})$, let $n \to \infty$, then we can draw our conclusion.

Construction by Jordan curve

The construction by squares is not good enough to describe the boundary of twindragon. So we give another construction.
The recursive construction is defined as follows:

$$\tilde{\gamma}_n(t) = \begin{cases} \left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_n(4t), & t \in \left[0, \frac{1}{4}\right], \\ \left(-\frac{1}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_n(2-4t)+1, & t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \left(-\frac{4}{5} - \frac{2}{5}\mathbf{i}\right)\gamma_n(4t-2)+1, & t \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \left(\frac{1}{5} - \frac{2}{5}\mathbf{i}\right)\gamma_n(4-4t), & t \in \left[\frac{3}{4}, 1\right], \end{cases} \tag{6}$$

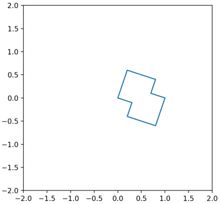
so $\tilde{\gamma}_n$ can be divided into 4 similar curves γ_n, where

$$\gamma_n(t) = \begin{cases} \frac{1}{2}(1+\mathbf{i})\gamma_{n-1}(2t), & t \in \left[0, \frac{1}{2}\right], \\ \frac{1}{4}(-1+\mathbf{i})\gamma_{n-1}(3-4t) + \frac{3}{4} + \frac{1}{4}\mathbf{i}, & t \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \frac{1}{4}(1-\mathbf{i})\gamma_{n-1}(4t-3) + \frac{3}{4} + \frac{1}{4}\mathbf{i}, & t \in \left[\frac{3}{4}, 1\right], \end{cases} \tag{7}$$

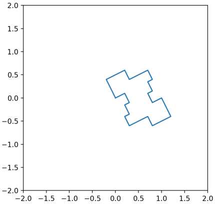
and

$$\gamma_0(t) = \begin{cases} (1+\mathbf{i})t, & t \in \left[0, \frac{1}{2}\right], \\ (1-\mathbf{i})t + \mathbf{i}, & t \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{8}$$

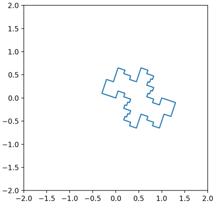
Here are the first 6 iterations.



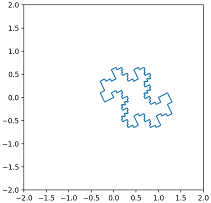
(f) $\tilde{\gamma}_0$



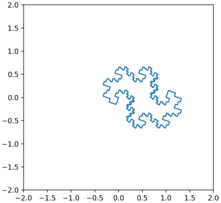
(g) $\tilde{\gamma}_1$



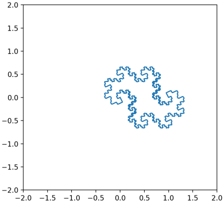
(h) $\tilde{\gamma}_2$



(i) $\tilde{\gamma}_3$



(j) $\tilde{\gamma}_4$



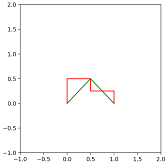
(k) $\tilde{\gamma}_5$

$\gamma(t)$ can be continuously parameterized

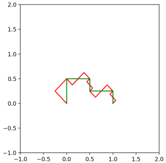
Proposition

By the upper definition, $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ is uniformly convergent.

Actually, $|\gamma_n(t) - \gamma_{n-1}(t)| \leq \left(\frac{\sqrt{2}}{2}\right)^{n+2}$. See the following figure.



(l) $\max|\gamma_1 - \gamma_0|$



(m) $\max|\gamma_2 - \gamma_1|$

γ_n and $-\gamma_n$ have no intersection except 0

Lemma

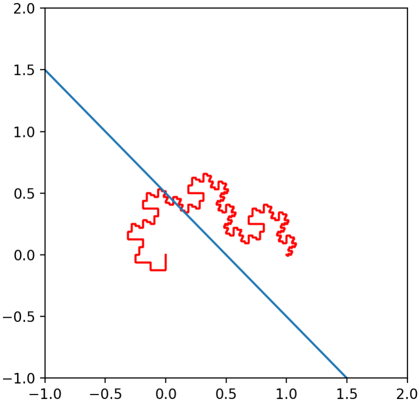
For all $n \in \mathbb{N}$, γ_n and $-\gamma_n$ do not intersect on any other point except 0.

By induction, we can prove that $\Re(\gamma_n(t)) + \Im(\gamma_n(t)) \in (-2, 2)$ and $\Re(\gamma_n(t)) - \Im(\gamma_n(t)) \in (-1, 3)$. As a corollary, if $t \in \left[0, \frac{1}{2}\right]$,

$$\Re(\gamma_n(t)) + \Im(\gamma_n(t)) > -\frac{1}{2}, \tag{9}$$

or if $t \in \left[\frac{1}{2}, 1\right]$,

$$\Re(\gamma_n(t)) + \Im(\gamma_n(t)) > \frac{1}{2}. \tag{10}$$


2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Now if there is $t_1, t_2 \in (0,1]$, such that $t_1 \neq t_2$ and $\gamma_n(t_1) = -\gamma_n(t_2)$. If $t_1 \in \left(0, \frac{1}{2}\right]$ and $t_2 \in \left[\frac{1}{2}, 1\right]$ (or $t_1 \in \left[\frac{1}{2}, 1\right]$ and $t_2 \in \left(0, \frac{1}{2}\right]$, without loss of generation we can consider one of them), we derive that $\Re(\gamma_n(t_1)) + \Im(\gamma_n(t_1)) > -\frac{1}{2}$ and $\Re(-\gamma_n(t_2)) + \Im(-\gamma_n(t_2)) < -\frac{1}{2}$, which cause a contradiction. If $t_1 \in \left[\frac{1}{2}, 1\right]$ and $t_2 \in \left[\frac{1}{2}, 1\right]$, we derive that $\Re(\gamma_n(t_1)) + \Im(\gamma_n(t_1)) > \frac{1}{2}$ and $\Re(-\gamma_n(t_2)) + \Im(-\gamma_n(t_2)) < -\frac{1}{2}$, which cause a contradiction. Finally, if $t_1 \in \left(0, \frac{1}{2}\right]$ and $t_2 \in \left(0, \frac{1}{2}\right]$, by the definition of γ_n, we derive that $\gamma_{n-1}(2t_1) = -\gamma_{n-1}(2t_2)$, so by induction we can prove that t_1 and t_2 which satisfy $\gamma_n(t_1) = -\gamma_n(t_2)$ can only be 0, a contradiction.

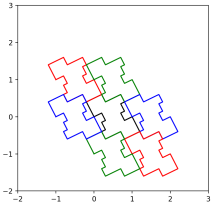
The two constructions are equivalent

Now we can prove that the two constructions are equivalent.

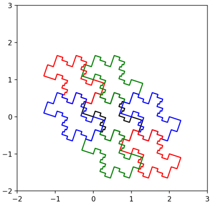
Theorem

$\tilde{\gamma} = \partial\psi$

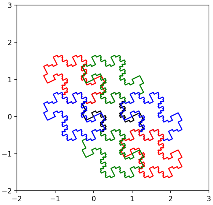
It is sufficient to prove that $\tilde{\gamma} = \overline{\bigcup_{n=1}^{\infty} A_n}$. We divide $\tilde{\gamma}$ into 6 subsets.



(n) $\tilde{\gamma}_1$



(o) $\tilde{\gamma}_2$



(p) $\tilde{\gamma}_3$

Denote $I_1 = \left[\frac{1}{16}, \frac{1}{8}\right], I_2 = \left[\frac{1}{8}, \frac{1}{4}\right], I_3 = \left[\frac{1}{4}, \frac{9}{16}\right], I_4 = \left[\frac{9}{16}, \frac{5}{8}\right],$ $I_5 = \left[\frac{5}{8}, \frac{3}{4}\right], I_6 = \left[\frac{3}{4}, 1\right] \cup \left[0, \frac{1}{16}\right]$. We claim that

$$((-1+\mathbf{i}) + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_1), \tag{11}$$

$$(\mathbf{i} + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_2), \tag{12}$$

$$(1 + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_3), \tag{13}$$

$$((1-\mathbf{i}) + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_4), \tag{14}$$

$$(-\mathbf{i} + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_5), \tag{15}$$

and

$$(-1 + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n(I_6). \tag{16}$$

This claim can be proved by induction. We give an example here.
Consider an arbitrary $z \in ((-1+\mathbf{i}) + \psi_{n+1}) \cap \psi_{n+1}$, which equals to $\left(\left(-\frac{1}{2} + \frac{1}{2}\mathbf{i}\right) + g_1(\psi_n)\right) \cap g_1(\psi_n)$. So $(1-\mathbf{i})z \in (\mathbf{i} + \psi_n) \cap \psi_n \subset \tilde{\gamma}_n\left[\frac{1}{8}, \frac{1}{4}\right]$. By the recursive construction, there is $t_0 \in \left[\frac{1}{8}, \frac{1}{4}\right]$ such that

$$(1-\mathbf{i})z = \left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_n(4t_0).$$

So we have

$$\begin{aligned} z &= \left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_{n+1}(2t_0) \\ &= \tilde{\gamma}_{n+1}\left(\frac{1}{2}t_0\right) \in \tilde{\gamma}_{n+1}\left[\frac{1}{16}, \frac{1}{8}\right], \end{aligned}$$

which proves that 11 holds for $n+1$.

By our claim, $A_n \subset \tilde{\gamma}_n$, thus noticing $A_n \subset A_{n+1}$ and $\tilde{\gamma}$ is closed, $\overline{\bigcup_{n=0}^{\infty} A_n} \subset \tilde{\gamma}$.

It is sufficient to prove that $\tilde{\gamma} \subset \overline{\bigcup_{n=0}^{\infty} A_n}$. First of all, for an arbitrary $\varepsilon > 0$ and $z = \tilde{\gamma}(t)$, there is $N \in \mathbb{N}$, such that for every $n > N$,

$$|\tilde{\gamma}_n(t) - z| < \frac{\varepsilon}{2}. \tag{17}$$

This can be derived by the uniform convergence of $\tilde{\gamma}_n$

Now, we claim that if $t \in I_k$ there is $t_0 \in I_k$, such that $\tilde{\gamma}_n(t_0) \in A_n$, and

$$|\tilde{\gamma}_n(t) - \tilde{\gamma}_n(t_0)| \leq \frac{2\sqrt{10}}{5}\left(\frac{\sqrt{2}}{2}\right)^{n+1}. \tag{18}$$

Still, we take I_1 as an example. By the conclusion of I_2, there is $t_0 \in I_1$, such that $|\tilde{\gamma}_n(2t) - \tilde{\gamma}_n(2t_0)| \leq \frac{2\sqrt{10}}{5}\left(\frac{\sqrt{2}}{2}\right)^{n+1}$ and $\left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_n(8t_0) = \tilde{\gamma}_n(2t_0) \in A_n$. We have

$$\begin{aligned}|\gamma_{n+1}(4t) - \gamma_{n+1}(4t_0)| &= |\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)(\gamma_n(8t) - \gamma_n(8t_0))| \\ &= \frac{\sqrt{2}}{2}|\gamma_n(8t) - \gamma_n(8t_0)| \\ &\leq \left(\frac{\sqrt{2}}{2}\right)^{n+2}.\end{aligned}$$

It has yet to be proven that t_0 satisfies $\tilde{\gamma}_{n+1}(t_0) \in A_{n+1}$. Notice that t_0 satisfies $\left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_n(8t_0) = \tilde{\gamma}_n(2t_0) \in A_n$. By 14(i)-(ii) and 11, $\tilde{\gamma}_n(2t_0) \in (\mathbf{i} + \psi_n) \cap \psi_n$. So

$$\begin{aligned}
\tilde{\gamma}_{n+1}(t_0) &= \left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\gamma_{n+1}(4t_0) \\
&= \left(\frac{4}{5} + \frac{2}{5}\mathbf{i}\right)\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)\gamma_n(8t_0) \\
&= \left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)\tilde{\gamma}_n(2t_0) \\
&\in \left(\left(-\frac{1}{2} + \frac{1}{2}\mathbf{i}\right) + g_1(\psi_n)\right) \cap g_1(\psi_n) \\
&\subset ((1 - \mathbf{i}) + \psi_{n+1}) \cap \psi_{n+1} \\
&\subset A_{n+1},
\end{aligned}$$

which proves the case I_1.

For a large enough n, combining 17 and 18, we have

$$|\tilde{\gamma}_n(t_0) - z| < \varepsilon,$$

where $\tilde{\gamma}_n(t_0) \in A_n$. Because z is picked arbitrarily, we derive $\tilde{\gamma} \subset \overline{\bigcup_{n=0}^{\infty} A_n}$.

Future research plan

- How to prove that $\tilde{\gamma} = \partial\psi$ is a Jordan curve?
  Maybe the proof of Lemma 17 is a probable way.
- What about the other case $\alpha = \frac{1 \pm \mathbf{i}\sqrt{7}}{2} = \sqrt{2}(\frac{\sqrt{2}}{4} \pm \mathbf{i}\frac{\sqrt{14}}{4})$?
- What about the lattice reptiles with k pieces?

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