PRIME ORBIT THEOREMS FOR EXPANDING THURSTON MAPS: LATTÈS MAPS AND SPLIT RUELLE OPERATORS

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ABSTRACT. We obtain an analog of the prime number theorem for a class of branched covering maps on the 2-sphere S^2 called expanding Thurston maps, which are topological models of some non-uniformly expanding rational maps without any smoothness or holomorphicity assumption. More precisely, we show that the number of primitive periodic orbits, ordered by a weight on each point induced by a non-constant (eventually) positive real-valued Hölder continuous function on S^2 satisfying some additional regularity conditions, is asymptotically the same as the well-known logarithmic integral, with an exponential error term. In particular, our results apply to postcritically-finite rational maps for which the Julia set is the whole Riemann sphere. Moreover, a stronger result is obtained for Lattès maps.

CONTENTS

1. Introduction

Complex dynamics is a vibrant field of dynamical systems, focusing on the study of iterations of polynomials and rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. It is closely connected, via *Sullivan's dictionary* [Su85, Su83], to geometric group theory, mainly concerning the study of Kleinian groups.

In complex dynamics, the lack of uniform expansion of a rational map arises from critical points in the Julia set. Rational maps for which each critical point is preperiodic (i.e., eventually periodic) are called *postcritically-finite rational maps* or *rational Thurston maps*. One natural class of non-uniformly expanding rational maps are called *topological Collet–Eckmann maps*, whose basic dynamical properties have been studied by S. Smirnov, F. Przytycki, J. Rivera-Letelier, Weixiao Shen, etc. (see [PRLS03, PRL07, PRL11, RLS14]). In this paper, we focus on a subclass of topological Collet–Eckmann maps for which each critical point is preperiodic and the Julia set is the whole Riemann sphere. Actually, the most general version of our results is established for topological models of these maps, called *expanding Thurston maps*. Thurston maps were studied by W. P. Thurston in his celebrated characterization theorem of postcritically-finite rational maps among such topological models [DH93]. Thurston maps and Thurson's theorem, sometimes known as a fundamental theorem of complex dynamics, are indispensable tools in the modern theory of complex dynamics. Expanding Thurston maps were studied extensively by M. Bonk, D. Meyer [BM10, BM17] and P. Haïssinsky, K. M. Pilgrim [HP09].

The investigations of the growth rate of the number of periodic orbits (e.g. closed geodesics) have been a recurring theme in dynamics and geometry.

Inspired by the seminal works of F. Naud [Na05] and H. Oh, D. Winter [OW17] on the growth rate of periodic orbits, known as Prime Orbit Theorems, for hyperbolic (uniformly expanding) polynomials and rational maps, we establish in this paper the first Prime Orbit Theorems (to the best of our knowledge) in a non-uniformly expanding setting in complex dynamics. On the other side of Sullivan's dictionary, see related works [MMO14, OW16, OP18]. For an earlier work on dynamical zeta functions for a class of sub-hyperbolic quadratic polynomials, see V. Baladi, Y. Jiang, and H. H. Rugh [BJR02]. See also related work of S. Waddington [Wad97] on strictly preperiodic points of hyperbolic rational maps.

Given a map $f \colon X \to X$ on a metric space (X, d) and a function $\phi \colon S^2 \to \mathbb{R}$, we define the weighted length $l_{f,\phi}(\tau)$ of a primitive periodic orbit

$$\tau := \{x,\, f(x),\, \cdots,\, f^{n-1}(x)\} \in \mathfrak{P}(f)$$

as

$$(1.1) \qquad l_{f,\phi}(\tau) := \phi(x) + \phi(f(x)) + \cdots + \phi(f^{n-1}(x)).$$

We denote by

$$(1.2) \qquad \pi_{f,\phi}(T) := \operatorname{card}\{\tau \in \mathfrak{P}(f) : l_{f,\phi}(\tau) \le T\}, \qquad T > 0,$$

the number of primitive periodic orbits with weighted lengths up to T. Here $\mathfrak{P}(f)$ denotes the set of all primitive periodic orbits of f (see Section 2).

Note that the Prime Orbit Theorems in [Na05, OW17] are established for the *geometric potential* $\phi = \log|f'|$. For hyperbolic rational maps, the Lipschitz continuity of the geometric potential plays a crucial role in [Na05, OW17]. In our non-uniform expanding setting, critical points destroy the continuity of $\log|f'|$. So we are left with two options to develop our theory, namely, considering

   (a) Hölder continuous ϕ or

   (b) the geometric potential $\log|f'|$.

Despite the lack of Hölder continuity of $\log|f'|$ in our setting, its value is closely related to the size of pull-backs of sets under backward iterations of the map f. This fact enables an investigation of the Prime Orbit Theorem in case (b), which will be investigated in an upcoming series of separate works starting with [LRL].

The current paper is the second of a series of three papers (together with [LZhe23a, LZhe23c]) focusing on case (a), in which the incompatibility of Hölder continuity of ϕ and non-uniform expansion of f calls for a close investigation of metric geometries associated to f.

Lattès maps are rational Thurston maps with parabolic orbifolds (c.f. Definition 8.4). They form a well-known class of rational maps. We first formulate our theorem for Lattès maps.

Theorem A (Prime Orbit Theorem for Lattès maps). *Let $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Lattès map on the Riemann sphere $\widehat{\mathbb{C}}$. Let $\phi \colon \widehat{\mathbb{C}} \to \mathbb{R}$ be eventually positive and continuously differentiable. Then there exists a unique positive number $s_0 > 0$ with $P(f, -s_0\phi) = 0$ and there exists $N_f \in \mathbb{N}$ depending only on f such that the following statements are equivalent:*

(i) *ϕ is not co-homologous to a constant in the space $C(\widehat{\mathbb{C}})$ of real-valued continuous functions on $\widehat{\mathbb{C}}$.*

(ii) *For each $n \in \mathbb{N}$ with $n \geq N_f$, we have $\pi_{F,\Phi}(T) \sim \mathrm{Li}(e^{s_0 T})$ as $T \to +\infty$, where $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$.*

(iii) *For each $n \in \mathbb{N}$ with $n \geq N_f$, there exists a constant $\delta \in (0, s_0)$ such that $\pi_{F,\Phi}(T) = \mathrm{Li}(e^{s_0 T}) + \mathcal{O}(e^{(s_0 - \delta)T})$ as $T \to +\infty$, where $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$.*

Here $P(f, \cdot)$ denotes the topological pressure, and $\mathrm{Li}(y) := \int_2^y \frac{1}{\log u}\, \mathrm{d}u$, $y > 0$, is the Eulerian logarithmic integral function.

See Definitions 3.1 and 3.19 for the definitions of co-homology and eventually positive functions, respectively.

The implication (i) $\implies$ (iii) relies crucially on some local properties of the metric geometry of Lattès maps, and is not expected (by the authors) in general. To establish the exponential error term similar to that in (iii) for a class of more general rational Thurston maps, we impose a condition called α-*strong non-integrability condition* (Definition 7.1), which turns out to be generic. The genericity of this condition will be the main theme of the third and last paper [LZhe23c] of the current series. An analog of this condition in the context of Anosov flows was first proposed by D. Dolgopyat in his seminal work [Do98].

The following theorem is an immediate consequence of a more general result in Theorem C.

Theorem B (Prime Orbit Theorems for rational expanding Thurston maps). *Let $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically-finite rational map without periodic critical points. Let σ be the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$, and $\phi \colon \widehat{\mathbb{C}} \to \mathbb{R}$ be an eventually positive real-valued Hölder continuous function. Then there exists a unique positive number $s_0 > 0$ with topological pressure $P(f, -s_0\phi) = 0$ and there exists $N_f \in \mathbb{N}$ depending only on f such that for each $n \in \mathbb{N}$ with $n \geq N_f$, the following statement holds for $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$:*

(i) *$\pi_{F,\Phi}(T) \sim \mathrm{Li}(e^{s_0 T})$ as $T \to +\infty$ if and only if ϕ is not co-homologous to a constant in $C(\widehat{\mathbb{C}})$.*

(ii) *Assume that ϕ satisfies the strong non-integrability condition (with respect to f and a visual metric). Then there exists $\delta \in (0, s_0)$ such that $\pi_{F,\Phi}(T) = \mathrm{Li}(e^{s_0 T}) + \mathcal{O}(e^{(s_0 - \delta)T})$ as $T \to +\infty$.*

Our strategy to overcome the obstacles presented by the incompatibility of the non-uniform expansion of our rational maps and the Hölder continuity of the weight ϕ (e.g. (a) the set of α-Hölder continuous functions is not invariant under the Ruelle operator $\mathcal{L}_\phi$, for each $\alpha \in (0, 1]$; (b) the weakening of the regularity of the temporal distance compared to that of the potential) is to investigate the metric geometry of various natural metrics associated to the dynamics such as visual metrics, the canonical orbifold metric, and the chordal metric. Such considerations lead us beyond conformal, or even smooth, dynamical settings and into the realm of topological dynamical systems. More precisely, we will work in the abstract setting of branched covering maps on the topological 2-sphere S^2 (c.f. Subsection 3.2) without any smoothness assumptions. A *Thurston map* is a postcritically-finite branched covering map on S^2. Thurston maps can be considered as topological models of the corresponding rational maps.

Via Sullivan's dictionary, the counterpart of Thurston's theorem [DH93] in the geometric group theory is Cannon's Conjecture [Ca94]. This conjecture predicts that an infinite, finitely presented Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is a topological 2-sphere is a Kleinian group. Gromov hyperbolic groups can be considered as metric-topological systems generalizing the conformal systems in the context, namely, convex-cocompact Kleinian groups. Inspired by Sullivan's dictionary and their interest in Cannon's Conjecture, M. Bonk and D. Meyer, along with others, studied a subclass of Thurston maps by imposing some additional condition of expansion. Roughly speaking, we say that a Thurston map is *expanding* if for any two points $x, y \in S^2$, their preimages under iterations of the map get closer and closer. For each expanding Thurston map, we can equip the 2-sphere S^2 with

a natural class of metrics called *visual metrics*. As the name suggests, these metrics are constructed in a similar fashion as the visual metrics on the boundary $\partial_\infty G$ of a Gromov hyperbolic group G. See Subsection 3.2 for a more detailed discussion on these notions. Various ergodic properties, including thermodynamic formalism, on which the current paper crucially relies, have been studied by the first-named author in [Li17] (c.f. [Li15, Li16, Li18]). Generalization of results in [Li17] to the more general branched covering maps studied by P. Haïssinsky, K. M. Pilgrim [HP09] has drawn significant interest recently [HRL19, DPTUZ19, LZheH23]. We believe that our ideas introduced in this paper can be used to establish Prime Orbit Theorems in their setting.

M. Bonk, D. Meyer [BM10, BM17] and P. Haïssinsky, K. M. Pilgrim [HP09] proved that an expanding Thurston map is conjugate to a rational map if and only if the sphere (S^2, d) equipped with a visual metric d is quasisymmetrically equivalent to the Riemann sphere $\widehat{\mathbb{C}}$ equipped with the chordal metric. The quasisymmetry cannot be promoted to Lipschitz equivalence due to the non-uniform expansion of Thurston maps. There exist expanding Thurston maps not conjugate to rational Thurston maps (e.g. ones with periodic critical points). Our theorems below apply to all expanding Thurston maps, which form the most general setting in this series of papers.

Theorem C (Prime Orbit Theorems for expanding Thurston maps)**.** *Let $f \colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f. Let $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Denote by s_0 the unique positive number with topological pressure $P(f, -s_0\phi) = 0$. Then there exists $N_f \in \mathbb{N}$ depending only on f such that for each $n \in \mathbb{N}$ with $n \geq N_f$, the following statements hold for $F \coloneqq f^n$ and $\Phi \coloneqq \sum_{i=0}^{n-1} \phi \circ f^i$:*

(i) *$\pi_{F,\Phi}(T) \sim \mathrm{Li}\big(e^{s_0 T}\big)$ as $T \to +\infty$ if and only if ϕ is not co-homologous to a constant in the space $C(S^2)$ of real-valued continuous functions on S^2.*

(ii) *Assume that ϕ satisfies the α-strong non-integrability condition. Then there exists a constant $\delta \in (0, s_0)$ such that $\pi_{F,\Phi}(T) = \mathrm{Li}\big(e^{s_0 T}\big) + \mathcal{O}\big(e^{(s_0 - \delta)T}\big)$ as $T \to +\infty$.*

Here $\mathrm{Li}(\cdot)$ is the Eulerian logarithmic integral function defined in Theorem A.

Note that $\lim_{y \to +\infty} \mathrm{Li}(y)/(y/\log y) = 1$, thus we also get $\pi_{F,\Phi}(T) \sim e^{s_0 T}/(s_0 T)$ as $T \to +\infty$.

We remark that our proofs can be modified to derive equidistribution of holonomies similar to the corresponding result in [OW17], but we choose to omit them in order to emphasize our new ideas and to limit the length of this paper.

In view of Remark 3.10, Theorem B is an immediate consequence of Theorem C.

Remark 1.1. The integer N_f can be chosen as the minimum of $N(f, \widetilde{\mathcal{C}})$ from Lemma 3.11 over all Jordan curves $\widetilde{\mathcal{C}}$ with post $f \subseteq \widetilde{\mathcal{C}} \subseteq S^2$, in which case $N_f = 1$ if there exists a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$, post $f \subseteq \mathcal{C}$, and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$ (see Definition 3.12). The same number N_f is used in other results in this paper. We also remark that many properties of expanding Thurston maps f can be established for f after being verified first for f^n for all $n \geq N_f$. However, some of the finer properties established for iterates of f still remain open for the map f itself; see for example, [Me13, Me12].

Note that due to the lack of algebraic structure of expanding Thurston maps, even the fact that there are only countably many periodic points is not apparent from the definition (c.f. [Li16]). Without any algebraic, differential, or conformal structures, the main tools we rely on are from the interplay between the metric properties of various natural metrics and the combinatorial information on the iterated preimages of certain Jordan curves $\mathcal{C}$ on S^2 (c.f. Subsection 3.2).

By well-known arguments of M. Pollicott and R. Sharp inspired from number theory [PS98], the counting result in Theorem C follows from some quantitative information on the holomorphic extension of certain dynamical zeta function $\zeta_{F, -\Phi}$ defined as formal infinite products over periodic orbits. We briefly recall dynamical zeta functions and define the dynamical Dirichlet series in our context below. See Subsection 3.4 for a more detailed discussion.

Let $f \colon S^2 \to S^2$ be an expanding Thurston map and $\psi \in C(S^2, \mathbb{C})$ be a complex-valued continuous function on S^2. We denote by the formal infinite product

$$\zeta_{f, -\psi}(s) \coloneqq \exp\left(\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{x = f^n(x)} e^{-s S_n \psi(x)}\right), \qquad s \in \mathbb{C},$$

the *dynamical zeta function* for the map f and the *potential* ψ. Here we write $S_n\psi(x) := \sum_{j=0}^{n-1} \psi(f^j(x))$ as defined in (2.4). We remark that $\zeta_{f,-\psi}$ is the Ruelle zeta function for the suspension flow over f with roof function ψ if ψ is positive. We define the *dynamical Dirichlet series* associated to f and ψ as the formal infinite product

$$\mathfrak{D}_{f,-\psi,\deg_f}(s) := \exp\left(\sum_{n=1}^{+\infty}\frac{1}{n}\sum_{x=f^n(x)} e^{-sS_n\psi(x)}\deg_{f^n}(x)\right), \qquad s \in \mathbb{C}.$$

Here $\deg_{f^n}$ is the *local degree* of f^n at $x \in S^2$.

Note that if $f \colon S^2 \to S^2$ is an expanding Thurston map, then so is f^n for each $n \in \mathbb{N}$.

Recall that a function is holomorphic on a set $A \subseteq \mathbb{C}$ if it is holomorphic on an open set containing A.

Theorem D (Holomorphic extensions of dynamical Dirichlet series and zeta functions for expanding Thurston maps). *Let $f \colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f. Fix $\alpha \in (0,1]$. Let $\phi \in C^{0,\alpha}(S^2,d)$ be an eventually positive real-valued Hölder continuous function that is not co-homologous to a constant in $C(S^2)$. Denote by s_0 the unique positive number with topological pressure $P(f,-s_0\phi) = 0$. Then there exists $N_f \in \mathbb{N}$ depending only on f such that for each $n \in \mathbb{N}$ with $n \geq N_f$, the following statements hold for $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$:*

(i) *Both the dynamical zeta function $\zeta_{F,-\Phi}(s)$ and the dynamical Dirichlet series $\mathfrak{D}_{F,-\Phi,\deg_F}(s)$ converge on $\{s \in \mathbb{C} : \Re(s) > s_0\}$ and extend to non-vanishing holomorphic functions on $\{s \in \mathbb{C} : \Re(s) \geq s_0\}$ except for the simple pole at $s = s_0$.*

(ii) *Assume in addition that ϕ satisfies the α-strong non-integrability condition. Then there exists a constant $\epsilon_0 \in (0,s_0)$ such that both $\zeta_{F,-\Phi}(s)$ and $\mathfrak{D}_{F,-\Phi,\deg_F}(s)$ converge on $\{s \in \mathbb{C} : \Re(s) > s_0\}$ and extend to non-vanishing holomorphic functions on $\{s \in \mathbb{C} : \Re(s) \geq s_0 - \epsilon_0\}$ except for the simple pole at $s = s_0$. Moreover, for each $\epsilon > 0$, there exist constants $C_\epsilon > 0$, $a_\epsilon \in (0,\epsilon_0]$, and $b_\epsilon \geq 2s_0 + 1$ such that*

(1.3) $$\exp\left(-C_\epsilon|\Im(s)|^{2+\epsilon}\right) \leq |\zeta_{F,-\Phi}(s)| \leq \exp\left(C_\epsilon|\Im(s)|^{2+\epsilon}\right),$$

(1.4) $$\exp\left(-C_\epsilon|\Im(s)|^{2+\epsilon}\right) \leq \left|\mathfrak{D}_{F,-\Phi,\deg_F}(s)\right| \leq \exp\left(C_\epsilon|\Im(s)|^{2+\epsilon}\right)$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < a_\epsilon$ and $|\Im(s)| \geq b_\epsilon$.

In order to get information about $\zeta_{F,-\Phi}$, we need to investigate the zeta function $\zeta_{\sigma_{A_\triangle},-\phi\circ\pi_\triangle}$ of a symbolic model of $\sigma_{A_\triangle} \colon \Sigma_{A_\triangle}^+ \to \Sigma_{A_\triangle}^+$ of F.

Theorem E (Holomorphic extensions of the symbolic zeta functions). *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$, post $f \subseteq \mathcal{C}$, and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Let d be a visual metric on S^2 for f. Fix $\alpha \in (0,1]$. Let $\phi \in C^{0,\alpha}(S^2,d)$ be an eventually positive real-valued Hölder continuous function that is not co-homologous to a constant in $C(S^2)$. Denote by s_0 the unique positive number with $P(f,-s_0\phi) = 0$. Let $\left(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle}\right)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 3.21, and let $\pi_\triangle \colon \Sigma_{A_\triangle}^+ \to S^2$ be the factor map as defined in (3.19).*

Then the dynamical zeta function $\zeta_{\sigma_{A_\triangle},-\phi\circ\pi_\triangle}(s)$ converges on the open half-plane $\{s \in \mathbb{C} : \Re(s) > s_0\}$, and the following statements hold:

(i) *The function $\zeta_{\sigma_{A_\triangle},-\phi\circ\pi_\triangle}(s)$ extends to a non-vanishing holomorphic function on the closed half-plane $\{s \in \mathbb{C} : \Re(s) \geq s_0\}$ except for the simple pole at $s = s_0$.*

(ii) *Assume in addition that ϕ satisfies the α-strong non-integrability condition. Then there exists a constant $\widetilde{\epsilon}_0 \in (0,s_0)$ such that $\zeta_{\sigma_{A_\triangle},-\phi\circ\pi_\triangle}(s)$ extends to a non-vanishing holomorphic function on the closed half-plane $\{s \in \mathbb{C} : \Re(s) \geq s_0 - \widetilde{\epsilon}_0\}$ except for the simple pole at $s = s_0$. Moreover, for each $\epsilon > 0$, there exist constants $\widetilde{C}_\epsilon > 0$, $\widetilde{a}_\epsilon \in (0,s_0)$, and $\widetilde{b}_\epsilon \geq 2s_0 + 1$ such that*

(1.5) $$\exp\left(-\widetilde{C}_\epsilon|\Im(s)|^{2+\epsilon}\right) \leq \left|\zeta_{\sigma_{A_\triangle},-\phi\circ\pi_\triangle}(s)\right| \leq \exp\left(\widetilde{C}_\epsilon|\Im(s)|^{2+\epsilon}\right)$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < \widetilde{a}_\epsilon$ and $|\Im(s)| \geq \widetilde{b}_\epsilon$.

We adapt D. Dolgopyat's cancellation argument developed in his landmark work [Do98] (building in part of work of Chernov [Ch98]) and arguments of M. Pollicott and R. Sharp [PS98] to establish a symbolic version of Theorem D as stated in Theorem E. The difficulties in adapting D. Dolgopyat's machinery in our metric-topological setting are purely technical, but overcoming these difficulties in any context is the heart of the matter (c.f. [Na05, OW17] as well as works on the decay of correlation and counting in [Liv04, AGY06, OW16, OW17, BDL18], etc.) We use the Hölder norm in the cancellation argument instead of the C^1-norm used in [Na05, OW17]. Another major technical difficulty comes from the fact that S^2 is connected and the usual Ruelle operator does not apply to characteristic functions on proper subsets of S^2, which is essential in Ruelle's estimate (see (6.2) in Proposition 6.1). Our approach is to adjust the definition of the Ruelle operator and to introduce what we call the *split Ruelle operator* (see Section 5). Such an approach should be useful in establishing Prime Orbit Theorems in other contexts.

We will now give a brief description of the structure of this paper.

After fixing some notation in Section 2, we give a review of basic definitions and results in Section 3. In Section 4, we state the assumptions on some of the objects in this paper, which we are going to repeatedly refer to later as *the Assumptions*. In Section 5, we define the *split Ruelle operator* $\mathbb{L}_{-s\phi}$ and study its properties including spectral gap. Section 6 contains arguments to bound the dynamical zeta function $\zeta_{\sigma_{A_\triangle}, -\phi \circ \pi_\triangle}$ with the bounds of the operator norm of $\mathbb{L}_{-s\phi}$. We provide a proof of Theorem D in Subsection 6.4 to deduce the holomorphic extension of $\mathfrak{D}_{F, -\Phi, \deg_F}$ from that of $\zeta_{\sigma_{A_\triangle}, -\Phi \circ \pi_\triangle}$, and ultimately to deduce the holomorphic extension of $\zeta_{F, -\Phi}$ from that of $\mathfrak{D}_{F, -\Phi, \deg_F}$. In Section 7, we adapt the arguments of D. Dolgopyat [Do98] in our metric-topological setting aiming to prove Theorem 6.2 at the end of this section, consequently establishing Theorems D, E, and C. Section 8 focuses on Lattès maps (recalled in Definition 8.4). We include the proof of Theorem A at the end of this section.

2. Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. For each complex number $z \in \mathbb{C}$, we denote by $\Re(z)$ the real part of z, and by $\Im(z)$ the imaginary part of z. We denote by $\mathbb{D}$ the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. For each $a \in \mathbb{R}$, we denote by $\mathbb{H}_a$ the open (right) half-plane $\mathbb{H}_a := \{z \in \mathbb{C} : \Re(z) > a\}$ on $\mathbb{C}$, and by $\overline{\mathbb{H}}_a$ the closed (right) half-plane $\overline{\mathbb{H}}_a := \{z \in \mathbb{C} : \Re(z) \geq a\}$. We follow the convention that $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, and $\widehat{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$, with the order relations $<, \leq, >, \geq$ defined in the obvious way. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leq x$, and $\lceil x \rceil$ the smallest integer $\geq x$. As usual, the symbol $\log$ denotes the logarithm to the base e, and $\log_c$ the logarithm to the base c for $c > 0$. The symbol $\mathbf{i}$ stands for the imaginary unit in the complex plane $\mathbb{C}$. For each $z \in \mathbb{C} \setminus \{0\}$, we denote by $\operatorname{Arg}(z)$ the principle argument of z, i.e., the unique real number in $(-\pi, \pi]$ with the property that $|z| e^{\mathbf{i} \operatorname{Arg}(z)} = z$. The cardinality of a set A is denoted by $\operatorname{card} A$.

Consider real-valued functions u, v, and w on $(0, +\infty)$. We write $u(T) \sim v(T)$ as $T \to +\infty$ if $\lim_{T \to +\infty} \frac{u(T)}{v(T)} = 1$, and write $u(T) = v(T) + \mathcal{O}(w(T))$ as $T \to +\infty$ if $\limsup_{T \to +\infty} \left| \frac{u(T) - v(T)}{w(T)} \right| < +\infty$.

Let $g \colon X \to Y$ be a map between two sets X and Y. We denote the restriction of g to a subset Z of X by $g|_Z$. Consider a map $f \colon X \to X$ on a set X. The inverse map of f is denoted by f^{-1}. We write

f^n for the n-th iterate of f, and $f^{-n} := (f^n)^{-1}$, for $n \in \mathbb{N}$. We set $f^0 := \mathrm{id}_X$, where the identity map $\mathrm{id}_X \colon X \to X$ sends each $x \in X$ to x itself. For each $n \in \mathbb{N}$, we denote by

$$(2.1) \qquad P_{n,f} := \{x \in X : f^n(x) = x, \; f^k(x) \neq x, \; k \in \{1, \, 2, \, \ldots, \, n-1\}\}$$

the *set of periodic points of f with periodic n*, and by

$$(2.2) \qquad \mathfrak{P}(n, f) := \{\{f^i(x) : i \in \{0, \, 1, \, \ldots, \, n-1\}\} : x \in P_{n,f}\}$$

the *set of primitive periodic orbits of f with period n*. The set of all primitive periodic orbits of f is denoted by

$$(2.3) \qquad \mathfrak{P}(f) := \bigcup_{n=1}^{+\infty} \mathfrak{P}(n, f).$$

Given a complex-valued function $\varphi \colon X \to \mathbb{C}$, we write

$$(2.4) \qquad S_n\varphi(x) = S_n^f\varphi(x) := \sum_{j=0}^{n-1} \varphi(f^j(x))$$

for $x \in X$ and $n \in \mathbb{N}_0$. The superscript f is often omitted when the map f is clear from the context. Note that when $n = 0$, by definition, we always have $S_0\varphi = 0$.

Let (X, d) be a metric space. For subsets $A, B \subseteq X$, we set $d(A, B) := \inf\{d(x, y) : x \in A, \, y \in B\}$, and $d(A, x) = d(x, A) := d(A, \{x\})$ for $x \in X$. For each subset $Y \subseteq X$, we denote the diameter of Y by $\mathrm{diam}_d(Y) := \sup\{d(x, y) : x, \, y \in Y\}$, the interior of Y by $\mathrm{int}\, Y$, and the characteristic function of Y by $\mathbb{1}_Y$, which maps each $x \in Y$ to $1 \in \mathbb{R}$ and vanishes otherwise. We use the convention that $\mathbb{1} = \mathbb{1}_X$ when the space X is clear from the context. For each $r > 0$ and each $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r)$ (resp. $\overline{B}_d(x, r)$).

We set $C(X)$ (resp. $B(X)$) to be the space of continuous (resp. bounded Borel) functions from X to $\mathbb{R}$, $\mathcal{M}(X)$ the set of finite signed Borel measures, and $\mathcal{P}(X)$ the set of Borel probability measures on X. We denote by $C(X, \mathbb{C})$ (resp. $B(X, \mathbb{C})$) the space of continuous (resp. bounded Borel) functions from X to $\mathbb{C}$. We adopt the convention that unless specifically referring to $\mathbb{C}$, we only consider real-valued functions. If we do not specify otherwise, we equip $C(X)$ and $C(X, \mathbb{C})$ with the uniform norm $\|\cdot\|_{C^0(X)}$. For a continuous map $g \colon X \to X$, $\mathcal{M}(X, g)$ is the set of g-invariant Borel probability measures on X.

The space of real-valued (resp. complex-valued) Hölder continuous functions with an exponent $\alpha \in (0, 1]$ on a compact metric space (X, d) is denoted by $C^{0,\alpha}(X, d)$ (resp. $C^{0,\alpha}((X, d), \mathbb{C})$). For each $\psi \in C^{0,\alpha}((X, d), \mathbb{C})$, we denote

$$(2.5) \qquad |\psi|_{\alpha, \, (X,d)} := \sup\{|\psi(x) - \psi(y)|/d(x, y)^\alpha : x, \, y \in X, \, x \neq y\},$$

and for $b \in \mathbb{R} \setminus \{0\}$, the *normalized Hölder norm* of ψ is defined as

$$(2.6) \qquad \|\psi\|_{C^{0,\alpha}(X,d)}^{[b]} := |b|^{-1} \, |\psi|_{\alpha, \, (X,d)} + \|\psi\|_{C^0(X)},$$

while the standard Hölder norm of ψ is denoted by

$$(2.7) \qquad \|\psi\|_{C^{0,\alpha}(X,d)} := \|\psi\|_{C^{0,\alpha}(X,d)}^{[1]}.$$

For a Lipschitz map $g \colon (X, d) \to (X, d)$ on a metric space (X, d), we denote the Lipschitz constant by

$$(2.8) \qquad \mathrm{LIP}_d(g) := \sup\{d(g(x), g(y))/d(x, y) : x, \, y \in X \text{ with } x \neq y\}.$$

3. Preliminaries

3.1. **Thermodynamic formalism.** We first review some basic concepts from dynamical systems. We refer the readers to [LZhe23a, Subsection 3.1] for more details and references.

Let (X, d) be a compact metric space and $g \colon X \to X$ a continuous map. For each real-valued continuous function $\phi \in C(X)$, the *measure-theoretic pressure* $P_\mu(g, \phi)$ of g for a g-invariant Borel probability measure μ and the potential ϕ is

$$(3.1) \qquad P_\mu(g, \phi) := h_\mu(g) + \int \phi \, \mathrm{d}\mu.$$

Here $h_\mu(g)$ denotes the usual measure-theoretic entropy of g for μ.

By the Variational Principle (see for example, [PU10, Theorem 3.4.1]), we have that for each $\phi \in C(X)$, the *topological pressure* $P(g, \phi)$ of g with respect to the potential ϕ satisfies

$$(3.2) \qquad\qquad P(g, \phi) = \sup\{P_\mu(g, \phi) : \mu \in \mathcal{M}(X, g)\}.$$

In particular, when ϕ is the constant function 0, the *topological entropy* $h_{\mathrm{top}}(g)$ of g satisfies

$$(3.3) \qquad\qquad h_{\mathrm{top}}(g) = \sup\{h_\mu(g) : \mu \in \mathcal{M}(X, g)\}.$$

A measure μ that attains the supremum in (3.2) is called an *equilibrium state* for the map g and the potential ϕ. A measure μ that attains the supremum in (3.3) is called a *measure of maximal entropy* of g.

Definition 3.1. Let $g\colon X \to X$ be a continuous map on a metric space (X, d). Let $\mathcal{K} \subseteq C(X, \mathbb{C})$ be a subspace of the space $C(X, \mathbb{C})$ of complex-valued continuous functions on X. Two functions $\phi, \psi \in C(X, \mathbb{C})$ are said to be *co-homologous (in $\mathcal{K}$)* if there exists $u \in \mathcal{K}$ such that $\phi - \psi = u \circ g - u$.

One of the main tools in the study of the existence, uniqueness, and other properties of equilibrium states is the *Ruelle operator*. We will postpone the discussion of the Ruelle operators of expanding Thurston maps to Subsection 3.2.

3.2. Thurston maps.

In this subsection, we go over some key concepts and results on Thurston maps, and expanding Thurston maps in particular. For a more thorough treatment of the subject, we refer to [BM17].

Let S^2 denote an oriented topological 2-sphere. A continuous map $f\colon S^2 \to S^2$ is called a *branched covering map* on S^2 if for each point $x \in S^2$, there exists a positive integer $d \in \mathbb{N}$, open neighborhoods U of x and V of $y = f(x)$, open neighborhoods U' and V' of 0 in $\widehat{\mathbb{C}}$, and orientation-preserving homeomorphisms $\varphi\colon U \to U'$ and $\eta\colon V \to V'$ such that $\varphi(x) = 0$, $\eta(y) = 0$, and

$$(\eta \circ f \circ \varphi^{-1})(z) = z^d$$

for each $z \in U'$. The positive integer d above is called the *local degree* of f at x and is denoted by $\deg_f(x)$.

The *degree* of f is

$$(3.4) \qquad\qquad \deg f = \sum_{x \in f^{-1}(y)} \deg_f(x)$$

for $y \in S^2$ and is independent of y. If $f\colon S^2 \to S^2$ and $g\colon S^2 \to S^2$ are two branched covering maps on S^2, then so is $f \circ g$, and

$$(3.5) \qquad\qquad \deg_{f \circ g}(x) = \deg_g(x) \deg_f(g(x)), \qquad \text{for each } x \in S^2,$$

and moreover,

$$(3.6) \qquad\qquad \deg(f \circ g) = (\deg f)(\deg g).$$

A point $x \in S^2$ is a *critical point* of f if $\deg_f(x) \geq 2$. The set of critical points of f is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. Note that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

Definition 3.2 (Thurston maps). A Thurston map is a branched covering map $f\colon S^2 \to S^2$ on S^2 with $\deg f \geq 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

We now recall the notation for cell decompositions of S^2 used in [BM17] and [Li17]. A *cell of dimension n* in S^2, $n \in \{1, 2\}$, is a subset $c \subseteq S^2$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{R}^n$. We define the *boundary of c*, denoted by ∂c, to be the set of points corresponding to $\partial \mathbb{B}^n$ under such a homeomorphism between c and $\overline{\mathbb{B}^n}$. The *interior of c* is defined to be $\operatorname{inte}(c) = c \setminus \partial c$. For each point $x \in S^2$, the set $\{x\}$ is considered as a *cell of dimension 0* in S^2. For a cell c of dimension 0, we adopt the convention that $\partial c = \emptyset$ and $\operatorname{inte}(c) = c$.

We record the following three definitions from [BM17].

Definition 3.3 (Cell decompositions). Let $\mathbf{D}$ be a collection of cells in S^2. We say that $\mathbf{D}$ is a *cell decomposition of S^2* if the following conditions are satisfied:

   (i) the union of all cells in $\mathbf{D}$ is equal to S^2,

  (ii) if $c \in \mathbf{D}$, then ∂c is a union of cells in $\mathbf{D}$,

 (iii) for c_1, $c_2 \in \mathbf{D}$ with $c_1 \neq c_2$, we have $\mathrm{inte}(c_1) \cap \mathrm{inte}(c_2) = \emptyset$,

 (iv) every point in S^2 has a neighborhood that meets only finitely many cells in $\mathbf{D}$.

Definition 3.4 (Refinements)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that $\mathbf{D}'$ is a *refinement* of $\mathbf{D}$ if the following conditions are satisfied:

   (i) every cell $c \in \mathbf{D}$ is the union of all cells $c' \in \mathbf{D}'$ with $c' \subseteq c$,

  (ii) for every cell $c' \in \mathbf{D}'$ there exits a cell $c \in \mathbf{D}$ with $c' \subseteq c$.

Definition 3.5 (Cellular maps and cellular Markov partitions)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that a continuous map $f \colon S^2 \to S^2$ is *cellular* for $(\mathbf{D}', \mathbf{D})$ if for every cell $c \in \mathbf{D}'$, the restriction $f|_c$ of f to c is a homeomorphism of c onto a cell in $\mathbf{D}$. We say that $(\mathbf{D}', \mathbf{D})$ is a *cellular Markov partition* for f if f is cellular for $(\mathbf{D}', \mathbf{D})$ and $\mathbf{D}'$ is a refinement of $\mathbf{D}$.

Let $f \colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing $\mathrm{post}\, f$. Then the pair f and $\mathcal{C}$ induces natural cell decompositions $\mathbf{D}^n(f, \mathcal{C})$ of S^2, for $n \in \mathbb{N}_0$, in the following way:

By the Jordan curve theorem, the set $S^2 \backslash \mathcal{C}$ has two connected components. We call the closure of one of them the *white 0-tile* for $(f, \mathcal{C})$, denoted by $X_{\mathfrak{w}}^0$, and the closure of the other the *black 0-tile* for $(f, \mathcal{C})$, denoted by $X_{\mathfrak{b}}^0$. The set of 0-*tiles* is $\mathbf{X}^0(f, \mathcal{C}) := \{ X_{\mathfrak{b}}^0,\, X_{\mathfrak{w}}^0 \}$. The set of 0-*vertices* is $\mathbf{V}^0(f, \mathcal{C}) := \mathrm{post}\, f$. We set $\overline{\mathbf{V}}^0(f, \mathcal{C}) := \{ \{x\} : x \in \mathbf{V}^0(f, \mathcal{C}) \}$. The set of 0-*edges* $\mathbf{E}^0(f, \mathcal{C})$ is the set of the closures of the connected components of $\mathcal{C} \setminus \mathrm{post}\, f$. Then we get a cell decomposition

$$\mathbf{D}^0(f, \mathcal{C}) := \mathbf{X}^0(f, \mathcal{C}) \cup \mathbf{E}^0(f, \mathcal{C}) \cup \overline{\mathbf{V}}^0(f, \mathcal{C})$$

of S^2 consisting of *cells of level* 0, or 0-*cells*.

We can recursively define unique cell decompositions $\mathbf{D}^n(f, \mathcal{C})$, $n \in \mathbb{N}$, consisting of n-*cells* such that f is cellular for $(\mathbf{D}^{n+1}(f, \mathcal{C}), \mathbf{D}^n(f, \mathcal{C}))$. We refer to [BM17, Lemma 5.12] for more details. We denote by $\mathbf{X}^n(f, \mathcal{C})$ the set of n-cells of dimension 2, called n-*tiles*; by $\mathbf{E}^n(f, \mathcal{C})$ the set of n-cells of dimension 1, called n-*edges*; by $\overline{\mathbf{V}}^n(f, \mathcal{C})$ the set of n-cells of dimension 0; and by $\mathbf{V}^n(f, \mathcal{C})$ the set $\{ x : \{x\} \in \overline{\mathbf{V}}^n(f, \mathcal{C}) \}$, called the set of n-*vertices*. The *k-skeleton*, for $k \in \{0, 1, 2\}$, of $\mathbf{D}^n(f, \mathcal{C})$ is the union of all n-cells of dimension k in this cell decomposition.

We record Proposition 5.16 of [BM17] here in order to summarize properties of the cell decompositions $\mathbf{D}^n(f, \mathcal{C})$ defined above.

Proposition 3.6 (M. Bonk & D. Meyer [BM17])**.** *Let* $k, n \in \mathbb{N}_0$, *let* $f \colon S^2 \to S^2$ *be a Thurston map,* $\mathcal{C} \subseteq S^2$ *be a Jordan curve with* $\mathrm{post}\, f \subseteq \mathcal{C}$, *and* $m = \mathrm{card}(\mathrm{post}\, f)$.

   (i) *The map* f^k *is cellular for* $(\mathbf{D}^{n+k}(f, \mathcal{C}), \mathbf{D}^n(f, \mathcal{C}))$. *In particular, if c is any $(n + k)$-cell, then $f^k(c)$ is an n-cell, and $f^k|_c$ is a homeomorphism of c onto $f^k(c)$.*

  (ii) *Let c be an n-cell. Then $f^{-k}(c)$ is equal to the union of all $(n + k)$-cells c' with $f^k(c') = c$.*

 (iii) *The 1-skeleton of $\mathbf{D}^n(f, \mathcal{C})$ is equal to $f^{-n}(\mathcal{C})$. The 0-skeleton of $\mathbf{D}^n(f, \mathcal{C})$ is the set $\mathbf{V}^n(f, \mathcal{C}) = f^{-n}(\mathrm{post}\, f)$, and we have $\mathbf{V}^n(f, \mathcal{C}) \subseteq \mathbf{V}^{n+k}(f, \mathcal{C})$.*

 (iv) $\mathrm{card}(\mathbf{X}^n(f, \mathcal{C})) = 2(\deg f)^n$, $\mathrm{card}(\mathbf{E}^n(f, \mathcal{C})) = m(\deg f)^n$, *and* $\mathrm{card}(\mathbf{V}^n(f, \mathcal{C})) \leq m(\deg f)^n$.

  (v) *The n-edges are precisely the closures of the connected components of $f^{-n}(\mathcal{C}) \backslash f^{-n}(\mathrm{post}\, f)$. The n-tiles are precisely the closures of the connected components of $S^2 \setminus f^{-n}(\mathcal{C})$.*

 (vi) *Every n-tile is an m-gon, i.e., the number of n-edges and the number of n-vertices contained in its boundary are equal to m.*

 (vii) *Let $F := f^k$ be an iterate of f with $k \in \mathbb{N}$. Then $\mathbf{D}^n(F, \mathcal{C}) = \mathbf{D}^{nk}(f, \mathcal{C})$.*

We also note that for each n-edge $e \in \mathbf{E}^n(f, \mathcal{C})$, $n \in \mathbb{N}_0$, there exist exactly two n-tiles $X, X' \in \mathbf{X}^n(f, \mathcal{C})$ such that $X \cap X' = e$.

For $n \in \mathbb{N}_0$, we define the *set of black n-tiles* as

$$\mathbf{X}_{\mathfrak{b}}^n(f, \mathcal{C}) := \left\{ X \in \mathbf{X}^n(f, \mathcal{C}) : f^n(X) = X_{\mathfrak{b}}^0 \right\},$$

and the *set of white n-tiles* as

$$\mathbf{X}_{\mathfrak{w}}^n(f,\mathcal{C}) := \left\{ X \in \mathbf{X}^n(f,\mathcal{C}) : f^n(X) = X_{\mathfrak{w}}^0 \right\}.$$

It follows immediately from Proposition 3.6 that

(3.7) $\mathrm{card}(\mathbf{X}_{\mathfrak{b}}^n(f,\mathcal{C})) = \mathrm{card}(\mathbf{X}_{\mathfrak{w}}^n(f,\mathcal{C})) = (\deg f)^n$

for each $n \in \mathbb{N}_0$.

From now on, if the map f and the Jordan curve $\mathcal{C}$ are clear from the context, we will sometimes omit $(f,\mathcal{C})$ in the notation above.

We denote, for each $x \in S^2$ and $n \in \mathbb{Z}$,

(3.8) $U^n(x) := \bigcup \{ Y^n \in \mathbf{X}^n : \text{there exists } X^n \in \mathbf{X}^n \text{ with } x \in X^n,\ X^n \cap Y^n \neq \emptyset \}$

if $n \geq 0$, and set $U^n(x) := S^2$ otherwise.

We can now recall a definition of expanding Thurston maps.

Definition 3.7 (Expansion). A Thurston map $f \colon S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing post f such that

$$\lim_{n \to +\infty} \max\{\mathrm{diam}_d(X) : X \in \mathbf{X}^n(f,\mathcal{C})\} = 0.$$

P. Haïssinsky and K. M. Pilgrim developed a notion of expansion in a more general context for finite branched coverings between topological spaces (see [HP09, Sections 2.1 and 2.2]). This applies to Thurston maps, and their notion of expansion is equivalent to our notion defined above in the context of Thurston maps (see [BM17, Proposition 6.4]). Our notion of expansion is not equivalent to classical notions such as forward-expansive maps or distance-expanding maps. One topological obstruction comes from the presence of critical points for (non-homeomorphic) branched covering maps on S^2.

For an expanding Thurston map f, we can fix a particular metric d on S^2 called a *visual metric for* f. For the existence and properties of such metrics, see [BM17, Chapter 8]. For a visual metric d for f, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of d (see [BM17, Chapter 8] for more details). One major advantage of a visual metric d is that in (S^2, d), we have good quantitative control over the sizes of the cells in the cell decompositions discussed above. We summarize several results of this type ([BM17, Proposition 8.4, Lemmas 8.10, 8.11]) in the lemma below.

Lemma 3.8 (M. Bonk & D. Meyer [BM17]). *Let $f \colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* post f. *Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Then there exist constants $C \geq 1$, $C' \geq 1$, $K \geq 1$, and $n_0 \in \mathbb{N}_0$ with the following properties:*

(i) $d(\sigma, \tau) \geq C^{-1}\Lambda^{-n}$ *whenever σ and τ are disjoint n-cells for $n \in \mathbb{N}_0$.*

(ii) $C^{-1}\Lambda^{-n} \leq \mathrm{diam}_d(\tau) \leq C\Lambda^{-n}$ *for all n-edges and all n-tiles τ for $n \in \mathbb{N}_0$.*

(iii) $B_d(x, K^{-1}\Lambda^{-n}) \subseteq U^n(x) \subseteq B_d(x, K\Lambda^{-n})$ *for $x \in S^2$ and $n \in \mathbb{N}_0$.*

(iv) $U^{n+n_0}(x) \subseteq B_d(x, r) \subseteq U^{n-n_0}(x)$ *where $n = \lceil -\log r / \log \Lambda \rceil$ for $r > 0$ and $x \in S^2$.*

(v) *For every n-tile $X^n \in \mathbf{X}^n(f,\mathcal{C})$, $n \in \mathbb{N}_0$, there exists a point $p \in X^n$ such that $B_d(p, C^{-1}\Lambda^{-n}) \subseteq X^n \subseteq B_d(p, C\Lambda^{-n})$.*

Conversely, if $\widetilde{d}$ is a metric on S^2 satisfying conditions (i) *and* (ii) *for some constant $C \geq 1$, then $\widetilde{d}$ is a visual metric with expansion factor $\Lambda > 1$.*

Recall that $U^n(x)$ is defined in (3.8).

In addition, we will need the fact that a visual metric d induces the standard topology on S^2 ([BM17, Proposition 8.3]) and the fact that the metric space (S^2, d) is linearly locally connected ([BM17, Proposition 18.5]). A metric space (X, d) is *linearly locally connected* if there exists a constant $L \geq 1$ such that the following conditions are satisfied:

(1) For all $z \in X$, $r > 0$, and $x, y \in B_d(z, r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \in E$ and $E \subseteq B_d(z, rL)$.

(2) For all $z \in X$, $r > 0$, and $x, y \in X \setminus B_d(z, r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \in E$ and $E \subseteq X \setminus B_d(z, r/L)$.

We call such a constant $L \geq 1$ a *linear local connectivity constant of d*.

In fact, visual metrics serve a crucial role in connecting the dynamical arguments with geometric properties for rational expanding Thurston maps, especially Lattès maps.

We first recall the following notions of equivalence between metric spaces.

Definition 3.9. Consider two metric spaces (X_1, d_1) and (X_2, d_2). Let $g \colon X_1 \to X_2$ be a homeomorphism. Then

(i) g is *bi-Lipschitz* if there exists a constant $C \geq 1$ such that for all u, $v \in X_1$,
$$C^{-1} d_1(u, v) \leq d_2(g(u), g(v)) \leq C d_1(u, v).$$

(ii) g is a *snowflake homeomorphism* if there exist constants $\alpha > 0$ and $C \geq 1$ such that for all u, $v \in X_1$,
$$C^{-1} d_1(u, v)^\alpha \leq d_2(g(u), g(v)) \leq C d_1(u, v)^\alpha.$$

(iii) g is a *quasisymmetric homeomorphism* or a *quasisymmetry* if there exists a homeomorphism $\eta \colon [0, +\infty) \to [0, +\infty)$ such that for all u, v, $w \in X_1$,
$$\frac{d_2(g(u), g(v))}{d_2(g(u), g(w))} \leq \eta \left(\frac{d_1(u, v)}{d_1(u, w)} \right).$$

Moreover, the metric spaces (X_1, d_1) and (X_2, d_2) are *bi-Lipschitz, snowflake,* or *quasisymmetrically equivalent* if there exists a homeomorphism from (X_1, d_1) to (X_2, d_2) with the corresponding property.

When $X_1 = X_2 =: X$, then we say the metrics d_1 and d_2 are *bi-Lipschitz, snowflake,* or *quasisymmetrically equivalent* if the identity map from (X, d_1) to (X, d_2) has the corresponding property.

Remark 3.10. If $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational expanding Thurston map (or equivalently, a postcritically-finite rational map without periodic critical points (see [BM17, Proposition 2.3])), then each visual metric is quasisymmetrically equivalent to the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$ (see [BM17, Lemma 18.10]). Here the chordal metric σ on $\widehat{\mathbb{C}}$ is given by $\sigma(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for z, $w \in \mathbb{C}$, and $\sigma(\infty, z) = \sigma(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$ for $z \in \mathbb{C}$. We also note that quasisymmetric embeddings of bounded connected metric spaces are Hölder continuous (see [**?**, Section 11.1 and Corollary 11.5]). Accordingly, the class of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and that on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for f are the same (up to a change of the Hölder exponent).

A Jordan curve $\mathcal{C} \subseteq S^2$ is *f-invariant* if $f(\mathcal{C}) \subseteq \mathcal{C}$. We are interested in f-invariant Jordan curves that contain $\operatorname{post} f$, since for such a Jordan curve $\mathcal{C}$, we get a cellular Markov partition $(\mathbf{D}^1(f, \mathcal{C}), \mathbf{D}^0(f, \mathcal{C}))$ for f. According to Example 15.11 in [BM17], such f-invariant Jordan curves containing $\operatorname{post} f$ need not exist. However, M. Bonk and D. Meyer [BM17, Theorem 15.1] proved that there exists an f^n-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$ for each sufficiently large n depending on f. A slightly stronger version of this result was proved in [Li16, Lemma 3.11], and we record it below.

Lemma 3.11 (M. Bonk & D. Meyer [BM17], Z. Li [Li16])**.** *Let* $f \colon S^2 \to S^2$ *be an expanding Thurston map, and* $\widetilde{\mathcal{C}} \subseteq S^2$ *be a Jordan curve with* $\operatorname{post} f \subseteq \widetilde{\mathcal{C}}$. *Then there exists an integer* $N(f, \widetilde{\mathcal{C}}) \in \mathbb{N}$ *such that for each* $n \geq N(f, \widetilde{\mathcal{C}})$ *there exists an* f^n-*invariant Jordan curve* $\mathcal{C}$ *isotopic to* $\widetilde{\mathcal{C}}$ *rel.* $\operatorname{post} f$ *such that no n-tile in* $\mathbf{X}^n(f, \mathcal{C})$ *joins opposite sides of* $\mathcal{C}$.

The phrase "joining opposite sides" has a specific meaning in our context.

Definition 3.12 (Joining opposite sides)**.** Fix a Thurston map f with $\operatorname{card}(\operatorname{post} f) \geq 3$ and an f-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$. A set $K \subseteq S^2$ *joins opposite sides* of $\mathcal{C}$ if K meets two disjoint 0-edges when $\operatorname{card}(\operatorname{post} f) \geq 4$, or K meets all three 0-edges when $\operatorname{card}(\operatorname{post} f) = 3$.

Note that $\operatorname{card}(\operatorname{post} f) \geq 3$ for each expanding Thurston map f [BM17, Lemma 6.1].

The following lemma proved in [Li18, Lemma 3.13] generalizes [BM17, Lemma 15.25].

Lemma 3.13 (M. Bonk & D. Meyer [BM17], Z. Li [Li18])**.** *Let* $f \colon S^2 \to S^2$ *be an expanding Thurston map, and* $\mathcal{C} \subseteq S^2$ *be a Jordan curve that satisfies* $\operatorname{post} f \subseteq \mathcal{C}$ *and* $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ *for some* $n_{\mathcal{C}} \in \mathbb{N}$. *Let* d *be a visual metric on* S^2 *for* f *with expansion factor* $\Lambda > 1$. *Then there exists a constant* $C_0 > 1$, *depending only on* f, d, $\mathcal{C}$, *and* $n_{\mathcal{C}}$, *with the following property:*

If $k,\, n \in \mathbb{N}_0$, $X^{n+k} \in \mathbf{X}^{n+k}(f, \mathcal{C})$, and $x,\, y \in X^{n+k}$, then

$$(3.9) \qquad C_0^{-1} d(x, y) \leq \Lambda^{-n} d(f^n(x), f^n(y)) \leq C_0 d(x, y).$$

We summarize the existence, uniqueness, and some basic properties of equilibrium states for expanding Thurston maps in the following theorem.

Theorem 3.14 (Z. Li [Li18]). *Let $f \colon S^2 \to S^2$ be an expanding Thurston map and d a visual metric on S^2 for f. Let $\phi,\, \gamma \in C^{0,\alpha}(S^2, d)$ be real-valued Hölder continuous functions with an exponent $\alpha \in (0, 1]$. Then the following statements are satisfied:*

   (i) *There exists a unique equilibrium state μ_ϕ for the map f and the potential ϕ.*

  (ii) *For each $t \in \mathbb{R}$, we have $\frac{\mathrm{d}}{\mathrm{d}t} P(f, \phi + t\gamma) = \int \gamma \, \mathrm{d}\mu_{\phi+t\gamma}$.*

 (iii) *If $\mathcal{C} \subseteq S^2$ is a Jordan curve containing $\operatorname{post} f$ with the property that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$, then $\mu_\phi\bigl(\bigcup_{i=0}^{+\infty} f^{-i}(\mathcal{C})\bigr) = 0$.*

Theorem 3.14 (i) is part of [Li18, Theorem 1.1]. Theorem 3.14 (ii) follows immediately from [Li18, Theorem 6.13] and the uniqueness of equilibrium states in Theorem 3.14 (i). Theorem 3.14 (iii) was established in [Li18, Proposition 7.1].

The following distortion lemma serves as the cornerstone in the development of thermodynamic formalism for expanding Thurston maps in [Li18] (see [Li18, Lemmas 5.1 and 5.2]).

Lemma 3.15 (Z. Li [Li18]). *Let $f \colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing $\operatorname{post} f$ with the property that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$ and a linear local connectivity constant $L \geq 1$. Fix $\phi \in C^{0,\alpha}(S^2, d)$ with $\alpha \in (0, 1]$. Then there exist constants $C_1 = C_1(f, \mathcal{C}, d, \phi, \alpha)$ and $C_2 = C_2(f, \mathcal{C}, d, \phi, \alpha) \geq 1$ depending only on f, $\mathcal{C}$, d, ϕ, and α such that*

$$(3.10) \qquad |S_n\phi(x) - S_n\phi(y)| \leq C_1 d(f^n(x), f^n(y))^\alpha,$$

$$(3.11) \qquad \frac{\sum_{z' \in f^{-n}(z)} \deg_{f^n}(z') \exp(S_n\phi(z'))}{\sum_{w' \in f^{-n}(w)} \deg_{f^n}(w') \exp(S_n\phi(w'))} \leq \exp\left(4C_1 L d(z, w)^\alpha\right) \leq C_2,$$

for $n,\, m \in \mathbb{N}_0$ with $n \leq m$, $X^m \in \mathbf{X}^m(f, \mathcal{C})$, $x,\, y \in X^m$, and $z,\, w \in S^2$. We choose

$$(3.12) \qquad C_1 := |\phi|_{\alpha,\, (S^2, d)}\, C_0 (1 - \Lambda^{-\alpha})^{-1} \qquad \text{and} \qquad C_2 := \exp\bigl(4C_1 L \bigl(\operatorname{diam}_d(S^2)\bigr)^\alpha\bigr)$$

where $C_0 > 1$ is a constant depending only on f, $\mathcal{C}$, and d from [Li18, Lemma 3.13].

Recall that the main tool used in [Li18] to develop the thermodynamic formalism for expanding Thurston maps is the Ruelle operator. We will need a complex version of the Ruelle operator in this paper discussed in [Li17]. We summarize relevant definitions and facts about the Ruelle operator below and refer the readers to [Li17, Chapter 3.3] for a detailed discussion.

Let $f \colon S^2 \to S^2$ be an expanding Thurston map and $\phi \in C(S^2, \mathbb{C})$ be a complex-valued continuous function. The *Ruelle operator* $\mathcal{L}_\phi$ (associated to f and ϕ) acting on $C(S^2, \mathbb{C})$ is defined as the following

$$(3.13) \qquad \mathcal{L}_\phi(u)(x) = \sum_{y \in f^{-1}(x)} \deg_f(y) u(y) \exp(\phi(y)),$$

for each $u \in C(S^2, \mathbb{C})$. Note that $\mathcal{L}_\phi$ is a well-defined and continuous operator on $C(S^2, \mathbb{C})$. The Ruelle operator $\mathcal{L}_\phi \colon C(S^2, \mathbb{C}) \to C(S^2, \mathbb{C})$ has an extension to the space of complex-valued bounded Borel functions $B(S^2, \mathbb{C})$ (equipped with the uniform norm) given by (3.13) for each $u \in B(S^2, \mathbb{C})$.

We observe that if $\phi \in C(S^2)$ is real-valued, then $\mathcal{L}_\phi(C(S^2)) \subseteq C(S^2)$ and $\mathcal{L}_\phi(B(S^2)) \subseteq B(S^2)$. The adjoint operator $\mathcal{L}_\phi^* \colon C^*(S^2) \to C^*(S^2)$ of $\mathcal{L}_\phi$ acts on the dual space $C^*(S^2)$ of the Banach space $C(S^2)$. We identify $C^*(S^2)$ with the space $\mathcal{M}(S^2)$ of finite signed Borel measures on S^2 by the Riesz representation theorem.

When $\phi \in C(S^2)$ is real-valued, we denote

$$(3.14) \qquad \overline{\phi} := \phi - P(f, \phi).$$

We record the following three technical results on the Ruelle operators in our context.

Lemma 3.16 (Z. Li [Li18]). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* post f *with the property that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then there exists a constant $C_3 = C_3(f, \mathcal{C}, d, \phi, \alpha)$ depending only on f, $\mathcal{C}$, d, ϕ, and α such that for each $x, y \in S^2$ and each $n \in \mathbb{N}_0$ the following equations are satisfied*

$$(3.15) \qquad \mathcal{L}_{\overline{\phi}}^n(\mathbb{1})(x)/\mathcal{L}_{\overline{\phi}}^n(\mathbb{1})(y) \le \exp\left(4C_1 L d(x,y)^\alpha\right) \le C_2,$$

$$(3.16) \qquad C_2^{-1} \le \mathcal{L}_{\overline{\phi}}^n(\mathbb{1})(x) \le C_2,$$

$$(3.17) \qquad \left|\mathcal{L}_{\overline{\phi}}^n(\mathbb{1})(x) - \mathcal{L}_{\overline{\phi}}^n(\mathbb{1})(y)\right| \le C_2(\exp(4C_1 L d(x,y)^\alpha) - 1) \le C_3 d(x,y)^\alpha,$$

where C_1, C_2 are constants in Lemma 3.15 depending only on f, $\mathcal{C}$, d, ϕ, and α.

Lemma 3.16 was proved in [Li18, Lemma 5.15]. The next theorem is part of [Li18, Theorem 5.16].

Theorem 3.17 (Z. Li [Li18]). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* post f *with the property that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then the sequence $\left\{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\overline{\phi}}^j(\mathbb{1})\right\}_{n\in\mathbb{N}}$ converges uniformly to a function $u_\phi \in C^{0,\alpha}(S^2, d)$, which satisfies $\mathcal{L}_{\overline{\phi}}(u_\phi) = u_\phi$, and $C_2^{-1} \le u_\phi(x) \le C_2$ for each $x \in S^2$, where $C_2 \ge 1$ is a constant from Lemma 3.15.*

Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then we denote

$$(3.18) \qquad \widetilde{\phi} := \phi - P(f, \phi) + \log u_\phi - \log(u_\phi \circ f),$$

where u_ϕ is the continuous function given by Theorem 3.17.

A measure $\mu \in \mathcal{P}(S^2)$ is an *eigenmeasure* of $\mathcal{L}_\phi^*$ if $\mathcal{L}_\phi^*(\mu) = c\mu$ for some $c \in \mathbb{R}$. See [Li18, Corollary 6.10] for the uniqueness of the measure m_ϕ. The reader is referred to the first paragraph of Subsection 5.3 for the notion of an abstract modulus of continuity and the definition of $C_h^b(S^2, d)$.

Lemma 3.18. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. We define a map $\tau\colon \mathbb{R} \to C^{0,\alpha}(S^2, d)$ by setting $\tau(t) = u_{t\phi}$. Then τ is continuous with respect to the uniform norm $\|\cdot\|_{C^0(S^2)}$ on $C^{0,\alpha}(S^2, d)$.*

Proof. Fix an arbitrary bounded open interval $I \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$, define $T_n\colon I \to C(S^2, d)$ by $T_n(t) := \mathcal{L}_{\overline{t\phi}}^n(\mathbb{1}_{S^2})$ for $t \in I$. Since $\overline{t\phi} = t\phi - P(f, t\phi)$, by (3.13) and the continuity of the topological pressure (see for example, [PU10, Theorem 3.6.1]), we know that T_n is a continuous function with respect to the uniform norm $\|\cdot\|_{C^0(S^2)}$ on $C(S^2, d)$. Applying [Li18, Theorem 6.8 and Corollary 6.10], we get that $T_n(t)$ converges to $\tau|_I(t)$ in the uniform norm on $C(S^2, d)$ uniformly in $t \in I$ as $n \to +\infty$. Hence $\tau(t)$ is continuous on I. Recall $u_{t\phi} \in C^{0,\alpha}(S^2, d)$ (see Theorem 3.17). Therefore $\tau(t)$ is continuous in $t \in \mathbb{R}$ with respect to the uniform norm on $C^{0,\alpha}(S^2, d)$. $\qquad \square$

The potentials that satisfy the following property are of particular interest in the considerations of Prime Orbit Theorems and the analytic study of dynamical zeta functions.

Definition 3.19 (Eventually positive functions). Let $g\colon X \to X$ be a map on a set X, and $\varphi\colon X \to \mathbb{C}$ be a complex-valued function on X. Then φ is *eventually positive* if there exists $N \in \mathbb{N}$ such that $S_n\varphi(x) > 0$ for each $x \in X$ and each $n \in \mathbb{N}$ with $n \ge N$.

Theorem 3.14 (ii) leads to the following corollary that we frequently use, often implicitly, throughout this paper. See [LZhe23a, Corollary 3.29] for a proof.

Corollary 3.20. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f. Let $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then the function $t \mapsto P(f, -t\phi)$, $t \in \mathbb{R}$, is strictly decreasing and there exists a unique number $s_0 \in \mathbb{R}$ such that $P(f, -s_0\phi) = 0$. Moreover, $s_0 > 0$.*

3.3. **Subshifts of finite type.** We give a brief review of the dynamics of one-sided subshifts of finite type in this subsection. We refer the readers to [Ki98] for a beautiful introduction to symbolic dynamics. For a discussion on results on subshifts of finite type in our context, see [PP90, Ba00].

Let S be a finite nonempty set, and $A \colon S \times S \to \{0, 1\}$ be a matrix whose entries are either 0 or 1. For $n \in \mathbb{N}_0$, we denote by A^n the usual matrix product of n copies of A. We denote the *set of admissible sequences defined by A* by

$$\Sigma_A^+ = \{\{x_i\}_{i \in \mathbb{N}_0} : x_i \in S, \, A(x_i, x_{i+1}) = 1, \text{ for each } i \in \mathbb{N}_0\}.$$

Given $\theta \in (0, 1)$, we equip the set Σ_A^+ with a metric d_θ given by $d_\theta(\{x_i\}_{i \in \mathbb{N}_0}, \{y_i\}_{i \in \mathbb{N}_0}) = \theta^N$ for $\{x_i\}_{i \in \mathbb{N}_0} \neq \{y_i\}_{i \in \mathbb{N}_0}$, where N is the smallest integer with $x_N \neq y_N$. The topology on the metric space (Σ_A^+, d_θ) coincides with that induced from the product topology, and is therefore compact.

The *left-shift operator* $\sigma_A \colon \Sigma_A^+ \to \Sigma_A^+$ (defined by A) is given by

$$\sigma_A(\{x_i\}_{i \in \mathbb{N}_0}) = \{x_{i+1}\}_{i \in \mathbb{N}_0} \qquad \text{for } \{x_i\}_{i \in \mathbb{N}_0} \in \Sigma_A^+.$$

The pair (Σ_A^+, σ_A) is called the *one-sided subshift of finite type* defined by A. The set S is called the *set of states* and the matrix $A \colon S \times S \to \{0, 1\}$ is called the *transition matrix*.

We say that a one-sided subshift of finite type (Σ_A^+, σ_A) is *topologically mixing* if there exists $N \in \mathbb{N}$ such that $A^n(x, y) > 0$ for each $n \geq N$ and each pair of $x, y \in S$.

Let X and Y be topological spaces, and $f \colon X \to X$ and $g \colon Y \to Y$ be continuous maps. We say that the topological dynamical system (X, f) is a *factor* of the topological dynamical system (Y, g) if there is a surjective continuous map $\pi \colon Y \to X$ such that $\pi \circ g = f \circ \pi$. We call the map $\pi \colon Y \to X$ a *factor map*.

The following proposition is established in [LZhe23a, Proposition 3.31].

Proposition 3.21. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Fix $\theta \in (0, 1)$. We set $S_\triangle := \mathbf{X}^1(f, \mathcal{C})$, and define a transition matrix $A_\triangle \colon S_\triangle \times S_\triangle \to \{0, 1\}$ by*

$$A_\triangle(X, X') = \begin{cases} 1 & \text{if } f(X) \supseteq X', \\ 0 & \text{otherwise} \end{cases}$$

for $X, X' \in \mathbf{X}^1(f, \mathcal{C})$. Then f is a factor of the one-sided subshift of finite type $(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle})$ defined by the transition matrix $A_\triangle$ with a surjective and Hölder continuous factor map $\pi_\triangle \colon \Sigma_{A_\triangle}^+ \to S^2$ given by

$$(3.19) \qquad \pi_\triangle(\{X_i\}_{i \in \mathbb{N}_0}) = x, \text{ where } \{x\} = \bigcap_{i \in \mathbb{N}_0} f^{-i}(X_i).$$

Here $\Sigma_{A_\triangle}^+$ is equipped with the metric d_θ defined in Subsection 3.3, and S^2 is equipped with the visual metric d.

Moreover, $(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle})$ is topologically mixing and $\pi_\triangle$ is injective on $\pi_\triangle^{-1}(S^2 \setminus \bigcup_{i \in \mathbb{N}_0} f^{-i}(\mathcal{C}))$.

Remark 3.22. We can show that if f has no periodic critical points, then π is uniformly bounded-to-one (i.e., there exists $N \in \mathbb{N}_0$ depending only on f such that card $(\pi_\triangle^{-1}(x)) \leq N$ for each $x \in S^2$); if f has at least one periodic critical point, then $\pi_\triangle$ is uncountable-to-one on a dense set. We will not use this fact in this paper.

3.4. **Dynamical zeta functions and Dirichlet series.** Let $g \colon X \to X$ be a map on a topological space X. Let $\psi \colon X \to \mathbb{C}$ be a complex-valued function on X. We write

$$(3.20) \qquad Z_{g, -\psi}^{(n)}(s) := \sum_{x \in P_{1, g^n}} e^{-s S_n \psi(x)}, \qquad n \in \mathbb{N} \text{ and } s \in \mathbb{C}.$$

Recall that P_{1, g^n} defined in (2.1) is the set of fixed points of g^n, and $S_n \psi$ is defined in (2.4). We denote by the formal infinite product

$$(3.21) \qquad \zeta_{g, -\psi}(s) := \exp\left(\sum_{n=1}^{+\infty} \frac{Z_{g, -\psi}^{(n)}(s)}{n}\right) = \exp\left(\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{x \in P_{1, g^n}} e^{-s S_n \psi(x)}\right), \qquad s \in \mathbb{C},$$

the *dynamical zeta function* for the map g and the potential ψ. More generally, we can define dynamical Dirichlet series as analogs of Dirichlet series in analytic number theory.

Definition 3.23. Let $g\colon X \to X$ be a map on a topological space X. Let $\psi\colon X \to \mathbb{C}$ and $w\colon X \to \mathbb{C}$ be complex-valued functions on X. We denote by the formal infinite product

$$(3.22) \qquad \mathfrak{D}_{g,\,-\psi,\,w}(s) \coloneqq \exp\!\left(\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{x \in P_{1,g^n}} e^{-sS_n\psi(x)} \prod_{i=0}^{n-1} w\big(g^i(x)\big)\right), \qquad s \in \mathbb{C},$$

the *dynamical Dirichlet series* with coefficient w for the map g and the potential ψ.

The following result is obtained in [LZhe23a, Section 5].

Proposition 3.24. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathrm{post}\,f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0,1]$. Denote by s_0 the unique positive number with $P(f, -s_0\phi) = 0$. Let $\big(\Sigma_{A_\triangle}^+, \sigma_{A_\triangle}\big)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 3.21, and let $\pi_\triangle\colon \Sigma_{A_\triangle}^+ \to S^2$ be the factor map defined in (3.19). Denote by $\deg_f(\cdot)$ the local degree of f. Then the following statements are satisfied:*

(i) *$P(\sigma_{A_\triangle}, \varphi \circ \pi_\triangle) = P(f, \varphi)$ for each $\varphi \in C^{0,\alpha}(S^2, d)$. In particular, for an arbitrary number $t \in \mathbb{R}$, we have $P(\sigma_{A_\triangle}, -t\phi \circ \pi_\triangle) = 0$ if and only if $t = s_0$.*

(ii) *All three infinite products $\zeta_{f,-\phi}$, $\zeta_{\sigma_{A_\triangle}, -\phi\circ\pi_\triangle}$, and $\mathfrak{D}_{f,-\phi,\deg_f}$ converge uniformly and absolutely to respective non-vanishing continuous functions on $\overline{\mathbb{H}}_a = \{s \in \mathbb{C} : \Re(s) \geq a\}$ that are holomorphic on $\mathbb{H}_a = \{s \in \mathbb{C} : \Re(s) > a\}$, for each $a \in \mathbb{R}$ satisfies $a > s_0$.*

(iii) *For all $s \in \mathbb{C}$ with $\Re(s) > s_0$, we have*

$$(3.23) \qquad \zeta_{f,-\phi}(s) = \prod_{\tau \in \mathfrak{P}(f)} \left(1 - \exp\!\left(-s\sum_{y \in \tau} \phi(y)\right)\right)^{-1},$$

$$(3.24) \qquad \mathfrak{D}_{f,-\phi,\deg_f}(s) = \prod_{\tau \in \mathfrak{P}(f)} \left(1 - \exp\!\left(-s\sum_{y \in \tau} \phi(y)\right) \prod_{z \in \tau} \deg_f(z)\right)^{-1},$$

$$(3.25) \qquad \zeta_{\sigma_{A_\triangle}, -\phi\circ\pi_\triangle}(s) = \prod_{\tau \in \mathfrak{P}(\sigma_{A_\triangle})} \left(1 - \exp\!\left(-s\sum_{y \in \tau} \phi \circ \pi_\triangle(y)\right)\right)^{-1}.$$

Recall that $\mathfrak{P}(g)$ denotes the set of all primitive periodic orbits of g (see (2.3)). We recall that an infinite product of the form $\exp\sum a_i$, $a_i \in \mathbb{C}$, converges uniformly (resp. absolutely) if $\sum a_i$ converges uniformly (resp. absolutely).

4. THE ASSUMPTIONS

We state below the hypotheses under which we will develop our theory in most parts of this paper. We will repeatedly refer to such assumptions in the later sections. We emphasize again that not all assumptions are assumed in all the statements in this paper.

The Assumptions.

(1) $f\colon S^2 \to S^2$ is an expanding Thurston map.

(2) $\mathcal{C} \subseteq S^2$ is a Jordan curve containing $\mathrm{post}\,f$ with the property that there exists $n_{\mathcal{C}} \in \mathbb{N}$ such that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ and $f^m(\mathcal{C}) \nsubseteq \mathcal{C}$ for each $m \in \{1, 2, \ldots, n_{\mathcal{C}} - 1\}$.

(3) d is a visual metric on S^2 for f with expansion factor $\Lambda > 1$ and a linear local connectivity constant $L \geq 1$.

(4) $\alpha \in (0, 1]$.

(5) $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ is a complex-valued Hölder continuous function with an exponent α.

(6) $\phi \in C^{0,\alpha}(S^2, d)$ is an eventually positive real-valued Hölder continuous function with an exponent α, and $s_0 \in \mathbb{R}$ is the unique positive real number satisfying $P(f, -s_0\phi) = 0$.

(7) μ_ϕ is the unique equilibrium state for the map f and the potential ϕ.

Note that the uniqueness of s_0 in (6) is guaranteed by Corollary 3.20. For a pair of f in (1) and ϕ in (6), we will say that a quantity depends on f and ϕ if it depends on s_0.

Observe that by Lemma 3.11, for each f in (1), there exists at least one Jordan curve $\mathcal{C}$ that satisfies (2). Since for a fixed f, the number $n_{\mathcal{C}}$ is uniquely determined by $\mathcal{C}$ in (2), in the remaining part of the paper, we will say that a quantity depends on f and $\mathcal{C}$ even if it also depends on $n_{\mathcal{C}}$.

Recall that the expansion factor Λ of a visual metric d on S^2 for f is uniquely determined by d and f. We will say that a quantity depends on f and d if it depends on Λ.

Note that even though the value of L is not uniquely determined by the metric d, in the remainder of this paper, for each visual metric d on S^2 for f, we will fix a choice of linear local connectivity constant L. We will say that a quantity depends on the visual metric d without mentioning the dependence on L, even though if we had not fixed a choice of L, it would have depended on L as well.

In the discussion below, depending on the conditions we will need, we will sometimes say "Let f, $\mathcal{C}$, d, ψ, α satisfy the Assumptions.", and sometimes say "Let f and d satisfy the Assumptions.", etc.

5. Ruelle operators and split Ruelle operators

In this section, we define appropriate variations of the Ruelle operator on the suitable function spaces in our context and establish some important inequalities that will be used later. More precisely, in Subsection 5.1, for an expanding Thurston map f with some forward invariant Jordan curve $\mathcal{C} \subseteq S^2$ and a complex-valued Hölder continuous function ψ, we "split" the Ruelle operator $\mathcal{L}_\psi \colon C(S^2, \mathbb{C}) \to C(S^2, \mathbb{C})$ into pieces $\mathcal{L}^{(n)}_{\psi,\mathfrak{c},E} \colon C(E, \mathbb{C}) \to C(X^0_{\mathfrak{c}}, \mathbb{C})$, for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $n \in \mathbb{N}_0$, and a union $E \subseteq S^2$ of an arbitrary collection of n-tiles in the cell decomposition $\mathbf{D}^n(f, \mathcal{C})$ of S^2 induced by f and $\mathcal{C}$. Such construction is crucial to the proof of Proposition 6.1 where the images of characteristic functions supported on n-tiles under $\mathcal{L}^{(n)}_{\psi,\mathfrak{c},E}$ are used to relate periodic points and preimage points of f. We then define the *split Ruelle operators* $\mathbb{L}_\psi$ on the product space $C(X^0_\mathfrak{b}, \mathbb{C}) \times C(X^0_\mathfrak{w}, \mathbb{C})$ by piecing together $\mathcal{L}^{(1)}_{\psi,\mathfrak{c}_1,\mathfrak{c}_2} = \mathcal{L}^{(1)}_{\psi,\mathfrak{c}_1,X^0_{\mathfrak{c}_2}}$, $\mathfrak{c}_1$, $\mathfrak{c}_2 \in \{\mathfrak{b}, \mathfrak{w}\}$. Subsection 5.2 is devoted to establishing various inequalities, among them the *basic inequalities* in Lemma 5.12, that are indispensable in the arguments in Section 7. In Subsection 5.3, we verify the spectral gap for $\mathbb{L}_\psi$ that is essential in the proof of Theorem 6.3.

5.1. Construction.

Lemma 5.1. *Let f, $\mathcal{C}$, d, Λ, α satisfy the Assumptions. Fix a constant $T > 0$. Then for all $n \in \mathbb{N}$, $X^n \in \mathbf{X}^n(f, \mathcal{C})$, x, $x' \in X^n$, and $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ with $|\Re(\psi)|_{\alpha,(S^2,d)} \leq T$, we have*

$$(5.1) \qquad |1 - \exp(S_n\psi(x) - S_n\psi(x'))| \leq C_{10} |\psi|_{\alpha,(S^2,d)} \, d(f^n(x), f^n(x'))^\alpha,$$

where the constant

$$(5.2) \qquad C_{10} = C_{10}(f, \mathcal{C}, d, \alpha, T) := \frac{2C_0}{1 - \Lambda^{-\alpha}} \exp\left(\frac{C_0 T}{1 - \Lambda^{-\alpha}} \big(\operatorname{diam}_d(S^2)\big)^\alpha\right) > 1$$

depends only on f, $\mathcal{C}$, d, α, and T. Here $C_0 > 1$ is a constant from Lemma 3.13 depending only on f, $\mathcal{C}$, and d.

Proof. Fix $T > 0$, $n \in \mathbb{N}$, $X^n \in \mathbf{X}^n(f, \mathcal{C})$, x, $x' \in X^n$, and $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ with $|\Re(\psi)|_{\alpha,(S^2,d)} \leq T$. By Lemma 3.15, for each $\phi \in C^{0,\alpha}(S^2, d)$,

$$(5.3) \qquad |S_n\phi(x) - S_n\phi(x')| \leq \frac{C_0 |\phi|_{\alpha,(S^2,d)}}{1 - \Lambda^{-\alpha}} d(f^n(x), f^n(x'))^\alpha.$$

Then by (5.3) and the fact that $|1 - e^y| \leq |y|e^{|y|}$ and $|1 - e^{\mathbf{i}y}| \leq |y|$ for $y \in \mathbb{R}$, we get

$$\left|1 - e^{S_n\psi(x) - S_n\psi(x')}\right| \leq \left|1 - e^{S_n\Re(\psi)(x) - S_n\Re(\psi)(x')}\right| + e^{S_n\Re(\psi)(x) - S_n\Re(\psi)(x')}\left|1 - e^{\mathbf{i}S_n\Im(\psi)(x) - \mathbf{i}S_n\Im(\psi)(x')}\right|$$

$$\leq \frac{C_0 |\Re(\psi)|_{\alpha,(S^2,d)}}{1 - \Lambda^{-\alpha}} d(f^n(x), f^n(x'))^\alpha \exp\left(\frac{C_0 T}{1 - \Lambda^{-\alpha}} \big(\operatorname{diam}_d(S^2)\big)^\alpha\right)$$

$$+ \exp\left(\frac{C_0 T}{1 - \Lambda^{-\alpha}} \big(\operatorname{diam}_d(S^2)\big)^\alpha\right) \frac{C_0 |\Im(\psi)|_{\alpha,(S^2,d)}}{1 - \Lambda^{-\alpha}} d(f^n(x), f^n(x'))^\alpha$$

$$\leq C_{10} |\psi|_{\alpha,(S^2,d)} \, d(f^n(x), f^n(x'))^\alpha.$$

Here the constant $C_{10} = C_{10}(f, \mathcal{C}, d, \alpha, T)$ is defined in (5.2). $\qquad\square$

Fix an expanding Thurston map $f \colon S^2 \to S^2$ with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\mathrm{post}\, f \subseteq \mathcal{C}$. Let d be a visual metric for f on S^2, and $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ a complex-valued Hölder continuous function.

Let $n \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and $x \in \mathrm{inte}(X_{\mathfrak{c}}^0)$, where $X_{\mathfrak{b}}^0$ (resp. $X_{\mathfrak{w}}^0$) is the black (resp. white) 0-tile. If $E \subseteq S^2$ is a union of n-tiles in $\mathbf{X}^n(f, \mathcal{C})$, $u \in C((E, d), \mathbb{C})$ a complex-valued continuous function defined on E, and if we define a function $v \in B(S^2, \mathbb{C})$ by

$$(5.4) \qquad v(y) = \begin{cases} u(y) & \text{if } y \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then by Proposition 3.6 (i) and (ii), the Ruelle operator associated to f and ψ (recalled in (3.13)) acting on $B(S^2, \mathbb{C})$ can be written in the following form:

$$(5.5) \qquad \mathcal{L}_\psi^n(v)(x) = \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} u\big((f^n|_{X^n})^{-1}(x)\big) \exp\big(S_n \psi\big((f^n|_{X^n})^{-1}(x)\big)\big).$$

Note that by default, a summation over an empty set is equal to 0. We will always use this convention in this paper. Inspired by (5.5), we give the following definition.

Definition 5.2. Let $f \colon S^2 \to S^2$ be an expanding Thurston map, $\mathcal{C} \subseteq S^2$ a Jordan curve containing $\mathrm{post}\, f$, and $\psi \in C(S^2, \mathbb{C})$ a complex-valued continuous function. Let $n \in \mathbb{N}_0$, and $E \subseteq S^2$ be a union of n-tiles in $\mathbf{X}^n(f, \mathcal{C})$. We define a map $\mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)} \colon C(E, \mathbb{C}) \to C(X_{\mathfrak{c}}^0, \mathbb{C})$, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, by

$$(5.6) \qquad \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u)(y) = \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} u\big((f^n|_{X^n})^{-1}(y)\big) \exp\big(S_n \psi\big((f^n|_{X^n})^{-1}(y)\big)\big),$$

for each complex-valued continuous function $u \in C(E, \mathbb{C})$ defined on E, and each point $y \in X_{\mathfrak{c}}^0$. When $E = X_{\mathfrak{c}'}^0$ for some $\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$, we often write

$$\mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(n)} := \mathcal{L}_{\psi, \mathfrak{c}, X_{\mathfrak{c}'}^0}^{(n)}.$$

Note that $\mathcal{L}_{\psi, \mathfrak{c}, E}^{(0)}(u) = \begin{cases} u & \text{if } X_{\mathfrak{c}}^0 \subseteq E \\ 0 & \text{otherwise} \end{cases}$, for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, whenever the expression on the left-hand side of the equation makes sense.

Lemma 5.3. Let f, $\mathcal{C}$, d, α satisfy the Assumptions. Let $\psi \in C(S^2, \mathbb{C})$ be a complex-valued continuous function. Fix numbers $n, m \in \mathbb{N}_0$ and a union $E \subseteq S^2$ of an arbitrary collection of n-tiles in $\mathbf{X}^n(f, \mathcal{C})$ (i.e., $E = \bigcup \{X^n \in \mathbf{X}^n(f, \mathcal{C}) : X^n \subseteq E\}$). Then for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each $u \in C(E, \mathbb{C})$, we have $\mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u) \in C(X_{\mathfrak{c}}^0, \mathbb{C})$, and

$$(5.7) \qquad \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n+m)}(u) = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(m)}\big(\mathcal{L}_{\psi, \mathfrak{c}', E}^{(n)}(u)\big).$$

If, in addition, $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ and $u \in C^{0,\alpha}((E, d), \mathbb{C})$ are Hölder continuous, then

$$(5.8) \qquad \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u) \in C^{0,\alpha}((X_{\mathfrak{c}}^0, d), \mathbb{C}).$$

Remark 5.4. In the above context, $\mathcal{L}_\psi^n(v) \in B(S^2, \mathbb{C})$ may not be continuous on S^2 if $E \neq S^2$, where v is defined in (5.4) extending u to S^2. If $E = S^2$, then it follows immediately from (5.6) that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u) = \big(\mathcal{L}_\psi^n(u)\big)\big|_{X_{\mathfrak{c}}^0}$. Hence by (5.8) and the linear local connectivity of (S^2, d), it can be shown that $\mathcal{L}_\psi^n(C^{0,\alpha}((S^2, d), \mathbb{C})) \subseteq C^{0,\alpha}((S^2, d), \mathbb{C})$. We will not use this fact in this paper.

Proof. Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $u \in C(E, \mathbb{C})$.

The cases of Lemma 5.3 when either $m = 0$ or $n = 0$ follow immediately from Definition 5.2. Thus, without loss of generality, we can assume $m, n \in \mathbb{N}$.

The continuity of $\mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u)$ follows trivially from (5.6) and Proposition 3.6 (i).

By (5.6), Proposition 3.6 (i) and (ii), and the fact that $f^{-m}(x) \cap \mathcal{C} \neq \emptyset$, we get

$$\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(m)}\left(\mathcal{L}_{\psi, \mathfrak{c}', E}^{(n)}(u)\right)(x) = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{y \in f^{-m}(x) \cap X_{\mathfrak{c}'}^0} e^{S_m \psi(y)} \sum_{z \in f^{-n}(y) \cap E} e^{S_n \psi(z)} u(z)$$

$$= \sum_{y \in f^{-m}(x)} \sum_{z \in f^{-n}(y) \cap E} e^{S_m \psi(y) + S_n \psi(z)} u(z) = \sum_{z \in f^{-(n+m)}(x) \cap E} e^{S_{n+m} \psi(z)} u(z) = \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n+m)}(u)(x).$$

Identity (5.7) is now established by the continuity of two sides of the equation above.

Finally, to prove (5.8), we first fix two distinct points $x, x' \in X_{\mathfrak{c}}^0$. Denote $y_{X^n} := (f^n|_{X^n})^{-1}(x)$ and $y'_{X^n} := (f^n|_{X^n})^{-1}(x')$ for each $X^n \in \mathbf{X}_{\mathfrak{c}}^n$.

By Lemmas 3.13, 3.15, and 5.1, we have

$$\frac{\left|\mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u)(x) - \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)}(u)(x')\right|}{d(x, x')^\alpha} \leq \frac{1}{d(x, x')^\alpha} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left|e^{S_n \psi(y_{X^n})} u(y_{X^n}) - e^{S_n \psi(y'_{X^n})} u(y'_{X^n})\right|$$

$$\leq \frac{1}{d(x, x')^\alpha} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left(\left|e^{S_n \psi(y_{X^n})}\right| |u(y_{X^n}) - u(y'_{X^n})| + \left|e^{S_n \psi(y_{X^n})} - e^{S_n \psi(y'_{X^n})}\right| |u(y'_{X^n})|\right)$$

$$\leq \frac{1}{d(x, x')^\alpha} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} e^{S_n \Re(\psi)(y_{X^n})} |u|_{\alpha, (E, d)} \, d(y_{X^n}, y'_{X^n})^\alpha$$

$$+ \frac{1}{d(x, x')^\alpha} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left|1 - e^{S_n \psi(y_{X^n}) - S_n \psi(y'_{X^n})}\right| e^{S_n \Re(\psi)(y'_{X^n})} |u(y'_{X^n})|$$

$$\leq |u|_{\alpha, (E, d)} C_0^\alpha \Lambda^{-\alpha n} \sum_{X^n \in \mathbf{X}^n} e^{S_n \Re(\psi)(y_{X^n})} + C_{10} |\psi|_{\alpha, (S^2, d)} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} e^{S_n \Re(\psi)(y'_{X^n})} |u(y'_{X^n})|$$

$$\leq C_0 \Lambda^{-\alpha n} |u|_{\alpha, (E, d)} \left\|\mathcal{L}_{\Re(\psi)}^n(\mathbb{1}_{S^2})\right\|_{C^0(S^2)} + C_{10} |\psi|_{\alpha, (S^2, d)} \left\|\mathcal{L}_{\Re(\psi), \mathfrak{c}, E}^{(n)}(|u|)\right\|_{C^0(S^2)},$$

where $C_0 > 1$ is a constant depending only on f, $\mathcal{C}$, and d from Lemma 3.13, and $C_{10} > 1$ is a constant depending only on f, $\mathcal{C}$, d, α, and ψ from Lemma 5.1. Therefore (5.8) holds. $\qquad\square$

Definition 5.5 (Split Ruelle operators). Let $f : S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$. Let d be a visual metric for f on S^2, and $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ a complex-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Let $X_{\mathfrak{b}}^0$, $X_{\mathfrak{w}}^0 \in \mathbf{X}^0(f, \mathcal{C})$ be the black 0-tile and the while 0-tile, respectively. The *split Ruelle operator* $\mathbb{L}_\psi : C(X_{\mathfrak{b}}^0, \mathbb{C}) \times C(X_{\mathfrak{w}}^0, \mathbb{C}) \to C(X_{\mathfrak{b}}^0, \mathbb{C}) \times C(X_{\mathfrak{w}}^0, \mathbb{C})$ on the product space $C(X_{\mathfrak{b}}^0, \mathbb{C}) \times C(X_{\mathfrak{w}}^0, \mathbb{C})$ is given by

$$\mathbb{L}_\psi(u_{\mathfrak{b}}, u_{\mathfrak{w}}) = \left(\mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{b}}^{(1)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{w}}^{(1)}(u_{\mathfrak{w}}), \, \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{b}}^{(1)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{w}}^{(1)}(u_{\mathfrak{w}})\right)$$

for $u_{\mathfrak{b}} \in C(X_{\mathfrak{b}}^0, \mathbb{C})$ and $u_{\mathfrak{w}} \in C(X_{\mathfrak{w}}^0, \mathbb{C})$.

Note that by Lemma 5.3, the operator $\mathbb{L}_\psi$ is well-defined. Moreover, by (5.8) in Lemma 5.3, we have

$$(5.9) \qquad \mathbb{L}_\psi\left(C^{0,\alpha}((X_{\mathfrak{b}}^0, d), \mathbb{C}) \times C^{0,\alpha}((X_{\mathfrak{w}}^0, d), \mathbb{C})\right) \subseteq C^{0,\alpha}((X_{\mathfrak{b}}^0, d), \mathbb{C}) \times C^{0,\alpha}((X_{\mathfrak{w}}^0, d), \mathbb{C}).$$

Note that it follows immediately from Definition 5.2 that $\mathbb{L}_\psi$ is a linear operator on the Banach space $C^{0,\alpha}((X_{\mathfrak{b}}^0, d), \mathbb{C}) \times C^{0,\alpha}((X_{\mathfrak{w}}^0, d), \mathbb{C})$ equipped with a norm given by

$$\|(u_{\mathfrak{b}}, u_{\mathfrak{w}})\| := \max\left\{\|u_{\mathfrak{b}}\|_{C^{0,\alpha}(X_{\mathfrak{b}}^0, d)}^{[b]}, \, \|u_{\mathfrak{w}}\|_{C^{0,\alpha}(X_{\mathfrak{w}}^0, d)}^{[b]}\right\},$$

for each $b \in \mathbb{R} \setminus \{0\}$. See (2.6) for the definition of the normalized Hölder norm $\|u\|_{C^{0,\alpha}(E, d)}^{[b]}$.

For each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, we define the projection $\pi_{\mathfrak{c}} : C(X_{\mathfrak{b}}^0, \mathbb{C}) \times C(X_{\mathfrak{w}}^0, \mathbb{C}) \to C(X_{\mathfrak{c}}^0, \mathbb{C})$ by

$$(5.10) \qquad\qquad \pi_{\mathfrak{c}}(u_{\mathfrak{b}}, u_{\mathfrak{w}}) = u_{\mathfrak{c}}, \qquad \text{for } (u_{\mathfrak{b}}, u_{\mathfrak{w}}) \in C(X_{\mathfrak{b}}^0, \mathbb{C}) \times C(X_{\mathfrak{w}}^0, \mathbb{C}).$$

Definition 5.6. Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$. Let d be a visual metric for f on S^2, and $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ a complex-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. For all $n \in \mathbb{N}_0$ and $b \in \mathbb{R} \setminus \{0\}$, we write the operator norm

$$\left\vert\!\left\vert\!\left\vert \mathbb{L}_\psi^n \right\vert\!\right\vert\!\right\vert_\alpha^{[b]} := \sup\Bigg\{ \left\| \pi_{\mathfrak{c}}\big(\mathbb{L}_\psi^n(u_{\mathfrak{b}}, u_{\mathfrak{w}}) \big) \right\|_{C^{0,\alpha}(X_{\mathfrak{c}}^0, d)}^{[b]} :$$

(5.11)
$$\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\},\ u_{\mathfrak{b}} \in C^{0,\alpha}((X_{\mathfrak{b}}^0, d), \mathbb{C}),\ u_{\mathfrak{w}} \in C^{0,\alpha}((X_{\mathfrak{w}}^0, d), \mathbb{C})$$
$$\text{with } \|u_{\mathfrak{b}}\|_{C^{0,\alpha}(X_{\mathfrak{b}}^0, d)}^{[b]} \le 1 \text{ and } \|u_{\mathfrak{w}}\|_{C^{0,\alpha}(X_{\mathfrak{w}}^0, d)}^{[b]} \le 1 \Bigg\}.$$

We write $\left\vert\!\left\vert\!\left\vert \mathbb{L}_\psi^n \right\vert\!\right\vert\!\right\vert_\alpha := \left\vert\!\left\vert\!\left\vert \mathbb{L}_\psi^n \right\vert\!\right\vert\!\right\vert_\alpha^{[1]}$.

Lemma 5.7. *Let f, $\mathcal{C}$, d, α, ψ satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $X_{\mathfrak{b}}^0$, $X_{\mathfrak{w}}^0 \in \mathbf{X}^0(f, \mathcal{C})$ be the black 0-tile and the white 0-tile, respectively. Then for all $n \in \mathbb{N}_0$, $u_{\mathfrak{b}} \in C(X_{\mathfrak{b}}^0, \mathbb{C})$, and $u_{\mathfrak{w}} \in C(X_{\mathfrak{w}}^0, \mathbb{C})$,*

(5.12)
$$\mathbb{L}_\psi^n(u_{\mathfrak{b}}, u_{\mathfrak{w}}) = \Big(\mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{b}}^{(n)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{w}}^{(n)}(u_{\mathfrak{w}}),\ \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{b}}^{(n)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{w}}^{(n)}(u_{\mathfrak{w}}) \Big).$$

Consequently,

$$\left\vert\!\left\vert\!\left\vert \mathbb{L}_\psi^n \right\vert\!\right\vert\!\right\vert_\alpha^{[b]} = \sup\Bigg\{ \frac{\left\| \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{b}}^{(n)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{w}}^{(n)}(u_{\mathfrak{w}}) \right\|_{C^{0,\alpha}(X_{\mathfrak{c}}^0, d)}^{[b]}}{\max\big\{ \|u_{\mathfrak{b}}\|_{C^{0,\alpha}(X_{\mathfrak{b}}^0, d)}^{[b]},\ \|u_{\mathfrak{w}}\|_{C^{0,\alpha}(X_{\mathfrak{w}}^0, d)}^{[b]} \big\}} :$$

(5.13)
$$\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\},\ u_{\mathfrak{b}} \in C^{0,\alpha}((X_{\mathfrak{b}}^0, d), \mathbb{C}),\ u_{\mathfrak{w}} \in C^{0,\alpha}((X_{\mathfrak{w}}^0, d), \mathbb{C})$$
$$\text{with } \|u_{\mathfrak{b}}\|_{C^0(X_{\mathfrak{b}}^0)} \|u_{\mathfrak{w}}\|_{C^0(X_{\mathfrak{w}}^0)} \ne 0 \Bigg\}.$$

Proof. We prove (5.12) by induction. The case where $n = 0$ and the case where $n = 1$ both hold by definition. Assume now (5.12) holds when $n = m$ for some $m \in \mathbb{N}$. Then by (5.7) in Lemma 5.3, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, we have

$$\pi_{\mathfrak{c}}\big(\mathbb{L}_\psi^{m+1}(u_{\mathfrak{b}}, u_{\mathfrak{w}}) \big) = \pi_{\mathfrak{c}}\Big(\mathbb{L}_\psi\Big(\mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{b}}^{(m)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{b}, \mathfrak{w}}^{(m)}(u_{\mathfrak{w}}),\ \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{b}}^{(m)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{w}, \mathfrak{w}}^{(m)}(u_{\mathfrak{w}}) \Big) \Big)$$

$$= \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(1)}\Big(\mathcal{L}_{\psi, \mathfrak{c}', \mathfrak{b}}^{(m)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{c}', \mathfrak{w}}^{(m)}(u_{\mathfrak{w}}) \Big)$$

$$= \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(1)}\Big(\mathcal{L}_{\psi, \mathfrak{c}', \mathfrak{b}}^{(m)}(u_{\mathfrak{b}}) \Big) + \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{c}'}^{(1)}\Big(\mathcal{L}_{\psi, \mathfrak{c}', \mathfrak{w}}^{(m)}(u_{\mathfrak{w}}) \Big)$$

$$= \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{b}}^{(m+1)}(u_{\mathfrak{b}}) + \mathcal{L}_{\psi, \mathfrak{c}, \mathfrak{w}}^{(m+1)}(u_{\mathfrak{w}}),$$

for $u_{\mathfrak{b}} \in C(X_{\mathfrak{b}}^0, \mathbb{C})$ and $u_{\mathfrak{w}} \in C(X_{\mathfrak{w}}^0, \mathbb{C})$. This completes the inductive step, establishing (5.12).

Identity (5.13) follows immediately from Definition 5.6 and (5.12). $\qquad\qquad\square$

5.2. **Basic inequalities.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\psi \in C^{0,\alpha}((S^2, d), \mathbb{C})$ be a complex-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. We define

(5.14)
$$\widetilde{\psi} := \widetilde{\Re(\psi)} + \mathbf{i}\Im(\psi) = \psi - P(f, \Re(\psi)) + \log u_{\Re(\psi)} - \log\big(u_{\Re(\psi)} \circ f \big),$$

where $u_{\Re(\psi)}$ is the continuous function given by Theorem 3.17 with $\phi := \Re(\psi)$. Then for each $u \in C(S^2, \mathbb{C})$ and each $x \in S^2$, we have

$$\mathcal{L}_{\widetilde{\psi}}(u)(x) = \sum_{y \in f^{-1}(x)} \deg_f(y) u(y) e^{\psi(y) - P(f, \Re(\psi)) + \log u_{\Re(\psi)}(y) - \log(u_{\Re(\psi)}(f(y)))}$$

(5.15)
$$= \frac{\exp(-P(f, \Re(\psi)))}{u_{\Re(\psi)}(x)} \sum_{y \in f^{-1}(x)} \deg_f(y) u(y) u_{\Re(\psi)}(y) \exp(\psi(y))$$

$$= \frac{\exp(-P(f, \Re(\psi)))}{u_{\Re(\psi)}(x)} \mathcal{L}_\psi\big(u_{\Re(\psi)} u \big)(x).$$

Given a Jordan curve $\mathcal{C} \subseteq S^2$ with post $f \subseteq \mathcal{C}$, then for each $n \in \mathbb{N}_0$, each union E of n-tiles in $\mathbf{X}^n(f, \mathcal{C})$, each $v \in C(E, \mathbb{C})$, each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and each $z \in X_{\mathfrak{c}}^0$,

$$
\mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, E}^{(n)}(v)(z) = \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n(f, \mathcal{C}) \\ X^n \subseteq E}} \left(v e^{S_n(\psi - P(f, \Re(\psi)) + \log u_{\Re(\psi)} - \log(u_{\Re(\psi)} \circ f))} \right) \left((f^n|_{X^n})^{-1}(z) \right)
$$

$$
\text{(5.16)} \qquad = \frac{\exp(-nP(f, \Re(\psi)))}{u_{\Re(\psi)}(z)} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n(f, \mathcal{C}) \\ X^n \subseteq E}} \left(v u_{\Re(\psi)} \exp(S_n \psi) \right) \left((f^n|_{X^n})^{-1}(z) \right)
$$

$$
= \frac{\exp(-nP(f, \Re(\psi)))}{u_{\Re(\psi)}(z)} \mathcal{L}_{\psi, \mathfrak{c}, E}^{(n)} \left(u_{\Re(\psi)} v \right)(z).
$$

Definition 5.8 (Cones). Let $f \colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Fix a constant $\alpha \in (0, 1]$. For each subset $E \subseteq S^2$ and each constant $B \in \mathbb{R}$ with $B > 0$, we define the *B-cone inside $C^{0,\alpha}(E, d)$* as

$$
K_B(E, d) = \left\{ u \in C^{0,\alpha}(E, d) : u(x) > 0, \ |u(x) - u(y)| \leq B(u(x) + u(y)) d(x, y)^\alpha \text{ for } x, y \in E \right\}.
$$

It is essential to define the B-cones inside $C^{0,\alpha}(E, d)$ in the form above in order to establish the following lemma, which will be used in the proof of Proposition 7.13.

Lemma 5.9. *Let (X, d) be a metric space and $\alpha \in (0, 1]$. Then for each $B > 0$ and each $u \in K_B(X, d)$, we have $u^2 \in K_{2B}(X, d)$.*

Proof. Fix arbitrary $B > 0$ and $u \in K_B(X, d)$. For any $x, y \in X$,

$$
\left| u^2(x) - u^2(y) \right| = |u(x) + u(y)| |u(x) - u(y)| \leq B |u(x) + u(y)|^2 d(x, y)^\alpha \leq 2B \left(u^2(x) + u^2(y) \right) d(x, y)^\alpha.
$$

Therefore $u^2 \in K_{2B}(X, d)$. $\qquad\qquad\square$

Lemma 5.10. *Let f, d, α, ψ satisfy the Assumptions. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent α. Then the operator norm of $\mathcal{L}_{\widetilde{\phi}}$ acting on $C(S^2)$ is given by $\left\| \mathcal{L}_{\widetilde{\phi}} \right\|_{C^0(S^2)} = 1$. In addition, $\mathcal{L}_{\widetilde{\phi}}(\mathbb{1}_{S^2}) = \mathbb{1}_{S^2}$.*

Moreover, consider a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then for all $n \in \mathbb{N}_0$, $\mathfrak{c}, \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_{\mathfrak{b}} \in C(X_{\mathfrak{b}}^0, \mathbb{C})$, and $u_{\mathfrak{w}} \in C(X_{\mathfrak{w}}^0, \mathbb{C})$, we have

$$
\text{(5.17)} \qquad\qquad \left\| \mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, \mathfrak{c}'}^{(n)}(u_{\mathfrak{c}'}) \right\|_{C^0(X_{\mathfrak{c}}^0)} \leq \|u_{\mathfrak{c}'}\|_{C^0(X_{\mathfrak{c}'}^0)} \qquad\text{and}
$$

$$
\text{(5.18)} \qquad \left\| \mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, \mathfrak{b}}^{(n)}(u_{\mathfrak{b}}) + \mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, \mathfrak{w}}^{(n)}(u_{\mathfrak{w}}) \right\|_{C^0(X_{\mathfrak{c}}^0)} \leq \max\left\{ \|u_{\mathfrak{b}}\|_{C^0(X_{\mathfrak{b}}^0)}, \ \|u_{\mathfrak{w}}\|_{C^0(X_{\mathfrak{w}}^0)} \right\}.
$$

Proof. The fact that $\left\| \mathcal{L}_{\widetilde{\phi}} \right\|_{C^0(S^2)} = 1$ and $\mathcal{L}_{\widetilde{\phi}}(\mathbb{1}_{S^2}) = \mathbb{1}_{S^2}$ is established in [Li17, Lemma 5.25].

To prove (5.18), we first fix arbitrary $n \in \mathbb{N}_0$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_{\mathfrak{b}} \in C(X_{\mathfrak{b}}^0)$, and $u_{\mathfrak{w}} \in C(X_{\mathfrak{w}}^0)$. Denote $M := \max\left\{ \|u_{\mathfrak{b}}\|_{C^0(X_{\mathfrak{b}}^0)}, \ \|u_{\mathfrak{w}}\|_{C^0(X_{\mathfrak{w}}^0)} \right\}$. Then by Definition 5.2, (5.14), and the fact that $\mathcal{L}_{\widetilde{\Re(\psi)}}(\mathbb{1}_{S^2}) = \mathbb{1}_{S^2}$, for each $y \in \operatorname{inte}(X_{\mathfrak{c}}^0)$,

$$
\left\| \mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, \mathfrak{b}}^{(n)}(u_{\mathfrak{b}}) + \mathcal{L}_{\widetilde{\psi}, \mathfrak{c}, \mathfrak{w}}^{(n)}(u_{\mathfrak{w}}) \right\|_{C^0(X_{\mathfrak{c}}^0)} \leq M \sum_{X^n \in \mathbf{X}_{\mathfrak{c}}^n} \left| \exp\left(S_n \widetilde{\psi} \left((f^n|_{X^n})^{-1}(y) \right) \right) \right|
$$

$$
= M \mathcal{L}_{\widetilde{\Re(\psi)}}^n(\mathbb{1}_{S^2})(y) = M.
$$

This establishes (5.18). Finally, (5.17) follows immediately from (5.18) and Definition 5.2 by setting one of the functions $u_{\mathfrak{b}}$ and $u_{\mathfrak{w}}$ to be 0. $\qquad\square$

Lemma 5.11. *Let f, $\mathcal{C}$, d, L, α, Λ satisfy the Assumptions. Then there exist constants $C_{13} > 1$ and $C_{14} > 0$ depending only on f, $\mathcal{C}$, d, and α such that the following is satisfied:*

For all K, M, T, $a \in \mathbb{R}$ with $K > 0$, $M > 0$, $T > 0$, and $|a| \leq T$, and all real-valued Hölder continuous function $\phi \in C^{0,\alpha}(S^2, d)$ with $|\phi|_{\alpha, (S^2, d)} \leq K$ and $\|\phi\|_{C^0(S^2)} \leq M$, we have

$$(5.19) \qquad \big\|\widetilde{a\phi}\big\|_{C^0(S^2)} \leq C_{13}(K + M)T + |\log(\deg f)|,$$

$$(5.20) \qquad \big|\widetilde{a\phi}\big|_{\alpha, (S^2, d)} \leq C_{13}KTe^{C_{14}KT},$$

$$(5.21) \qquad \|u_{a\phi}\|_{C^{0,\alpha}(S^2, d)} \leq \big(4TKC_0(1 - \Lambda^{-\alpha})^{-1}L + 1\big)e^{2C_{15}},$$

$$(5.22) \qquad \exp(-C_{15}) \leq u_{a\phi}(x) \leq \exp(C_{15})$$

for $x \in S^2$, where the constant $C_0 > 1$ depending only on f, d, and $\mathcal{C}$ is from Lemma 3.13, and the constant

$$(5.23) \qquad C_{15} = C_{15}(f, \mathcal{C}, d, \alpha, T, K) := 4TKC_0(1 - \Lambda^{-\alpha})^{-1}L\big(\mathrm{diam}_d(S^2)\big)^{\alpha} > 0$$

depends only on f, $\mathcal{C}$, d, α, T, and K.

Proof. Fix K, M, T, a, ϕ that satisfy the conditions in this lemma.

Recall $\widetilde{a\phi} = a\phi - P(f, a\phi) + \log u_{a\phi} - \log(u_{a\phi} \circ f)$, where the function $u_{a\phi}$ is defined as u_ϕ in Theorem 3.17.

By Theorem 3.17 and (3.12) in Lemma 3.15, we immediately get (5.22).

By Lemma 3.25 in [LZhe23a], (3.4), and (3.5), for each $x \in S^2$,

$$P(f, a\phi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} \deg_{f^n}(y)e^{aS_n\phi(y)} \leq \lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} \deg_{f^n}(y)e^{nTM}$$

$$= TM + \lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} \deg_{f^n}(y) = TM + \log(\deg f).$$

Similarly, $P(f, a\phi) \geq -TM + \log(\deg f)$. So $|P(f, a\phi)| \leq TM + |\log(\deg f)|$.

Thus by combining the above with (5.22) and (5.23), we get

$$(5.24) \qquad \big\|\widetilde{a\phi}\big\|_{C^0(S^2)} \leq TM + TM + |\log(\deg f)| + 2C_{15} \leq C_{16}T(K + M) + |\log(\deg f)|,$$

where $C_{16} := 2 + 8\frac{C_0}{1-\Lambda^{-\alpha}}L\big(\mathrm{diam}_d(S^2)\big)^{\alpha}$ is a constant depending only on f, $\mathcal{C}$, d, and α.

Note f is Lipschitz with respect to d (see [Li18, Lemma 3.12]). Thus by (5.22) and the fact that $|\log t_1 - \log t_2| \leq \frac{|t_1 - t_2|}{\min\{t_1, t_2\}}$ for all $t_1, t_2 > 0$, we get

$$\big|\widetilde{a\phi}\big|_{\alpha, (S^2, d)} \leq |a\phi|_{\alpha, (S^2, d)} + |\log u_{a\phi}|_{\alpha, (S^2, d)} + |\log(u_{a\phi} \circ f)|_{\alpha, (S^2, d)}$$

$$(5.25) \qquad\qquad \leq TK + e^{C_{15}}(1 + \mathrm{LIP}_d(f))\,|u_{a\phi}|_{\alpha, (S^2, d)}\,.$$

Here $\mathrm{LIP}_d(f)$ denotes the Lipschitz constant of f with respect to the visual metric d (see (2.8)).

By Theorem 3.17, (3.17) in Lemma 3.16, (3.12) in Lemma 3.15, (5.23), and the fact that $|1 - e^{-t}| \leq t$ for $t > 0$, we get

$$|u_{a\phi}(x) - u_{a\phi}(y)| = \left|\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Big(\mathcal{L}_{\widetilde{a\phi}}^j\big(\mathbb{1}_{S^2}\big)(x) - \mathcal{L}_{\widetilde{a\phi}}^j\big(\mathbb{1}_{S^2}\big)(y)\Big)\right|$$

$$\leq \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left|\mathcal{L}_{\widetilde{a\phi}}^j\big(\mathbb{1}_{S^2}\big)(x) - \mathcal{L}_{\widetilde{a\phi}}^j\big(\mathbb{1}_{S^2}\big)(y)\right|$$

$$\leq e^{2C_{15}}\left(1 - \exp\left(-4\frac{TKC_0}{1 - \Lambda^{-\alpha}}Ld(x,y)^{\alpha}\right)\right) \leq e^{2C_{15}}\frac{4TKC_0}{1 - \Lambda^{-\alpha}}Ld(x,y)^{\alpha},$$

for all $x, y \in S^2$. So

$$(5.26) \qquad |u_{a\phi}|_{\alpha, (S^2, d)} \leq 4\frac{TKC_0}{1 - \Lambda^{-\alpha}}Le^{2C_{15}}.$$

Thus by (5.25), (5.26), and (5.23), we get

$$\big|\widetilde{a\phi}\big|_{\alpha, (S^2, d)} \leq TKC_{13}e^{C_{14}TK},$$

where the constants $C_{13} := \max\{C_{16}, 1 + 4C_0(1 - \Lambda^{-\alpha})^{-1}L(1 + \operatorname{LIP}_d(f))\}$ and $C_{14} := 12C_0(1 - \Lambda^{-\alpha})^{-1}L(\operatorname{diam}_d(S^2))^\alpha$ depend only on f, $\mathcal{C}$, d, and α. Since $C_{13} \geq C_{16}$, (5.19) follows from (5.24).

Finally, (5.21) follows from (5.22) and (5.26). □

Lemma 5.12 (Basic inequalities). *Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. Then there exists a constant $A_0 = A_0\big(f, \mathcal{C}, d, |\phi|_{\alpha,\,(S^2,d)}, \alpha\big) \geq 2C_0 > 2$ depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,\,(S^2,d)}$, and α such that A_0 increases as $|\phi|_{\alpha,\,(S^2,d)}$ increases, and that for all $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, x, $x' \in X^0$, $n \in \mathbb{N}$, union $E \subseteq S^2$ of n-tiles in $\mathbf{X}^n(f, \mathcal{C})$, $B \in \mathbb{R}$ with $B > 0$, and $a, b \in \mathbb{R}$ with $|a| \leq 2s_0$ and $|b| \in \{0\} \cup [1, +\infty)$, the following statements are satisfied:*

(i) *For each $u \in K_B(E, d)$, we have*

$$(5.27) \qquad \frac{|\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x) - \mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x')|}{\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x) + \mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x')} \leq A_0\left(\frac{B}{\Lambda^{\alpha n}} + \frac{|\widetilde{a\phi}|_{\alpha,\,(E,d)}}{1 - \Lambda^{-\alpha}}\right)d(x, x')^\alpha.$$

(ii) *Denote $s := a + \mathbf{i}b$. Fix an arbitrary $v \in C^{0,\alpha}((E, d), \mathbb{C})$. Then*

$$(5.28) \qquad \left|\mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},E}(v)(x) - \mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},E}(v)(x')\right| \leq \left(C_0 \frac{|v|_{\alpha,\,(E,d)}}{\Lambda^{\alpha n}} + A_0 \max\{1, |b|\}\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(|v|)(x)\right)d(x, x')^\alpha,$$

where $C_0 > 1$ is a constant from Lemma 3.13 depending only on f, d, and $\mathcal{C}$.

If, in addition, there exists a non-negative real-valued Hölder continuous function $h \in C^{0,\alpha}(E, d)$ such that

$$|v(y) - v(y')| \leq B(h(y) + h(y'))d(y, y')^\alpha$$

when y, $y' \in E$, then

$$(5.29) \qquad \left|\mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},E}(v)(x) - \mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},E}(v)(x')\right|$$
$$\leq A_0\left(\frac{B}{\Lambda^{\alpha n}}\left(\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(h)(x) + \mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(h)(x')\right) + \max\{1, |b|\}\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(|v|)(x)\right)d(x, x')^\alpha.$$

Proof. Fix $\mathfrak{c}$, n, E, B, a, and b as in the statement of Lemma 5.12.

(i) Note that by Lemma 5.11,

$$(5.30) \qquad \sup\{|\widetilde{\tau\phi}|_{\alpha,\,(S^2,d)} : \tau \in \mathbb{R}, |\tau| \leq 2s_0 + 1\} \leq T_0,$$

where the constant

$$(5.31) \qquad T_0 = T_0\big(f, \mathcal{C}, d, |\phi|_{\alpha,\,(S^2,d)}, \alpha\big) := (2s_0 + 1)C_{13}|\phi|_{\alpha,\,(S^2,d)}\exp\big((2s_0 + 1)C_{14}|\phi|_{\alpha,\,(S^2,d)}\big) > 0$$

depends only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,\,(S^2,d)}$, and α. Here $C_{13} > 1$ and $C_{14} > 0$ are constants from Lemma 5.11 depending only on f, $\mathcal{C}$, d, and α.

Fix $u \in K_B(E, d)$ and x, $x' \in X^0_{\mathfrak{c}}$. For each $X^n \in \mathbf{X}^n_{\mathfrak{c}}$, denote $y_{X^n} := (f^n|_{X^n})^{-1}(x)$ and $y'_{X^n} := (f^n|_{X^n})^{-1}(x')$.

Then by Definition 5.8,

$$\left|\mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x) - \mathcal{L}^{(n)}_{\widetilde{a\phi},\mathfrak{c},E}(u)(x')\right| \leq \sum_{\substack{X^n \in \mathbf{X}^n_{\mathfrak{c}} \\ X^n \subseteq E}}\left|u(y_{X^n})e^{S_n\widetilde{a\phi}(y_{X^n})} - u(y'_{X^n})e^{S_n\widetilde{a\phi}(y'_{X^n})}\right|$$

$$\leq \sum_{\substack{X^n \in \mathbf{X}^n_{\mathfrak{c}} \\ X^n \subseteq E}}\left(\left|u(y_{X^n}) - u(y'_{X^n})\right|e^{S_n\widetilde{a\phi}(y'_{X^n})} + u(y_{X^n})\left|e^{S_n\widetilde{a\phi}(y_{X^n})} - e^{S_n\widetilde{a\phi}(y'_{X^n})}\right|\right)$$

$$\leq \sum_{\substack{X^n \in \mathbf{X}^n_{\mathfrak{c}} \\ X^n \subseteq E}}B\left(u(y_{X^n})e^{S_n\widetilde{a\phi}(y_{X^n})}e^{\left|S_n\widetilde{a\phi}(y'_{X^n}) - S_n\widetilde{a\phi}(y_{X^n})\right|} + u(y'_{X^n})e^{S_n\widetilde{a\phi}(y'_{X^n})}\right)d(y_{X^n}, y'_{X^n})^\alpha$$

$$+ \sum_{\substack{X^n \in \mathbf{X}^n_{\mathfrak{c}} \\ X^n \subseteq E}}u(y_{X^n})\left|1 - e^{S_n\widetilde{a\phi}(y'_{X^n}) - S_n\widetilde{a\phi}(y_{X^n})}\right|e^{S_n\widetilde{a\phi}(y_{X^n})}.$$

Combining the above with Lemmas 3.15, 3.13, 5.1, (5.30), and (5.31), we get

$$\frac{\left|\mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(u)(x) - \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(u)(x')\right|}{\mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(u)(x) + \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(u)(x')}$$

$$\leq B \exp\left(\frac{\left|\widetilde{a\phi}\right|_{\alpha,(S^2,d)} C_0 (\mathrm{diam}_d(S^2))^\alpha}{1 - \Lambda^{-\alpha}}\right) \frac{d(x,x')^\alpha C_0^\alpha}{\Lambda^{\alpha n}} + C_{10} \left|\widetilde{a\phi}\right|_{\alpha,(S^2,d)} d(x,x')^\alpha$$

$$\leq A_1 \left(\frac{B}{\Lambda^{\alpha n}} + \frac{\left|\widetilde{a\phi}\right|_{\alpha,(S^2,d)}}{1 - \Lambda^{-\alpha}}\right) d(x,x')^\alpha,$$

where

$$(5.32) \qquad C_{10} = C_{10}(f, \mathcal{C}, d, \alpha, T_0) = \frac{2C_0}{1 - \Lambda^{-\alpha}} \exp\left(\frac{C_0 T_0}{1 - \Lambda^{-\alpha}} (\mathrm{diam}_d(S^2))^\alpha\right)$$

is a constant from Lemma 5.1, and

$$(5.33) \qquad\qquad A_1 := (1 - \Lambda^{-\alpha}) C_{10}(f, \mathcal{C}, d, \alpha, T_0).$$

Both of these constants only depend on f, $\mathcal{C}$, d, $|\phi|_{\alpha,(S^2,d)}$ and α.

Define a constant

$$(5.34) \qquad A_0 = A_0\left(f, \mathcal{C}, d, |\phi|_{\alpha,(S^2,d)}, \alpha\right) := \frac{(1 + 2T_0)A_1}{1 - \Lambda^{-\alpha}} = (1 + 2T_0) C_{10}\left(f, \mathcal{C}, d, \alpha, T_0\right) > 2$$

depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,(S^2,d)}$, and α. By (5.34), (5.31), and (5.32), we see that A_0 increases as $|\phi|_{\alpha,(S^2,d)}$ increases. Now (5.27) follows from the fact that $A_0 \geq A_1$.

(ii) Fix x, $x' \in X_{\mathfrak{c}}^0$. For each $X^n \in \mathbf{X}_{\mathfrak{c}}^n$, denote $y_{X^n} := (f^n|_{X^n})^{-1}(x)$ and $y'_{X^n} := (f^n|_{X^n})^{-1}(x')$. Note that by (3.18) and (5.30), we have

$$(5.35) \qquad \left|\widetilde{s\phi}\right|_{\alpha,(S^2,d)} \leq \left|\widetilde{a\phi}\right|_{\alpha,(S^2,d)} + |b\phi|_{\alpha,(S^2,d)} \leq T_0 + |b| \, |\phi|_{\alpha,(S^2,d)} \leq 2T_0 \max\{1, |b|\},$$

since $T_0 \geq |\phi|_{\alpha,(S^2,d)}$ by (5.31) and the fact that $C_{13} > 1$ from Lemma 5.11.

Note that

$$\left|\mathcal{L}_{\widetilde{s\phi},\mathfrak{c},E}^{(n)}(v)(x) - \mathcal{L}_{\widetilde{s\phi},\mathfrak{c},E}^{(n)}(v)(x')\right| \leq \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left|v(y_{X^n}) e^{S_n \widetilde{s\phi}(y_{X^n})} - v(y'_{X^n}) e^{S_n \widetilde{s\phi}(y'_{X^n})}\right|$$

$$(5.36) \qquad \leq \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left(\left|v(y_{X^n}) - v(y'_{X^n})\right| \left|e^{S_n \widetilde{s\phi}(y'_{X^n})}\right| + |v(y_{X^n})| \left|e^{S_n \widetilde{s\phi}(y_{X^n})} - e^{S_n \widetilde{s\phi}(y'_{X^n})}\right|\right).$$

We bound the two terms in the last summation above separately.
By Lemmas 3.15, 5.1, (5.33), and (5.35),

$$\sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} |v(y_{X^n})| \left|e^{S_n \widetilde{s\phi}(y_{X^n})} - e^{S_n \widetilde{s\phi}(y'_{X^n})}\right|$$

$$= \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} |v(y_{X^n})| \left|1 - e^{S_n \widetilde{s\phi}(y'_{X^n}) - S_n \widetilde{s\phi}(y_{X^n})}\right| e^{S_n \widetilde{a\phi}(y_{X^n})}$$

$$(5.37) \qquad \leq C_{10}(f, \mathcal{C}, d, \alpha, T_0) \left|\widetilde{s\phi}\right|_{\alpha,(S^2,d)} d(x,x')^\alpha \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(|v|)(x)$$

$$\leq A_1 \frac{2T_0 \max\{1, |b|\} \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(|v|)(x)}{1 - \Lambda^{-\alpha}} d(x,x')^\alpha$$

$$= A_0 \max\{1, |b|\} \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(|v|)(x) d(x,x')^\alpha,$$

where the last inequality follows from (5.34).

By (5.14), Lemma 3.13, and (5.17) in Lemma 5.10,

$$\sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left| v(y_{X^n}) - v(y'_{X^n}) \right| \left| e^{S_n \widetilde{s\phi}(y'_{X^n})} \right| \leq \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} |v|_{\alpha,(E,d)} \, d(y_{X^n}, y'_{X^n})^{\alpha} e^{S_n \widetilde{a\phi}(y'_{X^n})}$$

$$(5.38) \qquad \leq |v|_{\alpha,(E,d)} \frac{d(x,x')^{\alpha} C_0^{\alpha}}{\Lambda^{\alpha n}} \sum_{X^n \in \mathbf{X}_{\mathfrak{c}}^n} e^{S_n \widetilde{a\phi}(y'_{X^n})} \leq C_0 \frac{|v|_{\alpha,(E,d)}}{\Lambda^{\alpha n}} d(x,x')^{\alpha}.$$

Thus (5.28) follows from (5.36), (5.37) and (5.38).

If, in addition, there exists a non-negative real-valued Hölder continuous function $h \in C^{0,\alpha}(E,d)$ such that $|v(y) - v(y')| \leq B(h(y) + h(y'))d(y,y')^{\alpha}$ when $y, y' \in E$, then by Lemmas 3.15, 3.13, (5.30), (5.33), and (5.32),

$$\sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} \left| v(y_{X^n}) - v(y'_{X^n}) \right| \left| e^{S_n \widetilde{s\phi}(y'_{X^n})} \right|$$

$$\leq \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}}^n \\ X^n \subseteq E}} B\left(h(y_{X^n}) e^{S_n \widetilde{a\phi}(y_{X^n})} e^{\left| S_n \widetilde{a\phi}(y'_{X^n}) - S_n \widetilde{a\phi}(y_{X^n}) \right|} + h(y'_{X^n}) e^{S_n \widetilde{a\phi}(y'_{X^n})} \right) d(y_{X^n}, y'_{X^n})^{\alpha}$$

$$\leq B \exp\left(\frac{\left| \widetilde{a\phi} \right|_{\alpha,(S^2,d)} C_0 (\operatorname{diam}_d(S^2))^{\alpha}}{1 - \Lambda^{-\alpha}} \right) \frac{d(x,x')^{\alpha} C_0^{\alpha}}{\Lambda^{\alpha n}} \left(\mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(h)(x) + \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(h)(x') \right)$$

$$\leq A_1 B \Lambda^{-\alpha n} \left(\mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(h)(x) + \mathcal{L}_{\widetilde{a\phi},\mathfrak{c},E}^{(n)}(h)(x') \right) d(x,x')^{\alpha}.$$

Therefore, (5.29) follows from (5.36), (5.37), the last inequality, and the fact that $A_0 \geq A_1$ from (5.34). $\qquad \square$

5.3. Spectral gap.

Let (X,d) be a metric space. A function $h \colon [0,+\infty) \to [0,+\infty)$ is an *abstract modulus of continuity* if it is continuous at 0, non-decreasing, and $h(0) = 0$. Given any constant $\tau \in [0,+\infty]$, and any abstract modulus of continuity g, we define the subclass $C_g^{\tau}((X,d),\mathbb{C})$ of $C(X,\mathbb{C})$ as

$$C_g^{\tau}((X,d),\mathbb{C}) = \left\{ u \in C(X,\mathbb{C}) : \|u\|_{C^0(X)} \leq \tau \text{ and for } x, y \in X, |u(x) - u(y)| \leq g(d(x,y)) \right\}.$$

We denote $C_g^{\tau}(X,d) := C_g^{\tau}((X,d),\mathbb{C}) \cap C(X)$.

Assume now that (X,d) is compact. Then by the Arzelà-Ascoli Theorem, each $C_g^{\tau}((X,d),\mathbb{C})$ (resp. $C_g^{\tau}(X,d)$) is precompact in $C(X,\mathbb{C})$ (resp. $C(X)$) equipped with the uniform norm. It is easy to see that each $C_g^{\tau}((X,d),\mathbb{C})$ (resp. $C_g^{\tau}(X,d)$) is actually compact. On the other hand, for $u \in C(X,\mathbb{C})$, we can define an abstract modulus of continuity by

$$(5.39) \qquad g(t) := \sup\{|u(x) - u(y)| : x, y \in X, d(x,y) \leq t\}$$

for $t \in [0,+\infty)$, so that $u \in C_g^{\iota}((X,d),\mathbb{C})$, where $\iota := \|u\|_{\infty}$.

The following lemma is easy to check (see also [Li17, Lemma 5.24]).

Lemma 5.13. *Let (X,d) be a metric space. For each pair of constants τ_1, $\tau_2 \geq 0$, each pair of abstract moduli of continuity g_1, g_2, and each real number $c > 0$, we have*

$$\left\{ u_1 u_2 : u_1 \in C_{g_1}^{\tau_1}((X,d),\mathbb{C}), \, u_2 \in C_{g_2}^{\tau_2}((X,d),\mathbb{C}) \right\} \subseteq C_{\tau_1 g_2 + \tau_2 g_1}^{\tau_1 \tau_2}((X,d),\mathbb{C}) \qquad and$$

$$\left\{ 1/u : u \in C_{g_1}^{\tau_1}((X,d),\mathbb{C}), \, u(x) \geq c \text{ for each } x \in X \right\} \subseteq C_{c^{-2} g_1}^{c^{-1}}((X,d),\mathbb{C}).$$

The following corollary follows immediately from Lemma 5.13. We leave the proof to the readers.

Corollary 5.14. *Let (X,d) be a metric space, and $\alpha \in (0,1]$ a constant. Then for all Hölder continuous functions u, $v \in C^{0,\alpha}((X,d),\mathbb{C})$, we have u, $v \in C^{0,\alpha}((X,d),\mathbb{C})$ with*

$$\|uv\|_{C^{0,\alpha}(X,d)} \leq \|u\|_{C^{0,\alpha}(X,d)} \|v\|_{C^{0,\alpha}(X,d)},$$

and if, in addition, $|u(x)| \geq c$, for each $x \in X$, for some constant $c > 0$, then $1/u \in C^{0,\alpha}((X,d),\mathbb{C})$ with $\|1/u\|_{C^{0,\alpha}(X,d)} \leq c^{-1} + c^{-2} \|u\|_{C^{0,\alpha}(X,d)}.$

Lemma 5.15. *Let f, $\mathcal{C}$, d, α satisfy the Assumptions. Assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent α, and μ_ϕ denote the unique equilibrium state for f and ϕ. Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $u \in C(X^0)$. Then for each $n \in \mathbb{N}$,*

$$\int_{X_\mathfrak{c}^0} u \, \mathrm{d}\mu_\phi = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}'}^0} \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{c}}^{(n)}(u) \, \mathrm{d}\mu_\phi.$$

Proof. We define a function $v \in B(S^2)$ by setting $v(x) = u(x)$ if $x \in \mathrm{inte}(X_\mathfrak{c}^0)$ and $v(x) = 0$ otherwise. We choose a pointwise increasing sequence of continuous non-negative functions $\tau_i \in C(S^2)$, $i \in \mathbb{N}$, such that $\lim_{i \to +\infty} \tau_i(x) = \mathbb{1}_{\mathrm{inte}(X_\mathfrak{c}^0)}$ for all $x \in S^2$. Then $\{v\tau_i\}_{i \in \mathbb{N}}$ is a bounded sequence of continuous functions on S^2, convergent pointwise to v.

Fix $n \in \mathbb{N}$. Since $\mu_\phi(\mathcal{C}) = 0$ by [Li17, Proposition 5.39], then by (5.6), Proposition 3.6 (i) and (ii), and the Dominated Convergence Theorem, we get

$$\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}'}^0} \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{c}}^{(n)}(u) \, \mathrm{d}\mu_\phi = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}'}^0} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}'}^n \\ X^n \subseteq X_\mathfrak{c}^0}} \left(e^{S_n\widetilde{\phi}} u\right)\left((f^n|_{X^n})^{-1}(x)\right) \mathrm{d}\mu_\phi(x)$$

$$= \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \lim_{i \to +\infty} \int_{\mathrm{inte}(X_{\mathfrak{c}'}^0)} \sum_{\substack{X^n \in \mathbf{X}_{\mathfrak{c}'}^n \\ X^n \subseteq X_\mathfrak{c}^0}} \left(e^{S_n\widetilde{\phi}} v\tau_i\right)\left((f^n|_{X^n})^{-1}(x)\right) \mathrm{d}\mu_\phi(x)$$

$$= \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \lim_{i \to +\infty} \int_{\mathrm{inte}(X_{\mathfrak{c}'}^0)} \mathcal{L}_{\widetilde{\phi}}^n(v\tau_i)(x) \, \mathrm{d}\mu_\phi(x) = \lim_{i \to +\infty} \int_{S^2} \mathcal{L}_{\widetilde{\phi}}^n(v\tau_i) \, \mathrm{d}\mu_\phi$$

$$= \lim_{i \to +\infty} \int_{S^2} v\tau_i \, \mathrm{d}\left(\mathcal{L}_{\widetilde{\phi}}^*\right)^n(\mu_\phi) = \lim_{i \to +\infty} \int_{S^2} v\tau_i \, \mathrm{d}\mu_\phi = \int_{S^2} v \, \mathrm{d}\mu_\phi = \int_{X_\mathfrak{c}^0} u \, \mathrm{d}\mu_\phi. \qquad \square$$

Lemma 5.16. *Let f, $\mathcal{C}$, d satisfy the Assumptions. Assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Fix an abstract modulus of continuity g. Then for each $\alpha \in (0, 1]$, $K \in (0, +\infty)$, and $\delta_1 \in (0, +\infty)$, there exist constants $\delta_2 \in (0, +\infty)$ and $N \in \mathbb{N}$ with the following property:*

For all $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_\mathfrak{b} \in C_g^{+\infty}(X_\mathfrak{b}^0, d)$, $u_\mathfrak{w} \in C_g^{+\infty}(X_\mathfrak{w}^0, d)$, and $\phi \in C^{0,\alpha}(S^2, d)$, if $\|\phi\|_{C^{0,\alpha}(S^2,d)} \leq K$, $\max\{\|u_\mathfrak{b}\|_{C^0(X_\mathfrak{b}^0)}, \|u_\mathfrak{w}\|_{C^0(X_\mathfrak{w}^0)}\} \geq \delta_1$, and $\int_{X_\mathfrak{b}^0} u_\mathfrak{b} \, \mathrm{d}\mu_\phi + \int_{X_\mathfrak{w}^0} u_\mathfrak{w} \, \mathrm{d}\mu_\phi = 0$ where μ_ϕ denotes the unique equilibrium state for f and ϕ, then

$$\left\| \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(N)}(u_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(N)}(u_\mathfrak{w}) \right\|_{C^0(X_\mathfrak{c}^0)} \leq \max\{\|u_\mathfrak{b}\|_{C^0(X_\mathfrak{b}^0)}, \|u_\mathfrak{w}\|_{C^0(X_\mathfrak{w}^0)}\} - \delta_2.$$

Proof. Fix arbitrary constants $\alpha \in (0, 1]$, $K \in (0, +\infty)$, and $\delta_1 \in (0, +\infty)$. Choose $\epsilon > 0$ small enough such that $g(\epsilon) < \frac{\delta_1}{2}$. Let $n_0 \in \mathbb{N}$ be the smallest number such that $f^{n_0}(\mathrm{inte}(X_\mathfrak{b}^0)) = S^2 = f^{n_0}(\mathrm{inte}(X_\mathfrak{w}^0))$.

By Lemma 3.8 (iv), there exists a number $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, g, and δ_1 such that $N \geq 2n_0$ and for each $z \in S^2$, we have $U^{N-n_0}(z) \subseteq B_d(z, \epsilon)$ (see (3.8)).

Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in C^{0,\alpha}(S^2, d)$ with $\|\phi\|_{C^{0,\alpha}(S^2,d)} \leq K$, and functions $u_\mathfrak{b} \in C_g^{+\infty}(X_\mathfrak{b}^0, d)$ and $u_\mathfrak{w} \in C_g^{+\infty}(X_\mathfrak{w}^0, d)$ with $\max\{\|u_\mathfrak{b}\|_{C^0(X_\mathfrak{b}^0)}, \|u_\mathfrak{w}\|_{C^0(X_\mathfrak{w}^0)}\} \geq \delta_1$ and $\int_{X_\mathfrak{b}^0} u_\mathfrak{b} \, \mathrm{d}\mu_\phi + \int_{X_\mathfrak{w}^0} u_\mathfrak{w} \, \mathrm{d}\mu_\phi = 0$. Without loss of generality, we assume that $\int_{X_\mathfrak{b}^0} u_\mathfrak{b} \, \mathrm{d}\mu_\phi \leq 0$ and $\int_{X_\mathfrak{w}^0} u_\mathfrak{w} \, \mathrm{d}\mu_\phi \geq 0$. So we can choose points $y_1 \in X_\mathfrak{b}^0$ and $y_2 \in X_\mathfrak{w}^0$ in such a way that $u_\mathfrak{b}(y_1) \leq 0$ and $u_\mathfrak{w}(y_2) \geq 0$.

We denote

$$M := \max\{\|u_\mathfrak{b}\|_{C^0(X_\mathfrak{b}^0)}, \|u_\mathfrak{w}\|_{C^0(X_\mathfrak{w}^0)}\}.$$

We fix a point $x \in X_\mathfrak{c}^0$. Since $f^N(U^{N-n_0}(y_1) \cap X_\mathfrak{b}^0) = S^2$, there exists $y \in f^{-N}(x) \cap X_\mathfrak{b}^0$ such that $y \in U^{N-n_0}(y_1) \subseteq B_d(y_1, \epsilon)$. Since $M \geq \delta_1$, $u_\mathfrak{b}(y) \leq u_\mathfrak{b}(y_1) + g(\epsilon) < \frac{\delta_1}{2} \leq M - \frac{\delta_1}{2}$. Choose $X_y^N \in \mathbf{X}_\mathfrak{c}^N$ such that $y \in X_y^N \subseteq X_\mathfrak{b}^0$. Denote $w_{X^N} := (f^N|_{X^N})^{-1}(x)$ for each $X^N \in \mathbf{X}_\mathfrak{c}^N$. So by Lemma 5.10, we

have

$$\mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b})(x) + \mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})(x)$$

$$= u_\mathfrak{b}(y)e^{S_N\widetilde{\phi}(y)} + \sum_{\substack{X^N \in \mathbf{X}^N_\mathfrak{c} \setminus \{X^N_y\} \\ X^N \subseteq X^0_\mathfrak{b}}} u_\mathfrak{b}(w_{X^N})e^{S_N\widetilde{\phi}(w_{X^N})} + \sum_{\substack{X^N \in \mathbf{X}^N_\mathfrak{c} \\ X^N \subseteq X^0_\mathfrak{w}}} u_\mathfrak{w}(w_{X^N})e^{S_N\widetilde{\phi}(w_{X^N})}$$

$$\leq \left(M - \frac{\delta_1}{2}\right)\exp\left(S_N\widetilde{\phi}(y)\right) + M \sum_{X^N \in \mathbf{X}^N_\mathfrak{c} \setminus \{X^N_y\}} \exp\left(S_N\widetilde{\phi}(w_{X^N})\right)$$

$$= M \sum_{X^N \in \mathbf{X}^N_\mathfrak{c}} \exp\left(S_N\widetilde{\phi}(w_{X^N})\right) - \frac{\delta_1}{2}\exp\left(S_N\widetilde{\phi}(y)\right) = M - \frac{\delta_1}{2}\exp\left(S_N\widetilde{\phi}(y)\right).$$

Similarly, there exists $z \in f^{-N}(x) \cap X^0_\mathfrak{w}$ such that $z \in U^{N-n_0}(y_2) \subseteq B_d(y_2, \epsilon)$ and

$$\mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b})(x) + \mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})(x) \geq -M + \frac{\delta_1}{2}\exp\left(S_N\widetilde{\phi}(z)\right).$$

Hence we get $\left\|\mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b}) + \mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} \leq M - \frac{\delta_1}{2}\inf\left\{\exp\left(S_N\widetilde{\phi}(w)\right) : w \in S^2\right\}.$

By (5.19) in Lemma 5.11 with $T := 1$, the definition of M above, and (2.7), we have

$$\left\|\mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b}) + \mathcal{L}^{(N)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} \leq \max\left\{\|u_\mathfrak{b}\|_{C^0(X^0_\mathfrak{b})}, \|u_\mathfrak{w}\|_{C^0(X^0_\mathfrak{w})}\right\} - \delta_2$$

with $\delta_2 := \frac{\delta_1}{2}\exp(-N(C_{13}K + |\log(\deg f)|))$, where C_{13} is a constant from Lemma 5.11 depending only on f, $\mathcal{C}$, d, and α. Therefore the constant δ_2 depends only on f, $\mathcal{C}$, d, α, g, K, and δ_1. $\qquad\square$

Theorem 5.17. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\operatorname{post} f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$ and $\alpha \in (0,1]$ be a constant. Let H, $H_\mathfrak{b}$, and $H_\mathfrak{w}$ be bounded subsets of $C^{0,\alpha}(S^2, d)$, $C^{0,\alpha}(X^0_\mathfrak{b}, d)$, and $C^{0,\alpha}(X^0_\mathfrak{w}, d)$, respectively (with respect to Hölder norms). Then for all $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in H$, $u_\mathfrak{b} \in H_\mathfrak{b}$, and $u_\mathfrak{w} \in H_\mathfrak{w}$, we have*

$$(5.40) \qquad\qquad \lim_{n \to +\infty}\left\|\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(\overline{u}_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} = 0,$$

where the pair of functions $\overline{u}_\mathfrak{b} \in C^{0,\alpha}(X^0_\mathfrak{b}, d)$ and $\overline{u}_\mathfrak{w} \in C^{0,\alpha}(X^0_\mathfrak{w}, d)$ are given by

$$\overline{u}_\mathfrak{b} := u_\mathfrak{b} - \int_{X^0_\mathfrak{b}} u_\mathfrak{b}\,\mathrm{d}\mu_\phi - \int_{X^0_\mathfrak{w}} u_\mathfrak{w}\,\mathrm{d}\mu_\phi \text{ and } \overline{u}_\mathfrak{w} := u_\mathfrak{w} - \int_{X^0_\mathfrak{b}} u_\mathfrak{b}\,\mathrm{d}\mu_\phi - \int_{X^0_\mathfrak{w}} u_\mathfrak{w}\,\mathrm{d}\mu_\phi$$

with μ_ϕ denoting the unique equilibrium state for f and ϕ.

Moreover, the convergence in (5.40) is uniform in $\phi \in H$, $u_\mathfrak{b} \in H_\mathfrak{b}$, and $u_\mathfrak{w} \in H_\mathfrak{w}$.

Proof. Without loss of generality, we assume that $H \neq \emptyset$, $H_\mathfrak{b} \neq \emptyset$, and $H_\mathfrak{w} \neq \emptyset$. Define constants $K := \sup\{\|\phi\|_{C^{0,\alpha}(S^2,d)} : \phi \in H\} \in [0, +\infty)$ and $K_\mathfrak{c} := \sup\{\|u_\mathfrak{c}\|_{C^{0,\alpha}(X^0_\mathfrak{b},d)} : u_\mathfrak{c} \in H_\mathfrak{c}\} \in [0, +\infty)$ for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$. Define for each $n \in \mathbb{N}_0$,

$$a_n := \sup\left\{\left\|\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(\overline{u}_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} : \mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}, \phi \in H, u_\mathfrak{b} \in H_\mathfrak{b}, u_\mathfrak{w} \in H_\mathfrak{w}\right\}.$$

Note that by Definition 5.2, $a_0 \leq 2K_\mathfrak{b} + 2K_\mathfrak{w} < +\infty$.

By (5.7) in Lemma 5.3 and (5.18) in Lemma 5.10, for all $n \in \mathbb{N}_0$, $\phi \in H$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $v_\mathfrak{b} \in C(X^0_\mathfrak{b})$, and $v_\mathfrak{w} \in C(X^0_\mathfrak{w})$, we have

$$\left\|\mathcal{L}^{(n+1)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(v_\mathfrak{b}) + \mathcal{L}^{(n+1)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(v_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c},d)} = \left\|\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}^{(1)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}\left(\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}(v_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}(v_\mathfrak{w})\right)\right\|_{C^0(X^0_\mathfrak{c},d)}$$

$$\leq \max\left\{\left\|\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}(v_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}(v_\mathfrak{w})\right\|_{C^0(X^0_{\mathfrak{c}'},d)} : \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}\right\}.$$

So $\{a_n\}_{n \in \mathbb{N}_0}$ is a non-increasing sequence of non-negative real numbers.

Suppose now that $\lim_{n \to +\infty} a_n =: a_* > 0$. By Lemma 5.10, (5.28) in Lemma 5.12 with $a := 1$ and $b := 0$, (5.34), (5.31), and (5.2), we get that $\mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(n)}(\overline{u}_\mathfrak{w}) \in C_g^{2(K_\mathfrak{b}+K_\mathfrak{w})}(X_\mathfrak{c}^0, d)$, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each pair of $u_\mathfrak{b} \in H_\mathfrak{b}$ and $u_\mathfrak{w} \in H_\mathfrak{w}$, with an abstract modulus of continuity g given by $g(t) := 2(C_0(K_\mathfrak{b} + K_\mathfrak{w}) + 2(K_\mathfrak{b} + K_\mathfrak{w})A)t^\alpha$, $t \in [0, +\infty)$, where the constant $A > 1$ is given by

$$A := (1 + 2T)\frac{2C_0}{1 - \Lambda^{-\alpha}} \exp\Big(\frac{C_0 T}{1 - \Lambda^{-\alpha}}\big(\mathrm{diam}_d(S^2)\big)^\alpha\Big),$$

and $T := (2s_0 + 1)C_{13}K \exp(2s_0 C_{14}K)$. Here the constant $C_0 > 1$ depending only on f, d, and $\mathcal{C}$ comes from Lemma 3.13, and $C_{13} > 1$, $C_{14} > 0$ are constants from Lemma 5.11 depending only on f, $\mathcal{C}$, d, and α. So g and A both depend only on f, $\mathcal{C}$, d, α, H, $H_\mathfrak{b}$, and $H_\mathfrak{w}$. By Lemma 5.15,

$$\sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_\mathfrak{c}^0} \Big(\mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(n)}(\overline{u}_\mathfrak{w})\Big) \, \mathrm{d}\mu_\phi = \int_{X_\mathfrak{b}^0} \overline{u}_\mathfrak{b} \, \mathrm{d}\mu_\phi + \int_{X_\mathfrak{w}^0} \overline{u}_\mathfrak{w} \, \mathrm{d}\mu_\phi = 0.$$

By (5.7) in Lemma 5.3, (5.18) in Lemma 5.10, and applying Lemma 5.16 with f, $\mathcal{C}$, d, g, α, K, and $\delta_1 := \frac{a_*}{2} > 0$, we find constants $N \in \mathbb{N}$ and $\delta_2 > 0$ such that

$$\Big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(N+n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(N+n)}(\overline{u}_\mathfrak{w})\Big\|_{C^0(X_\mathfrak{c}^0)} = \Big\|\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(N)}\Big(\mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(n)}(\overline{u}_\mathfrak{w})\Big)\Big\|_{C^0(X_\mathfrak{c}^0)}$$

$$\leq \max\Big\{\Big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(n)}(\overline{u}_\mathfrak{w})\Big\|_{C^0(X_{\mathfrak{c}'}^0)} : \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}\Big\} - \delta_2 \leq a_n - \delta_2,$$

for all $n \in \mathbb{N}_0$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in H$, $u_\mathfrak{b} \in H_\mathfrak{b}$, and $u_\mathfrak{w} \in H_\mathfrak{w}$ satisfying

$$(5.41) \qquad \max\Big\{\Big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(n)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(n)}(\overline{u}_\mathfrak{w})\Big\|_{C^0(X_{\mathfrak{c}'}^0)} : \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}\Big\} \geq a_*/2.$$

Since $\lim_{n \to +\infty} a_n = a_*$, we can fix $m \geq 1$ large enough so that $a_m \leq a_* + \frac{\delta_2}{2}$. Then for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each $\phi \in H$, and each pair $u_\mathfrak{b} \in H_\mathfrak{b}$ and $u_\mathfrak{w} \in H_\mathfrak{w}$ satisfying (5.41) with $n := m$, we have

$$\Big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(N+m)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(N+m)}(\overline{u}_\mathfrak{w})\Big\|_{C^0(X_\mathfrak{c}^0)} \leq a_m - \delta_2 \leq a_* - 2^{-1}\delta_2.$$

On the other hand, by (5.18) in Lemma 5.10, for all $\phi \in H$, $u_\mathfrak{b} \in H_\mathfrak{b}$, and $u_\mathfrak{w} \in H_\mathfrak{w}$ with $\max\big\{\big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(m)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(m)}(\overline{u}_\mathfrak{w})\big\|_{C^0(X_{\mathfrak{c}'}^0)} : \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}\big\} < a_*/2$, the following holds for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$:

$$\Big\|\mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}^{(N+m)}(\overline{u}_\mathfrak{b}) + \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}^{(N+m)}(\overline{u}_\mathfrak{w})\Big\|_{C^0(X_\mathfrak{c}^0)} < a_*/2.$$

Thus $a_{N+m} \leq \max\big\{a_* - \frac{\delta_2}{2}, \frac{a_*}{2}\big\} < a_*$, contradicting the fact that $\{a_n\}_{n \in \mathbb{N}_0}$ is a non-increasing sequence and the assumption that $\lim_{n \to +\infty} a_n = a_* > 0$. This proves the uniform convergence in (5.40). $\qquad \square$

Theorem 5.18. *Let $f: S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathrm{post}\, f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\alpha \in (0, 1]$ be a constant and H be a bounded subset of $C^{0,\alpha}(S^2, d)$ with respect to the Hölder norm. Then there exists a constant $\rho_1 \in (0, 1)$ depending on f, $\mathcal{C}$, d, α, and H such that the following property is satisfied:*

For all $\phi \in H$, $n \in \mathbb{N}_0$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_\mathfrak{b} \in C^{0,\alpha}(X_\mathfrak{b}^0, d)$, and $u_\mathfrak{w} \in C^{0,\alpha}(X_\mathfrak{w}^0, d)$, we have

$$(5.42) \qquad \Big\|\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(n)}(\overline{u}_{\mathfrak{c}'})\Big\|_{C^0(X_\mathfrak{c}^0)} \leq 6\rho_1^n \max_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}}\big\{\|u_{\mathfrak{c}'}\|_{C^{0,\alpha}(X_{\mathfrak{c}'}^0, d)}\big\},$$

where $\overline{u}_{\mathfrak{c}'} \in C^{0,\alpha}(X_{\mathfrak{c}'}^0, d)$ for $\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$ are given by $\overline{u}_{\mathfrak{c}'} := u_{\mathfrak{c}'} - \int_{X_\mathfrak{b}^0} u_\mathfrak{b} \, \mathrm{d}\mu_\phi - \int_{X_\mathfrak{w}^0} u_\mathfrak{w} \, \mathrm{d}\mu_\phi$, with μ_ϕ denoting the unique equilibrium state for f and ϕ. In particular,

$$(5.43) \qquad \Big\|\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(n)}(u_{\mathfrak{c}'})\Big\|_{C^0(X_\mathfrak{c}^0)} \leq \Big|\int_{X_\mathfrak{b}^0} u_\mathfrak{b} \, \mathrm{d}\mu_\phi + \int_{X_\mathfrak{w}^0} u_\mathfrak{w} \, \mathrm{d}\mu_\phi\Big| + 6\rho_1^n \max_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}}\big\{\|u_{\mathfrak{c}'}\|_{C^{0,\alpha}(X_{\mathfrak{c}'}^0, d)}\big\}.$$

Proof. Without loss of generality, we assume that $H \neq \emptyset$. Define a constant

$$(5.44) \qquad\qquad K := \sup\{\|\phi\|_{C^{0,\alpha}(S^2,d)} : \phi \in H\} \in [0, +\infty).$$

Denote, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$,

$$H_{\mathfrak{c}} := \{v_{\mathfrak{c}} \in C^{0,\alpha}(X_{\mathfrak{c}}^0, d) : \|v_{\mathfrak{c}}\|_{C^{0,\alpha}(X_{\mathfrak{c}}^0, d)} \leq 3\}.$$

Inequality (5.43) follows immediately from (5.42), the triangle inequality, and the fact that $\mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{b}}^{(1)}(\mathbb{1}_{X_{\mathfrak{b}}^0}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{w}}^{(1)}(\mathbb{1}_{X_{\mathfrak{w}}^0}) = \mathbb{1}_{X_{\mathfrak{c}}^0}$ by (5.6) and Lemma 5.10. So it suffices to establish (5.42).

We first consider the special case where $u_{\mathfrak{b}} \in H_{\mathfrak{b}}$ and $u_{\mathfrak{w}} \in H_{\mathfrak{w}}$.

By (5.28) in Lemma 5.12 with $s := 1$, (5.17) in Lemma 5.10, and (5.44), for all $j \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in H$, $u_{\mathfrak{b}} \in H_{\mathfrak{b}}$, and $u_{\mathfrak{w}} \in H_{\mathfrak{w}}$, we have

$$\left| \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(j)}(\overline{u}_{\mathfrak{c}'}) \right|_{\alpha, (X_{\mathfrak{c}}^0, d)} \leq \frac{C_0}{\Lambda^{\alpha j}} \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} |\overline{u}_{\mathfrak{c}'}|_{\alpha, (X_{\mathfrak{c}'}^0, d)} + A_0 \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \left\| \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(j)}(|\overline{u}_{\mathfrak{c}'}|) \right\|_{C^0(X_{\mathfrak{c}}^0)}$$

$$(5.45) \qquad\qquad \leq \frac{6C_0}{\Lambda^{\alpha j}} + A_0 \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \|\overline{u}_{\mathfrak{c}'}\|_{C^0(X_{\mathfrak{c}'}^0)} \leq C_{17},$$

where the constant C_{17} is given by $C_{17} := 6C_0 + 12A_0$, the constant $A_0 := A_0(f, \mathcal{C}, d, K, \alpha) > 2$ defined in (5.34) from Lemma 5.12 depends only on f, $\mathcal{C}$, d, H, and α, and the constant $C_0 > 1$ from Lemma 3.13 depends only on f, $\mathcal{C}$, and d. Thus $C_{17} > 1$ depends only on f, $\mathcal{C}$, d, and H.

So by (5.7) in Lemma 5.3, (5.28) in Lemma 5.12 with $s := 1$, (5.45), and (5.17) in Lemma 5.10, we get that for all $k \in \mathbb{N}$,

$$(5.46)$$

$$\left| \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{b}}^{(k+j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{w}}^{(k+j)}(\overline{u}_{\mathfrak{w}}) \right|_{\alpha, (X_{\mathfrak{c}}^0, d)} \leq \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \left| \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(k)} \left(\mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(j)}(\overline{u}_{\mathfrak{w}}) \right) \right|_{\alpha, (X_{\mathfrak{c}}^0, d)}$$

$$\leq \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \left(\frac{C_0}{\Lambda^{\alpha k}} \left| \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(j)}(\overline{u}_{\mathfrak{w}}) \right|_{\alpha, (X_{\mathfrak{c}'}^0, d)} + A_0 \left\| \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(k)} \left(\left| \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(j)}(\overline{u}_{\mathfrak{w}}) \right| \right) \right\|_{C^0(X_{\mathfrak{c}}^0)} \right)$$

$$\leq \frac{2C_0 C_{17}}{\Lambda^{\alpha k}} + A_0 \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \left\| \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{b}}^{(j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c}',\mathfrak{w}}^{(j)}(\overline{u}_{\mathfrak{w}}) \right\|_{C^0(X_{\mathfrak{c}'}^0)}.$$

By Theorem 5.17, we can choose $N_0 \in \mathbb{N}$ with the property that

$$(5.47) \qquad \frac{2C_0 C_{17}}{\Lambda^{\alpha j}} \leq \frac{1}{8} \quad \text{and} \quad (1 + A_0) \left\| \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{b}}^{(j)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{w}}^{(j)}(\overline{u}_{\mathfrak{w}}) \right\|_{C^0(X_{\mathfrak{c}}^0)} \leq \frac{1}{8},$$

for all $j \in \mathbb{N}$ with $j \geq N_0$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in H$, $u_{\mathfrak{b}} \in H_{\mathfrak{b}}$, and $u_{\mathfrak{w}} \in H_{\mathfrak{w}}$. We set $N_0 \in \mathbb{N}$ to be the smallest integer with this property. So N_0 depends only on f, $\mathcal{C}$, d, α, and H.

For each $m \in \mathbb{N}$, each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each $\phi \in H$, and each pair of functions $u_{\mathfrak{b}} \in H_{\mathfrak{b}}$ and $u_{\mathfrak{w}} \in H_{\mathfrak{w}}$, we denote

$$(5.48) \qquad\qquad v_{m,\mathfrak{c}} := \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{b}}^{(2N_0 m)}(\overline{u}_{\mathfrak{b}}) + \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{w}}^{(2N_0 m)}(\overline{u}_{\mathfrak{w}}).$$

Then by (5.46) and (5.47), the function $v_{m,\mathfrak{c}} \in C^{0,\alpha}(X_{\mathfrak{c}}^0, d)$ satisfies $\|v_{m,\mathfrak{c}}\|_{C^{0,\alpha}(X_{\mathfrak{c}}^0, d)} \leq 3/8$. So $2v_{m,\mathfrak{c}} \in H_{\mathfrak{c}}$. We also note that by Lemma 5.15,

$$\sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} v_{m,\mathfrak{c}} \, d\mu_\phi = \sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \mathcal{L}_{\tilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(2N_0 m)}(\overline{u}_{\mathfrak{c}'}) \, d\mu_\phi = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}'}^0} \overline{u}_{\mathfrak{c}'} \, d\mu_\phi = 0.$$

Next, we prove by induction that for each $m \in \mathbb{N}$, each $\phi \in H$, and each pair of functions $u_{\mathfrak{b}} \in H_{\mathfrak{b}}$ and $u_{\mathfrak{w}} \in H_{\mathfrak{w}}$, we have

$$(5.49) \qquad\qquad \max\{\|v_{m,\mathfrak{b}}\|_{C^{0,\alpha}(X_{\mathfrak{b}}^0, d)}, \|v_{m,\mathfrak{w}}\|_{C^{0,\alpha}(X_{\mathfrak{w}}^0, d)}\} \leq 3(1/2)^m.$$

We have already shown that (5.49) holds for $m = 1$.

Assume that (5.49) holds for $m = j$ for some $j \in \mathbb{N}$, then $2^j v_{j,\mathfrak{b}} \in H_\mathfrak{b}$ and $2^j v_{j,\mathfrak{w}} \in H_\mathfrak{w}$. By (5.7) in Lemma 5.3, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $2^j v_{j+1,\mathfrak{c}} = \mathcal{L}^{(2N_0)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(2^j v_{j,\mathfrak{b}}) + \mathcal{L}^{(2N_0)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(2^j v_{j,\mathfrak{w}})$. Thus $\big\|2^j v_{j+1,\mathfrak{c}}\big\|_{C^{0,\alpha}(X^0_\mathfrak{c},d)} \leq$
$3/8 < 1/2$. So $\|v_{j+1,\mathfrak{c}}\|_{C^{0,\alpha}(X^0_\mathfrak{c},d)} \leq (1/2)^{j+1} < 3(1/2)^{j+1}$.

The induction is now complete.

Then by (5.7) in Lemma 5.3, (5.18) in Lemma 5.10, (5.48), and (5.49), the following holds for all $j \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\phi \in H$, $u_\mathfrak{b} \in H_\mathfrak{b}$, and $u_\mathfrak{w} \in H_\mathfrak{w}$:

$$\left\|\mathcal{L}^{(j+2N_0 m)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(j+2N_0 m)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(\overline{u}_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} = \left\|\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}^{(j)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}\left(\mathcal{L}^{(2N_0 m)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(2N_0 m)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}(\overline{u}_\mathfrak{w})\right)\right\|_{C^0(X^0_\mathfrak{c})}$$

$$\leq \max\left\{\left\|\mathcal{L}^{(2N_0 m)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(2N_0 m)}_{\widetilde{\phi},\mathfrak{c}',\mathfrak{w}}(\overline{u}_\mathfrak{w})\right\|_{C^0(X^0_{\mathfrak{c}'})} : \mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}\right\} \leq 3(1/2)^m.$$

Hence, for each $n \in \mathbb{N}_0$,

$$(5.50) \qquad \left\|\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}(\overline{u}_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}(\overline{u}_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})} \leq 3(1/2)^{\lfloor \frac{n}{2N_0} \rfloor} \leq 6\rho_1^n,$$

where the constant $\rho_1 := 2^{-1/(2N_0)}$ depends only on f, $\mathcal{C}$, d, α, and H.

Finally, we consider the general case. For each pair of functions $w_\mathfrak{b} \in C^{0,\alpha}(X^0_\mathfrak{b}, d)$ and $w_\mathfrak{w} \in C^{0,\alpha}(X^0_\mathfrak{w}, d)$, we denote $M := \max\{\|w_\mathfrak{b}\|_{C^{0,\alpha}(X^0_\mathfrak{b},d)}, \|w_\mathfrak{w}\|_{C^{0,\alpha}(X^0_\mathfrak{w},d)}\}$ and $\overline{w}_{\mathfrak{c}'} := w_{\mathfrak{c}'} - \int_{X^0_\mathfrak{b}} w_\mathfrak{b} \, d\mu_\phi - \int_{X^0_\mathfrak{w}} w_\mathfrak{w} \, d\mu_\phi$ for each $\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$. Let $u_\mathfrak{b} := \frac{1}{M} w_\mathfrak{b}$ and $u_\mathfrak{w} := \frac{1}{M} w_\mathfrak{w}$. Then clearly $u_\mathfrak{b} \in H_\mathfrak{b}$, $u_\mathfrak{w} \in H_\mathfrak{w}$, $\overline{u}_\mathfrak{b} = \frac{1}{M}\overline{w}_\mathfrak{b}$, and $\overline{u}_\mathfrak{w} = \frac{1}{M}\overline{w}_\mathfrak{w}$. Therefore, by (5.50), for each $n \in \mathbb{N}_0$, each $\phi \in H$, and each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$,

$$\left\|\mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{b}}\left(\frac{\overline{w}_\mathfrak{b}}{M}\right) + \mathcal{L}^{(n)}_{\widetilde{\phi},\mathfrak{c},\mathfrak{w}}\left(\frac{\overline{w}_\mathfrak{w}}{M}\right)\right\|_{C^0(X^0_\mathfrak{c})} \leq 6\rho_1^n.$$

Now (5.42) follows. This completes the proof. $\qquad\qquad\square$

Remark 5.19. For $\phi \in C^{0,\alpha}(S^2, d)$, the existence of the spectral gap for the split Ruelle operator $\mathbb{L}_{\widetilde{\phi}}$ on $C^{0,\alpha}(X^0_\mathfrak{b}, d) \times C^{0,\alpha}(X^0_\mathfrak{w}, d)$ follows immediately from (5.12) in Lemma 5.7, Theorem 5.18, and Lemma 5.12 (ii).

Finally, we establish the following lemma that will be used in Section 6.

Lemma 5.20. Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. Assume in addition $f(\mathcal{C}) \subseteq \mathcal{C}$. Then for all $n \in \mathbb{N}$ and $s \in \mathbb{C}$ satisfying $|\Re(s)| \leq 2s_0$ and $|\Im(s)| \geq 1$, we have

$$(5.51) \qquad \left\|\left|\mathbb{L}^n_{\widetilde{s\phi}}\right|\right\|^{[\Im(s)]}_\alpha \leq 4A_0,$$

and more generally,

$$(5.52) \qquad \left\|\left(\mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})\right)^m\right\|^{[\Im(s)]}_{C^{0,\alpha}(X^0_\mathfrak{c},d)} \leq (3m+1)A_0$$

for all $m \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_\mathfrak{b} \in C^{0,\alpha}((X^0_\mathfrak{b}, d), \mathbb{C})$, and $u_\mathfrak{w} \in C^{0,\alpha}((X^0_\mathfrak{w}, d), \mathbb{C})$ satisfying

$$(5.53) \qquad \|u_\mathfrak{b}\|^{[\Im(s)]}_{C^{0,\alpha}(X^0_\mathfrak{b},d)} \leq 1 \quad and \quad \|u_\mathfrak{w}\|^{[\Im(s)]}_{C^{0,\alpha}(X^0_\mathfrak{w},d)} \leq 1.$$

Here $A_0 = A_0(f, \mathcal{C}, d, |\phi|_{\alpha,(S^2,d)}, \alpha) \geq 2C_0 > 2$ is a constant from Lemma 5.12 depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,(S^2,d)}$, and α, and $C_0 > 1$ is a constant depending only on f, $\mathcal{C}$, and d from Lemma 3.13.

Proof. Fix $n, m \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and $s = a + \mathbf{i}b$ with $a, b \in \mathbb{R}$ satisfying $|a| \leq 2s_0$ and $|b| \geq 1$. Choose arbitrary $u_\mathfrak{b} \in C^{0,\alpha}((X^0_\mathfrak{b}, d), \mathbb{C})$ and $u_\mathfrak{w} \in C^{0,\alpha}((X^0_\mathfrak{w}, d), \mathbb{C})$ satisfying (5.53). We denote $M := \left\|\mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{b}}(u_\mathfrak{b}) + \mathcal{L}^{(n)}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{w}}(u_\mathfrak{w})\right\|_{C^0(X^0_\mathfrak{c})}$. By (5.18) in Lemma 5.10, we have $M \leq 1$.

We then observe that for each Hölder continuous function $v \in C^{0,\alpha}((X, d_0), \mathbb{C})$ on a compact metric space (X, d_0), we have $|v^m|_{\alpha,(X,d_0)} \leq m\|v\|^{m-1}_{C^0(X)} |v|_{\alpha,(X,d_0)}$. Thus we get from (5.28) in Lemma 5.12,

(5.17) in Lemma 5.10, (5.53), and the observation above that

$$\left\|\left(\mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{b}}^{(n)}(u_\mathfrak{b}) + \mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{w}}^{(n)}(u_\mathfrak{w})\right)^m\right\|_{C^{0,\alpha}(X_\mathfrak{c}^0,d)}^{[b]} = M^m + \frac{1}{|b|}\left|\left(\mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{b}}^{(n)}(u_\mathfrak{b}) + \mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{w}}^{(n)}(u_\mathfrak{w})\right)^m\right|_{\alpha,\,(X_\mathfrak{c}^0,d)}$$

$$\leq 1 + mM^{m-1}|b|^{-1}\left|\mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{b}}^{(n)}(u_\mathfrak{b}) + \mathcal{L}_{\widetilde{s\phi},\mathfrak{c},\mathfrak{w}}^{(n)}(u_\mathfrak{w})\right|_{\alpha,\,(X_\mathfrak{c}^0,d)}$$

$$\leq 1 + mC_0\Lambda^{-\alpha n}\sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\|u_{\mathfrak{c}'}\|_{C^{0,\alpha}(X_{\mathfrak{c}'}^0,d)}^{[b]} + mA_0\sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\left\|\mathcal{L}_{a\phi,\mathfrak{c},\mathfrak{c}'}^{(n)}(|u_{\mathfrak{c}'}|)\right\|_{C^0(X_\mathfrak{c}^0)}$$

$$\leq 1 + 2mC_0 + mA_0\big(\|u_\mathfrak{b}\|_{C^0(X_\mathfrak{b}^0)} + \|u_\mathfrak{w}\|_{C^0(X_\mathfrak{w}^0)}\big) \leq (3m+1)A_0,$$

where $C_0 > 1$ is a constant depending only on f, $\mathcal{C}$, and d from Lemma 3.13, and the last inequality follows from the fact that $A_0 \geq 2C_0 > 2$ (see Lemma 5.12). The inequality (5.52) is now established, and (5.51) follows from (5.13) in Lemma 5.7 and (5.52). $\qquad\square$

6. BOUND THE ZETA FUNCTION WITH THE OPERATOR NORM

In this section, we bound the dynamical zeta function $\zeta_{\sigma_{A_\Delta},\,-\phi\circ\pi_\Delta}$ using some bounds of the operator norm of $\mathbb{L}_{-s\phi}$, for an expanding Thurston map f with some forward invariant Jordan curve $\mathcal{C}$ and an eventually positive real-valued Hölder continuous potential ϕ.

Subsection 6.1 focuses on Proposition 6.1, which provides a bound of the dynamical zeta function $\zeta_{\sigma_{A_\Delta},\,-\phi\circ\pi_\Delta}$ for the symbolic system $\big(\Sigma_{A_\Delta}^+, \sigma_{A_\Delta}\big)$ asscociated to f in terms of the operator norms of $\mathbb{L}_{-s\phi}^n$, $n \in \mathbb{N}$ and $s \in \mathbb{C}$ in some vertical strip with $|\Im(s)|$ large enough. The idea of the proof originated from D. Ruelle [Ru90]. In Subsection 6.2, we establish in Theorem 6.3 an exponential decay bound on the operator norm $\left\|\left\|\mathbb{L}_{-s\phi}^n\right\|\right\|_\alpha$ of $\mathbb{L}_{-s\phi}^n$, $n \in \mathbb{N}$, assuming the bound stated in Theorem 6.2. Theorem 6.2 will be proved at the end of Subsection 7.3. Combining the bounds in Proposition 6.1 and Theorem 6.3, we give a proof of Theorem E in Subsection 6.3. Finally, in Subsection 6.5, we deduce Theorem 6.5 from Theorem D following the ideas from [PS98] using basic complex analysis.

6.1. Ruelle's estimate.

Proposition 6.1. *Let f, $\mathcal{C}$, d, Λ, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Let $\big(\Sigma_{A_\Delta}^+, \sigma_{A_\Delta}\big)$ be the one-sided subshift of finite type associated to f and $\mathcal{C}$ defined in Proposition 3.21, and let $\pi_\Delta \colon \Sigma_{A_\Delta}^+ \to S^2$ be defined in (3.19). Then for each $\delta > 0$ there exists a constant $D_\delta > 0$ such that for all integers $n \geq 2$ and $k \in \mathbb{N}$, we have*

$$(6.1) \qquad \sum_{X^k\in\mathbf{X}^k(f,\mathcal{C})}\max_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}}\left\|\mathcal{L}_{-s\phi,\mathfrak{c},X^k}^{(k)}(\mathbb{1}_{X^k})\right\|_{C^{0,\alpha}(X_\mathfrak{c}^0,d)} \leq D_\delta|\Im(s)|\Lambda^{-\alpha}\exp(k(\delta + P(f,-\Re(s)\phi)))$$

and

$$(6.2) \qquad \left|Z_{\sigma_{A_\Delta},\,-\phi\circ\pi_\Delta}^{(n)}(s) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}}\sum_{\substack{X^1\in\mathbf{X}^1(f,\mathcal{C})\\X^1\subseteq X_\mathfrak{c}^0}}\mathcal{L}_{-s\phi,\mathfrak{c},X^1}^{(n)}(\mathbb{1}_{X^1})(x_{X^1})\right|$$

$$\leq D_\delta|\Im(s)|\sum_{m=2}^{n}\left\|\left\|\mathbb{L}_{-s\phi}^{n-m}\right\|\right\|_\alpha\left(\frac{1}{\Lambda^\alpha}\exp(\delta + P(f,-\Re(s)\phi))\right)^m$$

for any choice of a point $x_{X^1} \in \mathrm{inte}(X^1)$ for each $X^1 \in \mathbf{X}^1(f,\mathcal{C})$, and for all $s \in \mathbb{C}$ with $|\Im(s)| \geq 2s_0 + 1$ and $|\Re(s) - s_0| \leq s_0$, where $Z_{\sigma_{A_\Delta},\,-\phi\circ\pi_\Delta}^{(n)}(s)$ is defined in (3.20).

Proof. Fix the integer $n \geq 2$.

We first choose $x_{X^n} \in X^n$ for each n-tile $X^n \in \mathbf{X}^n$ in the following way. If $X^n \subseteq f^n(X^n)$, then let x_{X^n} be the unique point in $X^n \cap P_{1,f^n}$ (see [Li16, Lemmas 4.1 and 4.2]); otherwise X^n must be a black n-tile contained in the white 0-tile, or a white n-tile contained in the black 0-tile, in which case we choose an arbitrary point $x_{X^n} \in \mathrm{inte}(X^n)$. Next, for each $i \in \mathbb{N}_0$ with $i \leq n-1$, and each $X^i \in \mathbf{X}^i$, we fix an arbitrary point $x_{X^i} \in \mathrm{inte}(X^i)$.

By (5.6) and our construction, we get that for all $s \in \mathbb{C}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and $X^n \in \mathbf{X}^n$ with $X^n \subseteq X_{\mathfrak{c}}^0$,

$$(6.3) \qquad \mathcal{L}_{-s\phi,\mathfrak{c},X^n}^{(n)}(\mathbb{1}_{X^n})(x_{X^n}) = \begin{cases} \exp(-sS_n\phi(x_{X^n})) & \text{if } X^n \subseteq f^n(X^n), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that by (6.3), the function $Z_{\sigma_{A_\Delta}, -\phi\circ\pi_\Delta}^{(n)}(s)$ defined in (3.20) satisfies

$$(6.4) \qquad Z_{\sigma_{A_\Delta}, -\phi\circ\pi_\Delta}^{(n)}(s) = \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^n\in\mathbf{X}^n \\ X^n\subseteq X_{\mathfrak{c}}^0}} \mathcal{L}_{-s\phi,\mathfrak{c},X^n}^{(n)}(\mathbb{1}_{X^n})(x_{X^n}).$$

Thus, by the triangle inequality, we get

$$\left| Z_{\sigma_{A_\Delta}, -\phi\circ\pi_\Delta}^{(n)}(s) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X_{\mathfrak{c}}^0}} \mathcal{L}_{-s\phi,\mathfrak{c},X^1}^{(n)}(\mathbb{1}_{X^1})(x_{X^1}) \right|$$

$$(6.5) \qquad \leq \sum_{m=2}^{n} \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^{m-1}\in\mathbf{X}^{m-1} \\ X^{m-1}\subseteq X_{\mathfrak{c}}^0}} \left| \mathcal{L}_{-s\phi,\mathfrak{c},X^{m-1}}^{(n)}(\mathbb{1}_{X^{m-1}})(x_{X^{m-1}}) - \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X_{\mathfrak{c}}^0}} \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m})(x_{X^m}) \right|$$

$$\leq \sum_{m=2}^{n} \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^{m-1}\in\mathbf{X}^{m-1} \\ X^{m-1}\subseteq X_{\mathfrak{c}}^0}} \left| \mathcal{L}_{-s\phi,\mathfrak{c},X^{m-1}}^{(n)}(\mathbb{1}_{X^{m-1}})(x_{X^{m-1}}) - \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X^{m-1}}} \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m})(x_{X^m}) \right|.$$

Note that for all $s \in \mathbb{C}$, $2 \leq m \leq n$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and $X^{m-1} \in \mathbf{X}^{m-1}$ with $X^{m-1} \subseteq X_{\mathfrak{c}}^0$, by (5.6),

$$\mathcal{L}_{-s\phi,\mathfrak{c},X^{m-1}}^{(n)}(\mathbb{1}_{X^{m-1}})(x_{X^{m-1}}) = \sum_{\substack{X^n\in\mathbf{X}_{\mathfrak{c}}^n \\ X^n\subseteq X^{m-1}}} \exp\left(-sS_n\phi\left((f^n|_{X^n})^{-1}(x_{X^{m-1}})\right)\right)$$

$$(6.6) \qquad\qquad = \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X^{m-1}}} \sum_{\substack{X^n\in\mathbf{X}_{\mathfrak{c}}^n \\ X^n\subseteq X^m}} \exp\left(-sS_n\phi\left((f^n|_{X^n})^{-1}(x_{X^{m-1}})\right)\right)$$

$$= \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X^{m-1}}} \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m})(x_{X^{m-1}}).$$

Hence by (6.5), (6.6), and (5.8), we get

$$\left| Z_{\sigma_{A_\Delta}, -\phi\circ\pi_\Delta}^{(n)}(s) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X_{\mathfrak{c}}^0}} \mathcal{L}_{-s\phi,\mathfrak{c},X^1}^{(n)}(\mathbb{1}_{X^1})(x_{X^1}) \right|$$

$$\leq \sum_{m=2}^{n} \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^{m-1}\in\mathbf{X}^{m-1} \\ X^{m-1}\subseteq X_{\mathfrak{c}}^0}} \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X^{m-1}}} \left| \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m})(x_{X^{m-1}}) - \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m})(x_{X^m}) \right|$$

$$\leq \sum_{m=2}^{n} \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^{m-1}\in\mathbf{X}^{m-1} \\ X^{m-1}\subseteq X_{\mathfrak{c}}^0}} \sum_{\substack{X^m\in\mathbf{X}^m \\ X^m\subseteq X^{m-1}}} \left\| \mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(n)}(\mathbb{1}_{X^m}) \right\|_{C^{0,\alpha}(X_{\mathfrak{c}}^0,d)} d(x_{X^{m-1}}, x_{X^m})^\alpha.$$

Note that by (5.8), $\mathcal{L}_{-s\phi,\mathfrak{c},X^m}^{(m)}(\mathbb{1}_{X^m}) \in C^{0,\alpha}\big((X_{\mathfrak{c}}^0, d), \mathbb{C}\big)$ for $s \in \mathbb{C}$, $m \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $X^m \in \mathbf{X}^m$, and that by Lemma 3.8 (ii), $d(x_{X^{m-1}}, x_{X^m}) \leq \operatorname{diam}_d(X^{m-1}) \leq C\Lambda^{-m+1}$. Here $C \geq 1$ is a constant

from Lemma 3.8 depending only on f, $\mathcal{C}$, and d. So by (5.7) in Lemma 5.3 and (5.13) in Lemma 5.7,

$$
(6.7) \qquad \left| Z^{(n)}_{\sigma_{A_\triangle},\, -\phi\circ\pi_\triangle}(s) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\,\mathfrak{w}\}} \sum_{\substack{X^1\in\mathbf{X}^1 \\ X^1\subseteq X^0_\mathfrak{c}}} \mathcal{L}^{(n)}_{-s\phi,\mathfrak{c},X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right|
$$

$$
\leq \sum_{m=2}^{n} \sum_{X^m\in\mathbf{X}^m} \left\|\!\left\| \mathbb{L}^{n-m}_{-s\phi} \right\|\!\right\|_\alpha \left(\max_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}} \left\| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m}) \right\|_{C^{0,\alpha}(X^0,d)} \right) C^\alpha \Lambda^{\alpha(1-m)}.
$$

We now give an upper bound for $\sum_{X^m\in\mathbf{X}^m} \max_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}} \left\| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m}) \right\|_{C^{0,\alpha}(X^0_{\mathfrak{c}'},d)}$.

Fix an arbitrary point $y\in\mathcal{C}\setminus\operatorname{post} f$.

Consider arbitrary $s\in\mathbb{C}$ with $|\Re(s)-s_0|\leq s_0$, $m\in\mathbb{N}$, $X^m_\mathfrak{b}\in\mathbf{X}^m_\mathfrak{b}$, $X^m_\mathfrak{w}\in\mathbf{X}^m_\mathfrak{w}$, $X^m\in\mathbf{X}^m$, $x_\mathfrak{b}, x'_\mathfrak{b}\in X^0_\mathfrak{b}$, $x_\mathfrak{w}, x'_\mathfrak{w}\in X^0_\mathfrak{w}$, and $\mathfrak{c}, \mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}$ with $\mathfrak{c}\neq\mathfrak{c}'$. By (5.6), Lemmas 3.13, and 3.8 (ii), we have

$$
(6.8) \qquad \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_\mathfrak{c}}(\mathbb{1}_{X^m_\mathfrak{c}})(x_{\mathfrak{c}'}) = 0,
$$

and

$$
(6.9) \qquad
\begin{aligned}
\left| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x_{\mathfrak{c}'}) \right| &= \left| \exp\!\left(-s S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(x_{\mathfrak{c}'})\right)\right) \right| \\
&= \exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right) \frac{\exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(x_{\mathfrak{c}'})\right)\right)}{\exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right)} \\
&\leq \exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right) \exp\!\left(\Re(s)C_1\left(\operatorname{diam}_d(X^0_{\mathfrak{c}'})\right)^\alpha\right) \\
&\leq \exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right) \exp\!\left(\Re(s)C^\alpha C_1\right),
\end{aligned}
$$

where $C_1>0$ is a constant from Lemma 3.15 depending only on f, $\mathcal{C}$, d, ϕ, and α.

Hence by (6.8) and (6.9), we get

$$
(6.10) \qquad \left\| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m}) \right\|_{C^0(X^0_{\mathfrak{c}'})} \leq \exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m})^{-1}(y)\right)\right) \exp\left(\Re(s)C^\alpha C_1\right).
$$

By (5.6),

$$
(6.11) \qquad \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m_\mathfrak{c}})(x_{\mathfrak{c}'}) - \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m_\mathfrak{c}})(x'_{\mathfrak{c}'}) = 0.
$$

By (5.6) and Lemma 5.1 with $T:=2s_0|\phi|_{\alpha,\,(S^2,d)}$,

$$
\begin{aligned}
&\left| 1 - \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x_{\mathfrak{c}'}) \big/ \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x'_{\mathfrak{c}'}) \right| \\
&\quad = \left| 1 - \exp\!\left(-s\left(S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(x_{\mathfrak{c}'})\right) - S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(x'_{\mathfrak{c}'})\right)\right)\right) \right| \\
&\quad \leq C_{10} |s\phi|_{\alpha,\,(S^2,d)}\, d(x_{\mathfrak{c}'}, x'_{\mathfrak{c}'})^\alpha = C_{10}|s|\,|\phi|_{\alpha,\,(S^2,d)}\, d(x_{\mathfrak{c}'}, x'_{\mathfrak{c}'})^\alpha,
\end{aligned}
$$

where the constant $C_{10}=C_{10}(f,\mathcal{C},d,\alpha,T)>1$ depends only on f, $\mathcal{C}$, d, α, and ϕ in our context.

Thus by (6.9),

$$
\begin{aligned}
&\left| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x_{\mathfrak{c}'}) - \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x'_{\mathfrak{c}'}) \right| \\
&\quad \leq \left| 1 - \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x_{\mathfrak{c}'}) \big/ \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x'_{\mathfrak{c}'}) \right| \left| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}})(x'_{\mathfrak{c}'}) \right| \\
&\quad \leq 4^{-1}C_{11}|s|d(x_{\mathfrak{c}'}, x'_{\mathfrak{c}'})^\alpha \exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right),
\end{aligned}
$$

where we define the constant

$$
(6.12) \qquad C_{11} := \max\!\left\{2,\, 4C_{10}|\phi|_{\alpha,\,(S^2,d)}\right\} \exp\left(2s_0 C^\alpha C_1\right)
$$

depending only on f, $\mathcal{C}$, d, α, and ϕ.

So we get

$$
(6.13) \qquad \left| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m_{\mathfrak{c}'}}(\mathbb{1}_{X^m_{\mathfrak{c}'}}) \right|_{\alpha,\,(X^0_{\mathfrak{c}'},d)} \leq 4^{-1}C_{11}|s|\exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m_{\mathfrak{c}'}})^{-1}(y)\right)\right).
$$

Thus by (6.11) and (6.13), we have

$$
(6.14) \qquad \left| \mathcal{L}^{(m)}_{-s\phi,\mathfrak{c}',X^m}(\mathbb{1}_{X^m}) \right|_{\alpha,\,(X^0_{\mathfrak{c}'},d)} \leq 4^{-1}C_{11}|s|\exp\!\left(-\Re(s)S_m\phi\!\left((f^m|_{X^m})^{-1}(y)\right)\right).
$$

Hence by (6.10) and (6.14), for all $m \in \mathbb{N}$, $X^m \in \mathbf{X}^m$, $s \in \mathbb{C}$, and $\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$ satisfying $|\Im(s)| \geq 2s_0 + 1$ and $|\Re(s) - s_0| \leq s_0$, we have

$$(6.15) \qquad \big\|\mathcal{L}^{(m)}_{-s\phi, \mathfrak{c}', X^m}(\mathbb{1}_{X^m})\big\|_{C^{0,\alpha}(X^0_{\mathfrak{c}'}, d)} \leq C_{11}|\Im(s)| \exp\big(-\Re(s)S_m\phi\big((f^m|_{X^m})^{-1}(y)\big)\big).$$

So by (6.15) and the fact that $y \in \mathcal{C}$, we get

$$
\sum_{X^m \in \mathbf{X}^m} \max_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \big\|\mathcal{L}^{(m)}_{-s\phi, \mathfrak{c}', X^m}(\mathbb{1}_{X^m})\big\|_{C^{0,\alpha}(X^0_{\mathfrak{c}'}, d)}
$$

$$
(6.16) \qquad \leq C_{11}|\Im(s)| \sum_{X^m \in \mathbf{X}^m} \exp\big(-\Re(s)S_m\phi\big((f^m|_{X^m})^{-1}(y)\big)\big)
$$

$$
= 2C_{11}|\Im(s)|\mathcal{L}^m_{-\Re(s)\phi}(\mathbb{1}_{S^2})(y).
$$

We construct a sequence of continuous functions $p_m \colon \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, as

$$(6.17) \qquad p_m(a) := \big(\mathcal{L}^m_{-a\phi}(\mathbb{1}_{S^2})(y)\big)^{1/m}.$$

By Lemma 3.25 in [LZhe23a], the function $a \mapsto p_m(a) - e^{P(f, -a\phi)}$ converges to 0 as m tends to $+\infty$, uniformly in $a \in [0, 2s_0]$. Recall that $a \mapsto P(f, -a\phi)$ is continuous in $a \in \mathbb{R}$ (see for example, [PU10, Theorem 3.6.1]). Thus by (6.16), there exists a constant $C_{12} > 0$ depending only on f, $\mathcal{C}$, d, α, ϕ, and δ such that for all $m \in \mathbb{N}$ and $s \in \mathbb{C}$ with $|\Im(s)| \geq 2s_0 + 1$ and $|\Re(s) - s_0| \leq s_0$,

$$
(6.18) \qquad \sum_{X^m \in \mathbf{X}^m} \max_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \big\|\mathcal{L}^{(m)}_{-s\phi, \mathfrak{c}', X^m}(\mathbb{1}_{X^m})\big\|_{C^{0,\alpha}(X^0_{\mathfrak{c}'}, d)}
$$

$$
\leq 2C_{11}|\Im(s)|(p_m(\Re(s)))^m \leq C_{12}|\Im(s)|e^{m(\delta + P(f, -\Re(s)\phi))}.
$$

Combining (6.7) with the above inequality, we get for all $s \in \mathbb{C}$ with $|\Im(s)| \geq 2s_0 + 1$ and $|\Re(s) - s_0| \leq s_0$,

$$
\Bigg|Z^{(n)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) - \sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \mathcal{L}^{(n)}_{-s\phi, \mathfrak{c}, X^1}(\mathbb{1}_{X^1})(x_{X^1})\Bigg|
$$

$$
\leq D_\delta|\Im(s)| \sum_{m=2}^{n} \big\|\big|\mathbb{L}^{n-m}_{-s\phi}\big|\big\|_\alpha \Big(\frac{1}{\Lambda^\alpha} \exp(\delta + P(f, -\Re(s)\phi))\Big)^m,
$$

where $D_\delta := C^\alpha C_{12}\Lambda^\alpha > C_{12} > 0$ is a constant depending only on f, $\mathcal{C}$, d, ϕ, α, and δ.

Inequality (6.1) now follows from (6.18) and $D_\delta\Lambda^{-\alpha} \geq C_{12}$. $\qquad\square$

6.2. **Operator norm.** The following theorem is one of the main estimates we need to prove in this paper.

Theorem 6.2. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathrm{post}\, f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$, and $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$ that satisfies the α-strong non-integrability condition. Let $s_0 \in \mathbb{R}$ be the unique positive real number satisfying $P(f, -s_0\phi) = 0$.*

Then there exist constants $\iota \in \mathbb{N}$, $a_0 \in (0, s_0]$, $b_0 \in [2s_0 + 1, +\infty)$, and $\rho \in (0, 1)$ such that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each $n \in \mathbb{N}$, each $s \in \mathbb{C}$ with $|\Re(s) - s_0| \leq a_0$ and $|\Im(s)| \geq b_0$, and each pair of functions $u_{\mathfrak{b}} \in C^{0,\alpha}\big((X^0_{\mathfrak{b}}, d), \mathbb{C}\big)$ and $u_{\mathfrak{w}} \in C^{0,\alpha}\big((X^0_{\mathfrak{w}}, d), \mathbb{C}\big)$ satisfying $\|u_{\mathfrak{b}}\|^{[\Im(s)]}_{C^{0,\alpha}(X^0_{\mathfrak{b}}, d)} \leq 1$ and $\|u_{\mathfrak{w}}\|^{[\Im(s)]}_{C^{0,\alpha}(X^0_{\mathfrak{w}}, d)} \leq 1$, we have

$$(6.19) \qquad \int_{X^0_{\mathfrak{c}}} \Big|\mathcal{L}^{(n\iota)}_{\widetilde{-s\phi}, \mathfrak{c}, \mathfrak{b}}(u_{\mathfrak{b}}) + \mathcal{L}^{(n\iota)}_{\widetilde{-s\phi}, \mathfrak{c}, \mathfrak{w}}(u_{\mathfrak{w}})\Big|^2 \, \mathrm{d}\mu_{-s_0\phi} \leq \rho^n.$$

Here $\mu_{-s_0\phi}$ denotes the unique equilibrium state for the map f and the potential $-s_0\phi$.

We will prove the above theorem at the end of Section 7. Assuming Theorem 6.2, we can establish the following theorem.

Theorem 6.3. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathrm{post}\, f \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$, and $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$ that satisfies the α-strong non-integrability condition. Let $s_0 \in \mathbb{R}$ be the unique positive real number satisfying $P(f, -s_0\phi) = 0$.*

Then there exists a constant $D' = D'(f, \mathcal{C}, d, \alpha, \phi) > 0$ such that for each $\epsilon > 0$, there exist constants $\delta_\epsilon \in (0, s_0)$, $\widetilde{b}_\epsilon \geq 2s_0 + 1$, and $\rho_\epsilon \in (0, 1)$ with the following property:

For each $n \in \mathbb{N}$ and all $s \in \mathbb{C}$ satisfying $|\Re(s) - s_0| < \delta_\epsilon$ and $|\Im(s)| \geq \widetilde{b}_\epsilon$, we have

$$(6.20) \qquad \left\|\left\|\mathbb{L}_{-s\phi}^n\right\|\right\|_\alpha \leq D' |\Im(s)|^{1+\epsilon} \rho_\epsilon^n.$$

Proof. Fix an arbitrary number $\epsilon > 0$. Let $\iota \in \mathbb{N}$, $a_0 \in (0, s_0]$, $b_0 \in [2s_0 + 1, +\infty)$, and $\rho \in (0, 1)$ be constants from Theorem 6.2 depending only on f, $\mathcal{C}$, d, α, and ϕ.

We choose $\iota_0 \in \mathbb{N}$ to be the smallest integer satisfying $\frac{1}{2\iota_0} < \epsilon$, $\iota_0 \geq 2$, and $\frac{\iota_0}{\iota} \in \mathbb{N}$.

Denote

$$(6.21) \qquad \gamma := -\log \max\left\{\rho^{\iota_0/(2\iota)}, \rho_1^{1/2}, \Lambda^{-\alpha}\right\} > 0,$$

where $\rho_1 := \rho_1(f, \mathcal{C}, d, \alpha, H) \in (0, 1)$, with $H := \left\{\widetilde{-t\phi} : t \in \mathbb{R}, |t - s_0| \leq a_0\right\}$ a bounded subset of $C^{0,\alpha}(S^2, d)$, is a constant from Theorem 5.18 depending only on f, $\mathcal{C}$, d, and α in our context here.

We define

$$(6.22) \qquad \rho_\epsilon := e^{-\gamma/(32\iota_0)} \in (0, 1),$$

$$(6.23) \qquad \widetilde{b}_\epsilon := \max\left\{e^{\iota_0\gamma}, (21A_0^2)^{2\iota_0}, 2s_0 + 1\right\} > e.$$

Here $A_0 = A_0\left(f, \mathcal{C}, d, |\phi|_{\alpha,\,(S^2,d)}, \alpha\right) > 2$ is a constant from Lemma 5.12 depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,\,(S^2,d)}$, and α.

Moreover, note that by (3.18), $\left\|\widetilde{-a\phi} - \widetilde{-s_0\phi}\right\|_{C^0(S^2)} \leq |a - s_0| \|\phi\|_{C^0(S^2)} + |P(f, -a\phi) - P(f, -s_0\phi)| + 2\|\log u_{-a\phi} - \log u_{-s_0\phi}\|_{C^0(S^2)}$. Since the function $t \mapsto P(f, t\phi)$ is continuous (see for example, [PU10, Theorem 3.6.1]), $P(f, -s_0\phi) = 0$, and the map $t \mapsto u_{t\phi}$ is continuous on $C^{0,\alpha}(S^2, d)$ equipped with the uniform norm $\|\cdot\|_{C^0(S^2)}$ by Lemma 3.18, we can choose $\delta_\epsilon \in (0, a_0)$ sufficiently small so that if $a \in [s_0 - \delta_\epsilon, s_0 + \delta_\epsilon]$, then

$$(6.24) \qquad |P(f, -a\phi)| \leq -\log \rho_\epsilon \quad \text{and} \quad \left\|\widetilde{-a\phi} - \widetilde{-s_0\phi}\right\|_{C^0(S^2)} \leq \log \min\left\{\rho^{-1/(2\iota)}, \rho_1^{-1/2}\right\}.$$

Fix an arbitrary number $s = a + \mathbf{i}b \in \mathbb{C}$ with $a, b \in \mathbb{R}$ satisfying $|a - s_0| \leq \delta_\epsilon$ and $|b| \geq \widetilde{b}_\epsilon$, and fix an arbitrary pair of complex-valued Hölder continuous functions $u_\mathfrak{b} \in C^{0,\alpha}\left((X_\mathfrak{b}^0, d), \mathbb{C}\right)$ and $u_\mathfrak{w} \in C^{0,\alpha}\left((X_\mathfrak{w}^0, d), \mathbb{C}\right)$ satisfying $\|u_\mathfrak{b}\|_{C^{0,\alpha}(X_\mathfrak{b}^0, d)}^{[b]} \leq 1$ and $\|u_\mathfrak{w}\|_{C^{0,\alpha}(X_\mathfrak{w}^0, d)}^{[b]} \leq 1$.

We denote by $m \in \mathbb{N}$ the smallest integer satisfying

$$(6.25) \qquad m\iota_0\gamma \geq 2\log|b| \geq 0.$$

Then $m \geq 2$ by (6.23).

We first note that by (5.6), the Cauchy–Schwartz inequality, Lemma 5.10, (5.43) in Theorem 5.18, Theorem 6.2, (5.52) in Lemma 5.20, and (6.24), and by denoting $L_{\mathfrak{c}'} := \left|\mathcal{L}_{-s\phi, \mathfrak{c}', \mathfrak{b}}^{(m\iota_0)}(u_\mathfrak{b}) + \mathcal{L}_{-s\phi, \mathfrak{c}', \mathfrak{w}}^{(m\iota_0)}(u_\mathfrak{w})\right|$, we have for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each $x \in X_\mathfrak{c}^0$,

$$\left(\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{-a\phi, \mathfrak{c}, \mathfrak{c}'}^{(m\iota_0)}\left(\left|\mathcal{L}_{-s\phi, \mathfrak{c}', \mathfrak{b}}^{(m\iota_0)}(u_\mathfrak{b}) + \mathcal{L}_{-s\phi, \mathfrak{c}', \mathfrak{w}}^{(m\iota_0)}(u_\mathfrak{w})\right|\right)(x)\right)^2$$

$$= \left(\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{\substack{X \in \mathbf{X}_\mathfrak{c}^{m\iota_0} \\ X \in X_{\mathfrak{c}'}^0}} \left(e^{\frac{1}{2}S_{m\iota_0}\widetilde{-a\phi} + \frac{1}{2}S_{m\iota_0}(\widetilde{-a\phi} - \widetilde{-s_0\phi})} \cdot e^{\frac{1}{2}S_{m\iota_0}\widetilde{-s_0\phi}} L_{\mathfrak{c}'}\right)\left(\left(f^{m\iota_0}|_X\right)^{-1}(x)\right)\right)^2$$

$$\leq \left(\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{-a\phi, \mathfrak{c}, \mathfrak{c}'}^{(m\iota_0)}\left(e^{m\iota_0\|\widetilde{-a\phi} - \widetilde{-s_0\phi}\|_{C^0(S^2)}}\right)(x)\right)\left(\sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}_{-s_0\phi, \mathfrak{c}, \mathfrak{c}'}^{(m\iota_0)}\left(L_{\mathfrak{c}'}^2\right)(x)\right)$$

$$\leq e^{m\iota_0\|\widetilde{-a\phi}-\widetilde{-s_0\phi}\|_{C^0(S^2)}}\left(6\rho_1^{m\iota_0}\max_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\|L_{\mathfrak{c}'}^2\|_{C^{0,\alpha}(X_{\mathfrak{c}'}^0,d)}+\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\int_{X_{\mathfrak{c}'}^0}L_{\mathfrak{c}'}^2\,\mathrm{d}\mu_{-s_0\phi}\right)$$

$$\leq 42A_0\rho_1^{m\iota_0/2}|b|+2\rho^{m\iota_0/(2\iota)}.$$

Combining the above with (6.25), (6.21), and the fact that $\iota_0\geq 2$ and $A_0>2$, we get for each $\mathfrak{c}\in\{\mathfrak{b},\,\mathfrak{w}\}$,

(6.26)
$$\left\|\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\mathcal{L}_{\widetilde{-a\phi},\mathfrak{c},\mathfrak{c}'}^{(m\iota_0)}\left(\left|\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right|\right)\right\|_{C^0(X_{\mathfrak{c}}^0)}$$
$$\leq\left(42A_0|b|^{-2+1}+2|b|^{-2/\iota_0}\right)^{1/2}\leq 7A_0|b|^{-1/\iota_0}.$$

Thus by (5.10), (5.12), (5.7), and (6.26), we get that for each $\mathfrak{c}\in\{\mathfrak{b},\,\mathfrak{w}\}$,

(6.27)
$$\left\|\pi_{\mathfrak{c}}\left(\mathbb{L}_{\widetilde{-s\phi}}^{2m\iota_0}(u_{\mathfrak{b}},u_{\mathfrak{w}})\right)\right\|_{C^0(X_{\mathfrak{c}}^0)}=\left\|\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c},\mathfrak{c}'}^{(m\iota_0)}\left(\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right)\right\|_{C^0(X_{\mathfrak{c}}^0)}$$
$$\leq\left\|\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\mathcal{L}_{\widetilde{-a\phi},\mathfrak{c},\mathfrak{c}'}^{(m\iota_0)}\left(\left|\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right|\right)\right\|_{C^0(X_{\mathfrak{c}}^0)}$$
$$\leq 7A_0|b|^{-1/\iota_0}.$$

By (5.10), (5.12), (5.28) in Lemma 5.12, Lemma 5.20, (6.26), (6.25), and (6.21), we have for each $\mathfrak{c}\in\{\mathfrak{b},\,\mathfrak{w}\}$,

(6.28)
$$\frac{1}{|b|}\left|\pi_{\mathfrak{c}}\left(\mathbb{L}_{\widetilde{-s\phi}}^{2m\iota_0}(u_{\mathfrak{b}},u_{\mathfrak{w}})\right)\right|_{\alpha,\,(X_{\mathfrak{c}}^0,d)}$$
$$=\frac{1}{|b|}\left|\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c},\mathfrak{c}'}^{(m\iota_0)}\left(\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right)\right|_{\alpha,\,(X_{\mathfrak{c}}^0,d)}$$
$$\leq\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}\frac{C_0}{\Lambda^{\alpha m\iota_0}}\left\|\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right\|_{C^{0,\alpha}(X_{\mathfrak{c}'}^0,d)}^{[b]}$$
$$+\sum_{\mathfrak{c}'\in\{\mathfrak{b},\,\mathfrak{w}\}}A_0\left\|\mathcal{L}_{\widetilde{-a\phi},\mathfrak{c},\mathfrak{c}'}^{(m\iota_0)}\left(\left|\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{b}}^{(m\iota_0)}(u_{\mathfrak{b}})+\mathcal{L}_{\widetilde{-s\phi},\mathfrak{c}',\mathfrak{w}}^{(m\iota_0)}(u_{\mathfrak{w}})\right|\right)\right\|_{C^0(X_{\mathfrak{c}}^0)}$$
$$\leq 8A_0C_0\Lambda^{-\alpha m\iota_0}+A_0\left(7A_0|b|^{-1/\iota_0}\right)\leq 7A_0^2|b|^{-2}+7A_0^2|b|^{-1/\iota_0}\leq 14A_0^2|b|^{-1/\iota_0},$$

where $C_0>1$ is a constant depending only on f, $\mathcal{C}$, and d from Lemma 3.13, and $A_0\geq 2C_0>2$ (see Lemma 5.12).

Hence, for each $n\in\mathbb{N}$, by choosing $k\in\mathbb{N}_0$ and $r\in\{0,1,\ldots,2m\iota_0-1\}$ with $n=2m\iota_0 k+r$, we get from (6.27), (6.28), Definition 5.6, and (5.51) in Lemma 5.20 that since $|b|\geq\widetilde{b}_\epsilon$ and $m\geq 2$,

(6.29)
$$\left\|\left|\mathbb{L}_{\widetilde{-s\phi}}^n\right|\right\|_\alpha\leq|b|\left\|\left|\mathbb{L}_{\widetilde{-s\phi}}^{2m\iota_0 k+r}\right|\right\|_\alpha^{[b]}\leq|b|\left(\left\|\left|\mathbb{L}_{\widetilde{-s\phi}}^{2m\iota_0}\right|\right\|_\alpha^{[b]}\right)^k\left\|\left|\mathbb{L}_{\widetilde{-s\phi}}^r\right|\right\|_\alpha^{[b]}$$
$$\leq 4A_0|b|\left(7A_0|b|^{-1/\iota_0}+14A_0^2|b|^{-1/\iota_0}\right)^k\leq 4A_0|b|^{1-\frac{k}{2\iota_0}}$$
$$\leq 4A_0|b|^{1+\frac{1}{2\iota_0}-\frac{2m\iota_0 k+r}{2\iota_0}\frac{1}{2m\iota_0}}\leq 4A_0|b|^{1+\frac{1}{2\iota_0}}|b|^{-\frac{n}{4m\iota_0^2}}$$
$$\leq 4A_0|b|^{1+\epsilon}e^{-\frac{n\log|b|}{8(m-1)\iota_0^2}}\leq 4A_0|b|^{1+\epsilon}\rho_\epsilon^{2n},$$

where the last inequality follows from (6.22) and the fact that m is the smallest integer satisfying (6.25).

We now turn the upper bound for $\left\|\left|\mathbb{L}_{\widetilde{-s\phi}}^n\right|\right\|_\alpha$ in (6.29) into a bound for $\left\|\left|\mathbb{L}_{-s\phi}^n\right|\right\|_\alpha$.

By (5.13), (5.16), (5.22) in Lemma 5.11, and Corollary 5.14, we get

$$
\begin{aligned}
\left\vert\!\left\vert\!\left\vert \mathbb{L}_{-s\phi}^n \right\vert\!\right\vert\!\right\vert_\alpha &= \sup\left\{ \frac{\left\Vert \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}_{-s\phi,\mathfrak{c}',\mathfrak{c}}^{(n)}(v_\mathfrak{c}) \right\Vert_{C^{0,\alpha}(X_{\mathfrak{c}'}^0,d)}}{\max\{\Vert v_\mathfrak{c}\Vert_{C^{0,\alpha}(X_\mathfrak{c}^0,d)} : \mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\}} \right\} \\
&\le e^{nP(f,-a\phi)}\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\sup\left\{ \frac{\left\Vert \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}_{-s\phi,\mathfrak{c}',\mathfrak{c}}^{(n)}(v_\mathfrak{c}/u_{-a\phi}) \right\Vert_{C^{0,\alpha}(X_{\mathfrak{c}'}^0,d)}}{\max\{\Vert v_\mathfrak{c}\Vert_{C^{0,\alpha}(X_\mathfrak{c}^0,d)} : \mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\}} \right\} \\
&\le e^{nP(f,-a\phi)}\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\left\vert\!\left\vert\!\left\vert \mathbb{L}_{-\widetilde{s\phi}}^n \right\vert\!\right\vert\!\right\vert_\alpha \sup\left\{ \frac{\max\{\Vert v_\mathfrak{c}/u_{-a\phi}\Vert_{C^{0,\alpha}(X_\mathfrak{c}^0,d)} : \mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\}}{\max\{\Vert v_\mathfrak{c}\Vert_{C^{0,\alpha}(X_\mathfrak{c}^0,d)} : \mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\}} \right\} \\
&\le e^{nP(f,-a\phi)}\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\left\vert\!\left\vert\!\left\vert \mathbb{L}_{-\widetilde{s\phi}}^n \right\vert\!\right\vert\!\right\vert_\alpha \Vert 1/u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)} \\
&\le e^{nP(f,-a\phi)}\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\left\vert\!\left\vert\!\left\vert \mathbb{L}_{-\widetilde{s\phi}}^n \right\vert\!\right\vert\!\right\vert_\alpha e^{2C_{15}}\left(1+\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\right) \\
&\le \left\vert\!\left\vert\!\left\vert \mathbb{L}_{-\widetilde{s\phi}}^n \right\vert\!\right\vert\!\right\vert_\alpha e^{2C_{15}}\left(1+\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\right)^2 \exp(nP(f,-a\phi)),
\end{aligned}
$$

where the suprema are taken over all $v_\mathfrak{b}\in C^{0,\alpha}\big((X_\mathfrak{b}^0,d),\mathbb{C}\big)$, $v_\mathfrak{w}\in C^{0,\alpha}\big((X_\mathfrak{w}^0,d),\mathbb{C}\big)$, and $\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}$ with $\Vert v_\mathfrak{b}\Vert_{C^0(X_\mathfrak{b}^0)}\Vert v_\mathfrak{w}\Vert_{C^0(X_\mathfrak{w}^0)}\ne 0$. Here the constant $C_{15}=C_{15}(f,\mathcal{C},d,\alpha,T,K)$, with $T:=2s_0$ and $K:=\vert\phi\vert_{\alpha,(S^2,d)}>0$, is defined in (5.23) in Lemma 5.11 and depends only on f, $\mathcal{C}$, d, α, and $\vert\phi\vert_{\alpha,(S^2,d)}$ in our context.

Combining the above inequality with (6.29), (6.23), (6.24), and (5.21) in Lemma 5.11, we get that if $a\in(s_0-\delta_\epsilon,s_0+\delta_\epsilon)$ and $\vert b\vert\ge\widetilde{b}_\epsilon$, then

$$
\left\vert\!\left\vert\!\left\vert \mathbb{L}_{-s\phi}^n \right\vert\!\right\vert\!\right\vert_\alpha \le 4A_0\vert b\vert^{1+\epsilon}\rho_\epsilon^{2n}\rho_\epsilon^{-n}e^{2C_{15}}\left(1+\Vert u_{-a\phi}\Vert_{C^{0,\alpha}(S^2,d)}\right)^2 \le D'\vert b\vert^{1+\epsilon}\rho_\epsilon^n,
$$

where $D':=4A_0 e^{2C_{15}}\big(8\frac{s_0\vert\phi\vert_{\alpha,(S^2,d)}C_0}{1-\Lambda^{-\alpha}}L+2\big)^2\big(e^{2C_{15}}\big)^2>1$, which depends only on f, $\mathcal{C}$, d, α, and ϕ. $\square$

6.3. Bound the symbolic zeta function.

Using Proposition 6.1 and Theorem 6.3, we can get the following bound for the zeta function $\zeta_{\sigma_{A_\Delta},-\phi\circ\pi_\Delta}$ (c.f. (3.21)).

Proposition 6.4. *Let f, $\mathcal{C}$, d, Λ, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that ϕ satisfies the α-strong non-integrability condition, and that $f(\mathcal{C})\subseteq\mathcal{C}$ and no 1-tile in $\mathbf{X}^1(f,\mathcal{C})$ joins opposite sides of $\mathcal{C}$. Then for each $\epsilon>0$ there exist constants $\widetilde{C}_\epsilon>0$ and $\widetilde{a}_\epsilon\in(0,s_0)$ such that*

$$
(6.30) \qquad\qquad \left\vert \sum_{n=1}^{+\infty}\frac{1}{n}Z_{\sigma_{A_\Delta},-\phi\circ\pi_\Delta}^{(n)}(s) \right\vert \le \widetilde{C}_\epsilon\vert\Im(s)\vert^{2+\epsilon}
$$

for all $s\in\mathbb{C}$ with $\vert\Re(s)-s_0\vert<\widetilde{a}_\epsilon$ and $\vert\Im(s)\vert\ge\widetilde{b}_\epsilon$, where $\widetilde{b}_\epsilon\ge 2s_0+1$ is a constant depending only on f, $\mathcal{C}$, d, α, ϕ, and ϵ defined in Theorem 6.3.

Recall $Z_{\sigma_{A_\Delta},-\phi\circ\pi_\Delta}^{(n)}(s)$ defined in (3.20).

Proof. Let $\delta:=\frac{1}{3}\log(\Lambda^\alpha)>0$.

Since $t\mapsto P(f,-t\phi)$ is continuous on $\mathbb{R}$ (see for example, [PU10, Theorem 3.6.1]), we fix $\widetilde{a}_\epsilon\in(0,\delta_\epsilon)\subseteq(0,s_0)$ such that $\vert P(f,-t\phi)\vert<\frac{1}{3}\log(\Lambda^\alpha)$ for each $t\in\mathbb{R}$ with $\vert t-s_0\vert<\widetilde{a}_\epsilon$, where $\delta_\epsilon\in(0,s_0)$ is a constant defined in Theorem 6.3 depending only on f, $\mathcal{C}$, d, α, ϕ, and ϵ.

Fix an arbitrary point $x_{X^1} \in \mathrm{inte}(X^1)$ for each $X^1 \in \mathbf{X}^1$. By Lemmas 5.3, 5.7, and (6.1) in Proposition 6.1, for each $n \geq 2$ and each $s \in \mathbb{C}$ with $|\Re(s) - s_0| < \widetilde{a}_\epsilon$, we have

$$
\left| \sum_{\substack{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \mathcal{L}^{(n)}_{-s\phi, \mathfrak{c}, X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right|
$$

(6.31)
$$
\leq \sum_{\substack{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \left| \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}^{(n-1)}_{-s\phi, \mathfrak{c}, \mathfrak{c}'} \left(\mathcal{L}^{(1)}_{-s\phi, \mathfrak{c}', X^1}(\mathbb{1}_{X^1}) \right)(x_{X^1}) \right|
$$

$$
\leq \left\| \mathbb{L}^{n-1}_{-s\phi} \right\|_\alpha \sum_{\substack{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \max_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \left\| \mathcal{L}^{(1)}_{-s\phi, \mathfrak{c}', X^1}(\mathbb{1}_{X^1}) \right\|_{C^{0,\alpha}(X^0_{\mathfrak{c}'}, d)}
$$

$$
\leq \left\| \mathbb{L}^{n-1}_{-s\phi} \right\|_\alpha D_\delta |\Im(s)| \Lambda^{-\alpha} \exp(\delta + P(f, -\Re(s)\phi)),
$$

where $D_\delta > 0$ is a constant depending only on f, $\mathcal{C}$, d, α, ϕ, and δ from Proposition 6.1.

Hence by (3.20), Proposition 6.1, (6.31), Theorem 6.3, and the choices of δ and $\widetilde{a}_\epsilon$ above, we get that for each $s \in \mathbb{C}$ with $|\Re(s) - s_0| < \widetilde{a}_\epsilon$ and $|\Im(s)| \geq \widetilde{b}_\epsilon$,

$$
\sum_{n=2}^{+\infty} \frac{1}{n} \left| Z^{(n)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) \right| \leq \sum_{n=2}^{+\infty} \frac{1}{n} \left(\left| \sum_{\substack{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \mathcal{L}^{(n)}_{-s\phi, \mathfrak{c}, X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right| \right.
$$

$$
\left. + \left| Z^{(n)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) - \sum_{\substack{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}}} \sum_{\substack{X^1 \in \mathbf{X}^1 \\ X^1 \subseteq X^0_{\mathfrak{c}}}} \mathcal{L}^{(n)}_{-s\phi, \mathfrak{c}, X^1}(\mathbb{1}_{X^1})(x_{X^1}) \right| \right)
$$

$$
\leq \sum_{n=2}^{+\infty} \frac{1}{n} \left(\left\| \mathbb{L}^{n-1}_{-s\phi} \right\|_\alpha D_\delta |\Im(s)| \Lambda^{-\frac{\alpha}{3}} + D_\delta |\Im(s)| \sum_{m=2}^{n} \left\| \mathbb{L}^{n-m}_{-s\phi} \right\|_\alpha \Lambda^{-\frac{m\alpha}{3}} \right)
$$

$$
\leq |\Im(s)|^{2+\epsilon} \sum_{n=2}^{+\infty} \frac{D'}{n} D_\delta \sum_{m=1}^{n} \rho_\epsilon^{n-m} \Lambda^{-\frac{m\alpha}{3}} \leq D' D_\delta |\Im(s)|^{2+\epsilon} \sum_{n=2}^{+\infty} \widetilde{\rho}_\epsilon^n \leq \frac{D' D_\delta}{1 - \widetilde{\rho}_\epsilon} |\Im(s)|^{2+\epsilon},
$$

where the constant $\widetilde{\rho}_\epsilon := \max\{\rho_\epsilon, \Lambda^{-\alpha/3}\} < 1$ depends only on f, $\mathcal{C}$, d, α, ϕ, and ϵ. Here constants $D' \in (0, s_0)$ and $\rho_\epsilon \in (0, 1)$ are from Theorem 6.3 depending only on f, $\mathcal{C}$, d, α, ϕ, and ϵ.

Therefore, by Proposition A.1 (i) in [LZhe23a] and (3.20), we have

$$
\left| \sum_{n=1}^{+\infty} \frac{1}{n} Z^{(n)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) \right| \leq \left| Z^{(1)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) \right| + \sum_{n=2}^{+\infty} \frac{1}{n} \left| Z^{(n)}_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s) \right| \leq \widetilde{C}_\epsilon |\Im(s)|^{2+\epsilon}
$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < \widetilde{a}_\epsilon$ and $|\Im(s)| \geq \widetilde{b}_\epsilon$, where the constant

$$
\widetilde{C}_\epsilon := D' D_\delta (1 - \widetilde{\rho}_\epsilon)^{-1} + 2 \deg f \exp\left(2 s_0 \|\phi\|_{C^0(S^2)}\right)
$$

depends only on f, $\mathcal{C}$, d, α, ϕ, and ϵ. $\qquad \square$

It follows immediately from the above proposition that $\zeta_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s)$ has a non-vanishing holomorphic extension across the vertical line $\Re(s) = s_0$ for high frequency. In order to get a similar theorem for $\zeta_{\sigma_{A_\Delta}, -\phi \circ \pi_\Delta}(s)$ as Theorem D, we just need to establish its holomorphic extension for low frequency.

Proof of Theorem E. Statement (i) of Theorem E has been established in [LZhe23a, Theorem E].

To verify statement (ii) in Theorem E, we assume, in addition, that ϕ satisfies the α-strong non-integrability condition.

Fix an arbitrary $\epsilon > 0$. Let $\widetilde{C}_\epsilon > 0$ and $\widetilde{a}_\epsilon \in (0, s_0)$ be constants from Proposition 6.4, and $\widetilde{b}_\epsilon \geq 2 s_0 + 1$ be a constant from Theorem 6.3, all of which depend only on f, $\mathcal{C}$, d, α, ϕ, and ϵ. The inequality (1.5) follows immediately from (6.30) in Proposition 6.4.

Therefore, by the compactness of $\left[-\widetilde{b}_\epsilon, \widetilde{b}_\epsilon\right]$, we can choose $\widetilde{\epsilon}_0 \in (0, \widetilde{a}_\epsilon) \subseteq (0, s_0)$ small enough such that $\zeta_{\sigma_{A_\triangle}, -\phi \circ \pi_\triangle}(s)$ extends to a non-vanishing holomorphic function on the closed half-plane $\{s \in \mathbb{C} : \Re(s) \geq s_0 - \widetilde{\epsilon}_0\}$ except for a simple pole at $s = s_0$. $\qquad \square$

6.4. Proof of Theorem D.
In this subsection, we give a proof of Theorem D assuming Theorem E.

Proof of Theorem D. Statement (i) is established in [LZhe23a, Theorem D].

To verify statement (ii), we continue with the proof of [LZhe23a, Theorem D] and assume in addition that ϕ satisfies the α-strong non-integrability condition. By statement (ii) in Theorem E and the proof of Claim 1 (in the proof of [LZhe23a, Theorem D] in [LZhe23a, Section 8]), $\mathfrak{D}_{f, -\phi, \deg_f}$ extends to a non-vanishing holomorphic function on $\overline{\mathbb{H}}_{s_0 - \epsilon_0'}$ except for the simple pole at $s = s_0$. Moreover, for each $\epsilon > 0$, there exists a constant $C_\epsilon' > 0$ such that

$$\exp\left(-C_\epsilon' |\Im(s)|^{2+\epsilon}\right) \leq \left|\mathfrak{D}_{f, -\phi, \deg_f}(s)\right| \leq \exp\left(C_\epsilon' |\Im(s)|^{2+\epsilon}\right)$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < \epsilon_0'$ and $|\Im(s)| \geq b_\epsilon$, where $b_\epsilon := \widetilde{b}_\epsilon > 0$ is a constant from Theorem E depending only on f, $\mathcal{C}$, d, ϕ, and ϵ.

Therefore, statement (ii) in Theorem D holds for $a_\epsilon := \min\{\epsilon_0, \widetilde{a}_\epsilon\} > 0$, $b_\epsilon = \widetilde{b}_\epsilon > 0$, and some constant $C_\epsilon > C_\epsilon' > 0$ depending only on f, $\mathcal{C}$, d, ϕ, and ϵ. $\qquad \square$

6.5. Proof of Theorem C.
We first state the following theorem on the logarithmic derivative of the zeta function, which will be proved at the end of this subsection.

Theorem 6.5. *Let $f \colon S^2 \to S^2$ be an expanding Thurston map, and d be a visual metric on S^2 for f. Let $\phi \in C^{0,\alpha}(S^2, d)$ be an eventually positive real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$ that satisfies the α-strong non-integrability condition. Denote by s_0 the unique positive number with $P(f, -s_0\phi) = 0$.*

Then there exists $N_f \in \mathbb{N}$ depending only on f such that for each $n \in \mathbb{N}$ with $n \geq N_f$, the following statement holds for $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$:

There exist constants $a \in (0, s_0)$, $b \geq 2s_0 + 1$, and $D > 0$ such that

$$(6.32) \qquad \left|\frac{\zeta_{F, -\Phi}'(s)}{\zeta_{F, -\Phi}(s)}\right| \leq D|\Im(s)|^{\frac{1}{2}}$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < a$ and $|\Im(s)| \geq b$.

Statement (i) in Theorem C is established in [LZhe23a]. Once Theorem D and Theorem 6.5 are established, statement (ii) in Theorem C follows from standard number-theoretic arguments. More precisely, a proof of statement (ii) in Theorem C, relying on Proposition 3.24, Theorem 6.5, and statement (ii) in Theorem D, is verbatim the same as that of [PS98, Theorem 1] presented in [PS98, Section 3]. We omit this proof here and direct the interested readers to the references cited above.

To prove Theorem 6.5, following the ideas from [PS98], we convert the bounds of the zeta function for an expanding Thurston map from Theorem D to a bound of its logarithmic derivative.

We first record a standard result from complex analysis (see [EE85, Theorem 4.2]) as in [PS98, Section 2].

Lemma 6.6. *Consider $z \in \mathbb{C}$, $R > 0$, and $\delta > 0$. Let $F \colon \Delta \to \mathbb{C}$ is a holomorphic function on the closed disk $\Delta := \{s \in \mathbb{C} : |s - z| \leq R(1 + \delta)^3\}$. Assume that F satisfies the following two conditions:*

(i) *$F(s)$ has no zeros on the subset*

$$\{s \in \mathbb{C} : |s - z| \leq R(1 + \delta)^2, \Re(s) > \Re(z) - R(1 + \delta)\} \subseteq \Delta.$$

(ii) *There exists a constant $U \geq 0$ depending only on z, R, δ, and F such that*

$$\log|F(s)| \leq U + \log|F(z)|$$

for all $s \in \Delta$ with $|s - z| \leq R(1 + \delta)^3$.

Then for each $s \in \Delta$ with $|s - z| \leq R$, we have

$$\left|\frac{F'(s)}{F(s)}\right| \leq \frac{2 + \delta}{\delta}\left(\left|\frac{F'(z)}{F(z)}\right| + \frac{\left(2 + (1 + \delta)^{-2}\right)(1 + \delta)}{R\delta^2} U\right).$$

We will also need a version of the well-known Phragmén–Lindelöf theorem recorded below. See [Ti39, Section 5.65] for the statement and proof of this theorem.

Theorem 6.7 (The Phragmén–Lindelöf Theorem). *Consider real numbers $\delta_1 < \delta_2$. Let $h(s)$ be a holomorphic function on the strip $\{s \in \mathbb{C} : \delta_1 \leq \Re(s) \leq \delta_2\}$. Assume that the following conditions are satisfied:*

(i) *For each $\sigma > 0$, there exist real numbers $C_\sigma > 0$ and $T_\sigma > 0$ such that*

$$|h(\delta + \mathbf{i}t)| \leq C_\sigma e^{\sigma|t|}$$

for all $\delta, t \in \mathbb{R}$ with $\delta_1 \leq \delta \leq \delta_2$ and $|t| \geq T_\sigma$.

(ii) *There exist real numbers $C_0 > 0$, $T_0 > 0$, and $k_1, k_2 \in \mathbb{R}$ such that*

$$|h(\delta_1 + \mathbf{i}t)| \leq C_0|t|^{k_1} \qquad and \qquad |h(\delta_2 + \mathbf{i}t)| \leq C_0|t|^{k_2}$$

for all $t \in \mathbb{R}$ with $|t| \geq T_0$.

Then there exist real numbers $D > 0$ and $T > 0$ such that

$$|h(\delta + \mathbf{i}t)| \leq C|t|^{k(\delta)}$$

for all $\delta, t \in \mathbb{R}$ with $\delta_1 \leq \delta \leq \delta_2$ and $|t| \geq T$, where $k(\delta)$ is the linear function of δ which takes values k_1, k_2 for $\delta = \delta_1, \delta_2$, respectively.

Assuming Theorem D, we establish Theorem 6.5 as follows.

Proof of Theorem 6.5. We choose $N_f \in \mathbb{N}$ as in Remark 1.1. Note that $P\big(f^i, -s_0 S_i^f \phi\big) = iP(f, -s_0\phi) = 0$ for each $i \in \mathbb{N}$ (see for example, [Wal82, Theorem 9.8]). We observe that by Lemma 3.11, it suffices to prove the case $n = N_f = 1$. In this case, $F = f$, $\Phi = \phi$, and there exists a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$, post $f \subseteq \mathcal{C}$, and no 1-tile in $\mathbf{X}^1(f, \mathcal{C})$ joins opposite sides of $\mathcal{C}$.

Let C_ϵ, $a_\epsilon \in (0, s_0)$, and $b_\epsilon \geq 2s_0 + 1$ be constants from Theorem D depending only on f, $\mathcal{C}$, d, α, ϕ, and ϵ. We fix $\epsilon := 1$ throughout this proof.

Define $R := \frac{a_\epsilon}{3}$, $\beta := b_\epsilon + \frac{a_\epsilon}{2}$, and $\delta := \left(\frac{3}{2}\right)^{1/3} - 1$. Note that $R(1 + \delta)^3 = \frac{a_\epsilon}{2}$.

Fix an arbitrary $z \in \mathbb{C}$ with $\Re(z) = s_0 + \frac{a_\epsilon}{4}$ and $|\Im(z)| \geq \beta$. The closed disk

$$\Delta := \left\{s \in \mathbb{C} : |s - z| \leq R(1 + \delta)^3\right\} = \left\{s \in \mathbb{C} : |s - z| \leq a_\epsilon/2\right\}$$

is a subset of $\{s \in \mathbb{C} : |\Re(s) - s_0| < a_\epsilon, |\Im(s)| \geq b_\epsilon\}$. Thus by Theorem D, inequality (1.3) holds for all $s \in \Delta$, and the zeta function $\zeta_{f, -\phi}$ has no zeros in Δ.

For each $s \in \Delta$, by (1.3) in Theorem D and the fact that $|\Im(z)| \geq \beta = b_\epsilon + \frac{a_\epsilon}{2}$,

$$\left|\log\left|\zeta_{f, -\phi}(s)\right| - \log\left|\zeta_{f, -\phi}(z)\right|\right| \leq 2C_\epsilon\left(|\Im(z)| + 2^{-1}a_\epsilon\right)^3 \leq 2^4 C_\epsilon|\Im(z)|^3 =: U.$$

Claim. For each $a \in \mathbb{R}$ with $a > s_0$, there exists a real number $\mathcal{K}(a) > 0$ depending only on f, $\mathcal{C}$, d, ϕ, and a such that $|\zeta'_{f, -\phi}(a + \mathbf{i}t)/\zeta_{f, -\phi}(a + \mathbf{i}t)| \leq \mathcal{K}(a)$ for all $t \in \mathbb{R}$.

To establish the claim, we first fix an arbitrary $a \in \mathbb{R}$ with $a > s_0$. By Corollary 3.20, the topological pressure $P(f, -a\phi) < 0$. It follows from [Li15, Proposition 6.8] that there exist numbers $N_a \in \mathbb{N}$ and $\tau_a \in (0, 1)$ such that for each integer $n \in \mathbb{N}$ with $n \geq N_a$,

$$\sum_{x \in P_{1, f^n}} \exp(-aS_n\phi(x)) \leq \tau_a^n.$$

Since the zeta function $\zeta_{f, -\phi}$ converges uniformly and absolutely to a non-vanishing holomorphic function on $\left\{s \in \mathbb{C} : \Re(s) \geq \frac{a + s_0}{2}\right\}$ (see Proposition 3.24), we get from (3.21), Theorem 3.20 (ii) in

[LZhe23a], and (3.6) that

$$\left|\frac{\zeta'_{f,-\phi}(a+\mathbf{i}t)}{\zeta_{f,-\phi}(a+\mathbf{i}t)}\right| = \left|\sum_{n=1}^{+\infty}\frac{1}{n}\sum_{x\in P_{1,f^n}}(S_n\phi(x))\exp(-(a+\mathbf{i}t)S_n\phi(x))\right|$$

$$\leq \|\phi\|_{C^0(S^2)}\sum_{n=1}^{+\infty}\sum_{x\in P_{1,f^n}}\exp(-aS_n\phi(x))$$

$$\leq \|\phi\|_{C^0(S^2)}\left(\sum_{n=N_a+1}^{+\infty}\tau_a^n + \sum_{n=1}^{N_a}\operatorname{card}P_{1,f^n}\right) \leq \mathcal{K}(a),$$

for all $t \in \mathbb{R}$, where $\mathcal{K}(a) := \|\phi\|_{C^0(S^2)}\left(\frac{1}{1-\tau_a} + N_a + \sum_{n=1}^{N_a}(\deg f)^n\right)$ is a constant depending only on f, $\mathcal{C}$, d, ϕ, and a. This establishes the claim.

Hence by Lemma 6.6, the claim with $a := s_0 + \frac{a_\epsilon}{4}$, and the choices of U, R, and δ above, we get that for all $s \in \Delta$ with $\Im(s) = \Im(z)$ and $\left|\Re(s) - \left(s_0 + \frac{a_\epsilon}{4}\right)\right| \leq R = \frac{a_\epsilon}{3}$, we have

$$(6.33) \qquad \left|\frac{\zeta'_{f,-\phi}(s)}{\zeta_{f,-\phi}(s)}\right| \leq \frac{2+\delta}{\delta}\left(\mathcal{K}\left(s_0 + \frac{a_\epsilon}{4}\right) + \frac{2^4 C_\epsilon\big(2+(1+\delta)^{-2}\big)(1+\delta)}{R\delta^2}|\Im(z)|^3\right) \leq C_{19}|\Im(s)|^3,$$

where $C_{19} := \frac{2+\delta}{\delta}\big(\mathcal{K}\big(s_0 + \frac{a_\epsilon}{4}\big) + \frac{2^4 C_\epsilon(2+(1+\delta)^{-2})(1+\delta)}{R\delta^2}\big)$ is a constant depending only on f, $\mathcal{C}$, d, α, and ϕ. Recall that the only restriction on $\Im(z)$ is that $|\Im(z)| \geq \beta$. Thus (6.33) holds for all $s \in \mathbb{C}$ with $\left|\Re(s) - \left(s_0 + \frac{a_\epsilon}{4}\right)\right| \leq \frac{a_\epsilon}{3}$ and $|\Im(s)| \geq \beta$.

By Theorem D, $h(s) := \frac{\zeta'_{f,-\phi}(s)}{\zeta_{f,-\phi}(s)} + \frac{1}{s-s_0}$ is holomorphic on $\{s \in \mathbb{C} : |\Re(s) - s_0| < a_\epsilon\}$. Applying the Phragmén–Lindelöf theorem (Theorem 6.7) to $h(s)$ on the strip $\{s \in \mathbb{C} : \delta_1 \leq \Re(s) \leq \delta_2\}$ with $\delta_1 := s_0 - \frac{a_\epsilon}{12}$ and $\delta_2 := s_0 + \frac{a_\epsilon}{200}$. It follows from (6.33) that condition (i) of Theorem 6.7 holds. On the other hand, (6.33) and the claim above guarantees condition (ii) of Theorem 6.7 with $k_1 := 3$ and $k_2 := 0$. Hence by Theorem 6.7, there exist constants $\widetilde{D} > 0$ and $b \geq 2s_0 + 1$ depending only on f, $\mathcal{C}$, d, α, and ϕ such that $|h(s)| \leq \widetilde{D}|\Im(s)|^{1/2}$ for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| \leq \frac{a_\epsilon}{200}$ and $|\Im(s)| \geq b$.

Therefore inequality (6.32) holds for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| \leq \frac{a_\epsilon}{200} =: a$ and $|\Im(s)| \geq b$, where $a \in (0, s_0)$, $b \geq 2s_0 + 1$, and $D := \widetilde{D} + 1$ are constants depending only on f, $\mathcal{C}$, d, α, and ϕ.   □

7. The Dolgopyat cancellation estimate

We adapt the arguments of D. Dolgopyat [Do98] in our metric-topological setting, aiming to prove Theorem 6.2 at the end of this section. In Subsection 7.1, we first give a formulation of the α-*strong non-integrability condition*, $\alpha \in (0, 1]$, for our setting and then show its independence on the choice of the Jordan curve $\mathcal{C}$. In Subsection 7.2, a consequence of the α-strong non-integrability condition that we will use in the remaining part of this section is formulated in Proposition 7.5. We remark that it is crucial for the arguments in Subsection 7.3 to have the same exponent $\alpha \in (0, 1]$ in both the lower bound and the upper bound in (7.25). The definition of the Dolgopyat operator $\mathbb{M}_{J,s,\phi}$ in our context is given in Definition 7.7 after important constants in the construction are carefully chosen. In Subsection 7.3, we adapt the cancellation arguments of D. Dolgopyat to establish the l^2-bound in Theorem 6.2.

7.1. **Strong non-integrability.**

Definition 7.1 (Strong non-integrability condition). Let $f: S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. Fix $\alpha \in (0, 1]$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent α.

  (1) We say that ϕ satisfies the $(\mathcal{C}, \alpha)$-*strong non-integrability condition* (with respect to f and d), for a Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} f \subseteq \mathcal{C}$, if there exist numbers $N_0, M_0 \in \mathbb{N}$, $\varepsilon \in (0, 1)$, and M_0-tiles $Y_{\mathfrak{b}}^{M_0} \in \mathbf{X}_{\mathfrak{b}}^{M_0}(f, \mathcal{C})$, $Y_{\mathfrak{w}}^{M_0} \in \mathbf{X}_{\mathfrak{w}}^{M_0}(f, \mathcal{C})$ such that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each integer $M \geq M_0$, and each M-tile $X \in \mathbf{X}^M(f, \mathcal{C})$ with $X \subseteq Y_{\mathfrak{c}}^{M_0}$, there exist two points $x_1(X), x_2(X) \in X$ with the following properties:

   (i) $\min\{d(x_1(X), S^2 \setminus X), d(x_2(X), S^2 \setminus X), d(x_1(X), x_2(X))\} \geq \varepsilon \operatorname{diam}_d(X)$, and

(ii) for each integer $N \geq N_0$, there exist two $(N + M_0)$-tiles $X_{\mathfrak{c},1}^{N+M_0}$, $X_{\mathfrak{c},2}^{N+M_0} \in \mathbf{X}^{N+M_0}(f, \mathcal{C})$ such that $Y_{\mathfrak{c}}^{M_0} = f^N(X_{\mathfrak{c},1}^{N+M_0}) = f^N(X_{\mathfrak{c},2}^{N+M_0})$, and that

$$(7.1) \qquad \frac{|S_N\phi(\varsigma_1(x_1(X))) - S_N\phi(\varsigma_2(x_1(X))) - S_N\phi(\varsigma_1(x_2(X))) + S_N\phi(\varsigma_2(x_2(X)))|}{d(x_1(X), x_2(X))^\alpha} \geq \varepsilon,$$

where we write $\varsigma_1 := \left(f^N|_{X_{\mathfrak{c},1}^{N+M_0}}\right)^{-1}$ and $\varsigma_2 := \left(f^N|_{X_{\mathfrak{c},2}^{N+M_0}}\right)^{-1}$.

(2) We say that ϕ satisfies the α-*strong non-integrability condition* (with respect to f and d) if ϕ satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition with respect to f and d for some Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} f \subseteq \mathcal{C}$.

(3) We say that ϕ satisfies the *strong non-integrability condition* (with respect to f and d) if ϕ satisfies the α'-strong non-integrability condition with respect to f and d for some $\alpha' \in (0, \alpha]$.

For given f, d, and α as in Definition 7.1, if $\phi \in C^{0,\alpha}(S^2, d)$ satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition for some Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} f \subseteq \mathcal{C}$, then we fix the choices of N_0, M_0, ε, $Y_{\mathfrak{b}}^{M_0}$, $Y_{\mathfrak{w}}^{M_0}$, $x_1(X)$, $x_2(X)$, $X_{\mathfrak{b},1}^{N+M_0}$, $X_{\mathfrak{w},1}^{N+M_0}$ as in Definition 7.1, and say that something depends only on f, d, α, and ϕ even if it also depends on some of these choices.

We will see in the following lemma that the strong non-integrability condition is independent of the Jordan curve $\mathcal{C}$.

Lemma 7.2. *Let f, d, α satisfies the Assumptions. Let $\mathcal{C}$ and $\widehat{\mathcal{C}}$ be Jordan curves on S^2 with $\operatorname{post} f \subseteq \mathcal{C} \cap \widehat{\mathcal{C}}$. Let $\phi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent α. Fix arbitrary integers $n, \widehat{n} \in \mathbb{N}$. Let $F := f^n$ and $\widehat{F} := f^{\widehat{n}}$ be iterates of f. Then $\Phi := S_n^f \phi$ satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition with respect to F and d if and only if $\widehat{\Phi} := S_{\widehat{n}}^f \phi$ satisfies the $(\widehat{\mathcal{C}}, \alpha)$-strong non-integrability condition with respect to $\widehat{F}$ and d.*

In particular, if ϕ satisfies the α-strong non-integrability condition with respect to f and d, then it satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition with respect to f and d.

Proof. Let $\Lambda > 1$ be the expansion factor of the visual metric d for f. Note that $\operatorname{post} f = \operatorname{post} F = \operatorname{post} \widehat{F}$, and that it follows immediately from Lemma 3.8 that d is a visual metric for both F and $\widehat{F}$.

By Lemma 3.8 (ii) and (v), there exist numbers $C_{20} \in (0, 1)$ and $l \in \mathbb{N}$ such that for each $\widehat{m} \in \mathbb{N}_0$, each $\widehat{X} \in \mathbf{X}^{\widehat{m}}(\widehat{F}, \widehat{\mathcal{C}})$, there exists $X \in \mathbf{X}^{\lceil \widehat{m}\widehat{n}/n \rceil + l}(F, \mathcal{C})$ such that $X \subseteq \widehat{X}$ and $\operatorname{diam}_d(X) \geq C_{20} \operatorname{diam}_d(\widehat{X})$.

By symmetry, it suffices to show the forward implication in the first statement of Lemma 7.2.

We assume that Φ satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition with respect to F and d. We use the choices of numbers N_0, M_0, ε, tiles $Y_{\mathfrak{b}}^{M_0} \in \mathbf{X}_{\mathfrak{b}}^{M_0}(F, \mathcal{C})$, $Y_{\mathfrak{w}}^{M_0} \in \mathbf{X}_{\mathfrak{w}}^{M_0}(F, \mathcal{C})$, $X_{\mathfrak{c},1}^{N+M_0}, X_{\mathfrak{c},2}^{N+M_0} \in \mathbf{X}^{N+M_0}(F, \mathcal{C})$, points $x_1(X)$, $x_2(X)$, and functions ς_1, ς_2 as in Definition 7.1 (with f and ϕ replaced by F and Φ, respectively).

It follows from Lemma 3.8 (ii) and (v) again that we can choose an integer $\widehat{M_0} \in \mathbb{N}$ large enough such that the following statements hold:

(1) $\lceil \widehat{M_0}\widehat{n}/n \rceil + l \geq M_0$.

(2) There exist $\widehat{M_0}$-tiles $\widehat{Y}_{\mathfrak{b}}^{\widehat{M_0}} \in \mathbf{X}_{\mathfrak{b}}^{\widehat{M_0}}(\widehat{F}, \widehat{\mathcal{C}})$ and $\widehat{Y}_{\mathfrak{w}}^{\widehat{M_0}} \in \mathbf{X}_{\mathfrak{w}}^{\widehat{M_0}}(\widehat{F}, \widehat{\mathcal{C}})$ such that $\widehat{Y}_{\mathfrak{b}}^{\widehat{M_0}} \subseteq \operatorname{inte}(Y_{\mathfrak{b}}^{M_0})$ and $\widehat{Y}_{\mathfrak{w}}^{\widehat{M_0}} \subseteq \operatorname{inte}(Y_{\mathfrak{w}}^{M_0})$.

We define the following constants:

$$(7.2) \qquad \widehat{N_0} := \left\lceil \frac{1}{\alpha\widehat{n}} \log_\Lambda \frac{2\,|\phi|_{\alpha,\,(S^2, d)}\, C_0 C^{2\alpha}}{(1 - \Lambda^{-\alpha})\varepsilon^{1+\alpha}(1 - C_{20})} \right\rceil.$$

$$(7.3) \qquad \widehat{\varepsilon} := \varepsilon C_{20} \in (0, \varepsilon).$$

For each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each integer $\widehat{M} \geq \widehat{M_0}$, and each $\widehat{M}$-tile $\widehat{X} \in \mathbf{X}^{\widehat{M}}(\widehat{F}, \widehat{\mathcal{C}})$ with $\widehat{X} \subseteq \widehat{Y}_{\mathfrak{c}}^{\widehat{M_0}}$, we denote $M := \lceil \widehat{M}\widehat{n}/n \rceil + l \geq M_0$, and choose an M-tile $X \in \mathbf{X}^M(F, \mathcal{C})$ with

$$(7.4) \qquad X \subseteq \widehat{X} \qquad \text{and} \qquad \operatorname{diam}_d(X) \geq C_{20} \operatorname{diam}_d(\widehat{X}).$$

Define, for each $i \in \{1, 2\}$,

$$(7.5) \qquad\qquad \widehat{x}_i(\widehat{X}) := x_i(X).$$

We need to verify Properties (i) and (ii) in Definition 7.1 for the $(\widehat{\mathcal{C}}, \alpha)$-strong non-integrability condition of $\widehat{\Phi}$ with respect to $\widehat{F}$ and d.

Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $\widehat{M} \in \mathbb{N}$, and $\widehat{X} \in \mathbf{X}^{\widehat{M}}(\widehat{F}, \widehat{\mathcal{C}})$ with $\widehat{M} \geq \widehat{M}_0$ and $\widehat{X} \subseteq \widehat{Y}_{\mathfrak{c}}^{\widehat{M}_0}$.

Property (i). By (7.4), (7.5), (7.3), and Property (i) for the $(\mathcal{C}, \alpha)$-strong non-integrability condition of Φ with respect to F and d, we get

$$d(\widehat{x}_1(\widehat{X}), \widehat{x}_2(\widehat{X})) / \operatorname{diam}_d(\widehat{X}) \geq d(x_1(X), x_2(X)) / \big(C_{20}^{-1} \operatorname{diam}_d(X)\big) \geq \varepsilon C_{20} = \widehat{\varepsilon},$$

and for each $i \in \{1, 2\}$,

$$d(\widehat{x}_i(\widehat{X}), S^2 \setminus \widehat{X}) / \operatorname{diam}_d(\widehat{X}) \geq d(x_i(X), S^2 \setminus X)\big(C_{20}^{-1} \operatorname{diam}_d(X)\big) \geq \varepsilon C_{20} = \widehat{\varepsilon}.$$

Property (ii). Fix an arbitrary integer $\widehat{N} \geq \widehat{N}_0$. Choose an integer $N \geq N_0$ large enough so that $Nn > \widehat{N}\widehat{n}$.

By Proposition 3.6 (i) and (vii), for each $i \in \{1, 2\}$, since F^N maps $X_{\mathfrak{c},i}^{N+M_0}$ injectively onto $Y_{\mathfrak{c}}^{M_0}$ and $\widehat{Y}_{\mathfrak{c}}^{\widehat{M}_0} \subseteq \operatorname{inte}\big(Y_{\mathfrak{c}}^{M_0}\big)$, we have

$$\varsigma_i\big(\widehat{Y}_{\mathfrak{c}}^{\widehat{M}_0}\big) \in \mathbf{X}^{\widehat{M}_0\widehat{n}+Nn}(f, \widehat{\mathcal{C}}),$$

where $\varsigma_i = \big(F^N\big|_{X_{\mathfrak{c},i}^{N+M_0}}\big)^{-1}$. Define, for each $i \in \{1, 2\}$,

$$\widehat{X}_{\mathfrak{c},i}^{\widehat{N}+\widehat{M}_0} := f^{Nn-\widehat{N}\widehat{n}}\big(\varsigma_i\big(\widehat{Y}_{\mathfrak{c}}^{\widehat{M}_0}\big)\big) \in \mathbf{X}^{\widehat{N}\widehat{n}+\widehat{M}_0\widehat{n}}(f, \widehat{\mathcal{C}}) = \mathbf{X}^{\widehat{N}+\widehat{M}_0}(\widehat{F}, \widehat{\mathcal{C}}),$$

and write $\widehat{\varsigma}_i = \big(\widehat{F}^{\widehat{N}}\big|_{\widehat{X}_{\mathfrak{c},i}^{\widehat{N}+\widehat{M}_0}}\big)^{-1} = \big(f^{\widehat{N}\widehat{n}}\big|_{\widehat{X}_{\mathfrak{c},i}^{\widehat{N}+\widehat{M}_0}}\big)^{-1}$. Note that $f^{Nn-\widehat{N}\widehat{n}} \circ \varsigma_i = \widehat{\varsigma}_i$.

By (7.4), (7.5), Properties (i) and (ii) for the $(\mathcal{C}, \alpha)$-strong non-integrability condition of Φ with respect to F and d, Lemmas 3.13, 3.8 (ii), (7.2), and (7.3), we have

$$\frac{\left| S_{\widehat{N}}^{\widehat{F}}\widehat{\Phi}(\widehat{\varsigma}_1(\widehat{x}_1(\widehat{X}))) - S_{\widehat{N}}^{\widehat{F}}\widehat{\Phi}(\widehat{\varsigma}_2(\widehat{x}_1(\widehat{X}))) - S_{\widehat{N}}^{\widehat{F}}\widehat{\Phi}(\widehat{\varsigma}_1(\widehat{x}_2(\widehat{X}))) + S_{\widehat{N}}^{\widehat{F}}\widehat{\Phi}(\widehat{\varsigma}_2(\widehat{x}_2(\widehat{X}))) \right|}{d(\widehat{x}_1(\widehat{X}), \widehat{x}_2(\widehat{X}))^\alpha}$$

$$= \frac{\left| S_{\widehat{N}\widehat{n}}^{f}\phi(\widehat{\varsigma}_1(x_1(X))) - S_{\widehat{N}\widehat{n}}^{f}\phi(\widehat{\varsigma}_2(x_1(X))) - S_{\widehat{N}\widehat{n}}^{f}\phi(\widehat{\varsigma}_1(x_2(X))) + S_{\widehat{N}\widehat{n}}^{f}\phi(\widehat{\varsigma}_2(x_2(X))) \right|}{d(x_1(X), x_2(X))^\alpha}$$

$$\geq \frac{\left| S_{Nn}^{f}\phi(\varsigma_1(x_1(X))) - S_{Nn}^{f}\phi(\varsigma_2(x_1(X))) - S_{Nn}^{f}\phi(\varsigma_1(x_2(X))) + S_{Nn}^{f}\phi(\varsigma_2(x_2(X))) \right|}{d(x_1(X), x_2(X))^\alpha}$$

$$- \sum_{i \in \{1,2\}} \frac{\left| S_{Nn-\widehat{N}\widehat{n}}^{f}\phi(\varsigma_i(x_1(X))) - S_{Nn-\widehat{N}\widehat{n}}^{f}\phi(\varsigma_i(x_2(X))) \right|}{d(x_1(X), x_2(X))^\alpha}$$

$$\geq \frac{\left| S_{N}^{F}\Phi(\varsigma_1(x_1(X))) - S_{N}^{F}\Phi(\varsigma_2(x_1(X))) - S_{N}^{F}\Phi(\varsigma_1(x_2(X))) + S_{N}^{F}\Phi(\varsigma_2(x_2(X))) \right|}{d(x_1(X), x_2(X))^\alpha}$$

$$- \sum_{i \in \{1,2\}} \frac{|\phi|_{\alpha, (S^2, d)}\, C_0}{1 - \Lambda^{-\alpha}} \cdot \frac{d\big(\big(f^{Nn-\widehat{N}\widehat{n}} \circ \varsigma_i\big)(x_1(X)), \big(f^{Nn-\widehat{N}\widehat{n}} \circ \varsigma_i\big)(x_2(X))\big)}{\varepsilon^\alpha(\operatorname{diam}_d(X))^\alpha}$$

$$\geq \varepsilon - \sum_{i \in \{1,2\}} \frac{|\phi|_{\alpha, (S^2, d)}\, C_0}{1 - \Lambda^{-\alpha}} \cdot \frac{\operatorname{diam}_d\big(\big(f^{Nn-\widehat{N}\widehat{n}} \circ \varsigma_i\big)(X)\big)}{\varepsilon^\alpha(\operatorname{diam}_d(X))^\alpha}$$

$$\geq \varepsilon - \frac{2\,|\phi|_{\alpha, (S^2, d)}\, C_0}{1 - \Lambda^{-\alpha}} \cdot \frac{C^\alpha \Lambda^{-\alpha(Mn+Nn-(Nn-\widehat{N}\widehat{n}))}}{\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha Mn}}$$

$$\geq \varepsilon - \frac{2\,|\phi|_{\alpha, (S^2, d)}\, C_0 C^{2\alpha}}{(1 - \Lambda^{-\alpha})\varepsilon^\alpha}\Lambda^{-\alpha \widehat{N}_0\widehat{n}} \geq \varepsilon - \varepsilon(1 - C_{20}) = \widehat{\varepsilon},$$

where $C \geq 1$ is a constant from Lemma 3.8 and $C_0 > 1$ is a constant from Lemma 3.13, both of which depend only on f, $\mathcal{C}$, and d.

The first statement of Lemma 7.2 is now established. The second statement is a special case of the first statement. $\qquad\square$

Proposition 7.3. *Let f, d, α satisfy the Assumptions. Fix $\phi \in C^{0,\alpha}(S^2, d)$. If ϕ satisfies the α-strong non-integrability condition (in the sense of Definition 7.1), then ϕ is non-locally integrable (in the sense of Definition 8.3).*

Proof. We argue by contradiction and assume that ϕ is locally integrable and satisfies the α-strong non-integrability condition.

Let $\Lambda > 1$ be the expansion factor of d for f. We first fix a Jordan curve $\mathcal{C} \subseteq S^2$ containing post f. Then we fix N_0, M_0, $Y_{\mathfrak{b}}^{M_0}$, and $Y_{\mathfrak{w}}^{M_0}$ as in Definition 7.1. We choose $M := M_0$ and consider an arbitrary M-tile $X \in \mathbf{X}^M(f, \mathcal{C})$ with $X \subseteq Y_{\mathfrak{b}}^{M_0}$. We fix $x_1(X), x_2(X) \in X$ satisfying Properties (i) and (ii) in Definition 7.1 (1). By Theorem F in [LZhe23a], $\phi = K + \beta \circ f - \beta$ for some constant $K \in \mathbb{C}$ and some Hölder continuous function $\beta \in C^{0,\alpha}((S^2, d), \mathbb{C})$.

Then by Property (ii) in Definition 7.1 (1), for each $N \geq N_0$,

$$|\beta(\varsigma_1(x_1(X))) - \beta(\varsigma_2(x_1(X))) - \beta(\varsigma_1(x_2(X))) + \beta(\varsigma_2(x_2(X)))| / d(x_1(X), x_2(X))^\alpha \geq \varepsilon > 0,$$

where $\varsigma_1 := \left(f^N \big|_{X_{\mathfrak{c},1}^{N+M_0}} \right)^{-1}$ and $\varsigma_2 := \left(f^N \big|_{X_{\mathfrak{c},2}^{N+M_0}} \right)^{-1}$. Combining the above with Property (i) in Definition 7.1 and Proposition 3.6 (i), we get

$$\frac{2 |\beta|_{\alpha, (S^2, d)} \left(\max\{ \mathrm{diam}_d (Y^{N+M_0}) : Y^{N+M_0} \in \mathbf{X}^{N+M_0}(f, \mathcal{C}) \} \right)^\alpha}{\varepsilon^\alpha (\mathrm{diam}_d(X))^\alpha} \geq \varepsilon > 0.$$

Thus by Lemma 3.8 (ii), $2 |\beta|_{\alpha, (S^2, d)} \frac{C^\alpha \Lambda^{-\alpha N - \alpha M_0}}{C^{-\alpha} \Lambda^{-\alpha M_0}} \geq \varepsilon^{1+\alpha} > 0$, where $C \geq 1$ is a constant from Lemma 3.8 depending only on f, $\mathcal{C}$, and d. This is impossible since $N \geq N_0$ is arbitrary. $\qquad\square$

7.2. Dolgopyat operator.

We now fix an expanding Thurston map $f \colon S^2 \to S^2$, a visual metric d on S^2 for f with expansion factor $\Lambda > 1$, a Jordan curve $\mathcal{C} \subseteq S^2$ with $f(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$, and an eventually positive real-valued Hölder continuous function $\phi \in C^{0,\alpha}(S^2, d)$ that satisfies the $(\mathcal{C}, \alpha)$-strong non-integrability condition. We use the notations from Definition 7.1 below.

We set the following constants that will be repeatedly used in this section. We will see that all these constants defined from (7.6) to (7.12) below depend only on f, $\mathcal{C}$, d, α, and ϕ.

$$(7.6) \qquad m_0 := \max\left\{ \lceil \alpha^{-1} \log_\Lambda (8C_1 \varepsilon^{\alpha-1}) \rceil, \lceil \log_\Lambda (10\varepsilon^{-1} C^2) \rceil \right\} \geq 1.$$

$$(7.7) \qquad \delta_0 := \min\left\{ (2C_1)^{-1}, \varepsilon^2 C^{-2}/20 \right\} \in (0, 1).$$

$$(7.8) \qquad b_0 := \max\left\{ 2s_0 + 1, C_0 T_0 / (1 - \Lambda^{-\alpha}), 2A_0 |\widetilde{-s_0\phi}|_{\alpha, (S^2, d)} / (1 - \Lambda^{-\alpha}) \right\}.$$

$$(7.9) \qquad A := \max\{ 3C_{10} T_0, 4A_0 \}.$$

$$(7.10) \qquad \epsilon_1 := \min\left\{ \pi\delta_0/16, (4A)^{-1} \Lambda^{-M_0} \right\} \in (0, 1).$$

$$(7.11) \qquad N_1 := \max\left\{ N_0, \lceil \alpha^{-1} \log_\Lambda (\max\{ 2^{10} A, 1280 A \Lambda C^2 / (\varepsilon\delta_0), 4A_0, 4C_{10} \}) \rceil \right\}.$$

$$(7.12) \qquad \eta := \min\left\{ 2^{-12}, \left(\frac{\varepsilon\delta_0\epsilon_1}{1280\Lambda C^2} \right)^2, \frac{A\epsilon_1\varepsilon^\alpha}{240 C_{10} C^2} \Lambda^{-2\alpha m_0 - 1} (\mathrm{LIP}_d(f))^{-\alpha N_1} \right\}.$$

Here the constants $M_0 \in \mathbb{N}$, $N_0 \in \mathbb{N}$, and $\varepsilon \in (0, 1)$ depending only on f, d, $\mathcal{C}$, and ϕ are from Definition 7.1; the constant s_0 is the unique positive real number satisfying $P(f, -s_0\phi) = 0$; the constant $C \geq 1$ depending only on f, d, and $\mathcal{C}$ is from Lemma 3.8; the constant $C_0 > 1$ depending only on f, d, and $\mathcal{C}$ is from Lemma 3.13; the constant $C_1 > 0$ depending only on f, d, $\mathcal{C}$, ϕ, and α is from Lemma 3.15; the constant $A_0 > 2$ depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha, (S^2, d)}$, and α is from Lemma 5.12; the constant $C_{10} = C_{10}(f, \mathcal{C}, d, \alpha, T_0) > 1$ depending only on f, $\mathcal{C}$, d, α, and ϕ is defined in (5.2) from Lemma 5.1; and the constant $T_0 > 0$ depending only on f, $\mathcal{C}$, d, ϕ, and α is defined in (5.31), and according to Lemma 5.11 satisfies

$$(7.13) \qquad \sup\{ |\widetilde{a\phi}|_{\alpha, (S^2, d)} : a \in \mathbb{R}, |a| \leq 2s_0 \} \leq T_0.$$

We denote for each $b \in \mathbb{R}$ with $|b| \geq 1$,

$$(7.14) \qquad \mathfrak{C}_b := \left\{ X \in \mathbf{X}^{m(b)}(f, \mathcal{C}) : X \subseteq Y_{\mathfrak{b}}^{M_0} \cup Y_{\mathfrak{w}}^{M_0} \right\},$$

where we write

$$(7.15) \qquad m(b) := \left\lceil \alpha^{-1} \log_\Lambda (C|b|/\epsilon_1) \right\rceil.$$

Note that by (7.10),

$$m(b) \geq \log_\Lambda(1/\epsilon_1) \geq M_0,$$

and if $X \in \mathfrak{C}_b$, then $\operatorname{diam}_d(X) \leq \left(\frac{\epsilon_1}{|b|} \right)^{1/\alpha}$ by Lemma 3.8 (ii).

For each $X \in \mathfrak{C}_b$, we now fix choices of tiles $\mathfrak{X}_1(X), \mathfrak{X}_2(X) \in \mathbf{X}^{m(b)+m_0}(f, \mathcal{C})$ and $\mathfrak{X}_1'(X), \mathfrak{X}_2'(X) \in \mathbf{X}^{m(b)+2m_0}(f, \mathcal{C})$ in such a way that for each $i \in \{1, 2\}$,

$$(7.16) \qquad x_i(X) \in \mathfrak{X}_i'(X) \subseteq \mathfrak{X}_i(X).$$

By Property (i) in Definition 7.1, (7.6), and Lemma 3.8 (ii) and (v), it is easy to see that the constant m_0 we defined in (7.6) is large enough so that the following inequalities hold:

$$(7.17) \qquad d(\mathfrak{X}_i(X), S^2 \setminus X) \geq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)},$$

$$(7.18) \qquad \operatorname{diam}_d(\mathfrak{X}_i(X)) \leq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)},$$

$$(7.19) \qquad d(\mathfrak{X}_i'(X), S^2 \setminus \mathfrak{X}_i(X)) \geq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)-m_0},$$

$$(7.20) \qquad \operatorname{diam}_d(\mathfrak{X}_i'(X)) \leq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)-m_0}$$

for $i \in \{1, 2\}$, and that

$$(7.21) \qquad d(\mathfrak{X}_1(X), \mathfrak{X}_2(X)) \geq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)}.$$

For each $X \in \mathfrak{C}_b$ and each $i \in \{1, 2\}$, we define a function $\psi_{i,X} \colon S^2 \to \mathbb{R}$ by

$$(7.22) \qquad \psi_{i,X}(x) := \frac{d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha}{d(x, \mathfrak{X}_i'(X))^\alpha + d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha}$$

for $x \in S^2$. Note that

$$(7.23) \qquad \psi_{i,X}(x) = 1 \text{ if } x \in \mathfrak{X}_i'(X), \quad \text{and} \quad \psi_{i,X}(x) = 0 \text{ if } x \notin \mathfrak{X}_i(X).$$

Definition 7.4. We say that a subset $J \subseteq \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$ has a *full projection* if $\pi_3(J) = \mathfrak{C}_b$, where $\pi_3 \colon \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b \to \mathfrak{C}_b$ is the projection $\pi_3(j, i, X) = X$. We write $\mathcal{F}$ for the collection of all subsets of $\{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$ that have full projections.

For a subset $J \subseteq \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$, we define a function $\beta_J \colon S^2 \to \mathbb{R}$ as

$$(7.24) \qquad \beta_J(x) := \begin{cases} 1 - \frac{\eta}{4} \sum_{i \in \{1,2\}} \sum_{\substack{X \in \mathfrak{C}_b \\ (1,i,X) \in J}} \psi_{i,X}\big(f^{N_1}(x)\big) & \text{if } x \in \operatorname{inte}\big(X_{\mathfrak{b},1}^{N_1+M_0}\big) \cup \operatorname{inte}\big(X_{\mathfrak{w},1}^{N_1+M_0}\big), \\ 1 - \frac{\eta}{4} \sum_{i \in \{1,2\}} \sum_{\substack{X \in \mathfrak{C}_b \\ (2,i,X) \in J}} \psi_{i,X}\big(f^{N_1}(x)\big) & \text{if } x \in \operatorname{inte}\big(X_{\mathfrak{b},2}^{N_1+M_0}\big) \cup \operatorname{inte}\big(X_{\mathfrak{w},2}^{N_1+M_0}\big), \\ 1 & \text{otherwise,} \end{cases}$$

for $x \in S^2$.

The only properties of potentials that satisfy α-strong non-integrability used in this section are summarized in the following proposition.

Proposition 7.5. *Let f, $\mathcal{C}$, d, α, ϕ satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. Let $b \in \mathbb{R}$ with $|b| \geq 1$. Using the notation above, the following statement holds:*

For each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each $X \in \mathfrak{C}_b$, each $x \in \mathfrak{X}_1'(X)$, and each $y \in \mathfrak{X}_2'(X)$,

$$(7.25) \qquad \delta_0 d(x,y)^\alpha \leq |S_{N_1}\phi(\tau_1(x)) - S_{N_1}\phi(\tau_2(x)) - S_{N_1}\phi(\tau_1(y)) + S_{N_1}\phi(\tau_2(y))| \leq \delta_0^{-1} d(x,y)^\alpha,$$

where we write $\tau_1 := \big(f^{N_1}\big|_{X_{\mathfrak{c},1}^{N_1+M_0}}\big)^{-1}$ and $\tau_2 := \big(f^{N_1}\big|_{X_{\mathfrak{c},2}^{N_1+M_0}}\big)^{-1}$.

Proof. We first observe that the second inequality in (7.25) follows immediately from the triangle inequality, Lemma 3.15, and (7.7).

It suffices to prove the first inequality in (7.25). Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $X \in \mathfrak{C}_b$, $x \in \mathfrak{X}'_1(X)$, and $y \in \mathfrak{X}'_2(X)$. By (7.16), (7.21), Lemmas 3.8 (ii), 3.15, and (7.19),

$$
\left| S_{N_1}\phi(\tau_1(x)) - S_{N_1}\phi(\tau_2(x)) - S_{N_1}\phi(\tau_1(y)) + S_{N_1}\phi(\tau_2(y)) \right| / d(x,y)^\alpha
$$

$$
\geq \frac{\left| S_{N_1}\phi(\tau_1(x)) - S_{N_1}\phi(\tau_2(x)) - S_{N_1}\phi(\tau_1(y)) + S_{N_1}\phi(\tau_2(y)) \right|}{d(x_1(X), x_2(X))^\alpha} \cdot \frac{d(\mathfrak{X}_1(X), \mathfrak{X}_2(X))^\alpha}{(\mathrm{diam}_d(X))^\alpha}
$$

$$
\geq \left(\varepsilon - \frac{2C_1(\mathrm{diam}_d(\mathfrak{X}'_1(X))^\alpha + 2C_1(\mathrm{diam}_d(\mathfrak{X}'_2(X))^\alpha}{(\mathrm{diam}_d(X))^\alpha} \right) \frac{10^{-\alpha}\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha m(b)}}{(\mathrm{diam}_d(X))^\alpha}
$$

$$
\geq \left(\varepsilon - \frac{4C_1 10^{-\alpha}\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha m(b) - \alpha m_0}}{C^{-\alpha}\Lambda^{-\alpha m(b)}} \right) \frac{10^{-\alpha}\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha m(b)}}{C^\alpha \Lambda^{-\alpha m(b)}} \geq \frac{\varepsilon^{1+\alpha}}{2C^{2\alpha}10^\alpha} \geq \delta_0,
$$

where the last two inequalities follow from (7.6) and (7.7). $\qquad\square$

Lemma 7.6. *Let f, $\mathcal{C}$, d, Λ, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. We use the notation in this section.*

Fix $b \in \mathbb{R}$ with $|b| \geq 2s_0 + 1$. Then for each $X \in \mathfrak{C}_b$ and each $i \in \{1, 2\}$, the function $\psi_{i,X} \colon S^2 \to \mathbb{R}$ defined in (7.22) is Hölder with an exponent α and

$$(7.26) \qquad |\psi_{i,X}|_{\alpha,\,(S^2,d)} \leq 20\varepsilon^{-\alpha}C\Lambda^{\alpha(m(b)+2m_0)}.$$

Moreover, for each subset $J \subseteq \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$, the function $\beta_J \colon S^2 \to \mathbb{R}$ defined in (7.24) satisfies

$$(7.27) \qquad 1 \geq \beta_J(x) \geq 1 - \eta > 1/2$$

for $x \in S^2$. In addition, $\beta_J \in C^{0,\alpha}(S^2, d)$ with $|\beta_J|_{\alpha,\,(S^2,d)} \leq L_\beta$, where

$$(7.28) \qquad L_\beta := 40\varepsilon^{-\alpha}C\Lambda^{\alpha(m(b)+2m_0)}(\mathrm{LIP}_d(f))^{\alpha N_1}\eta$$

is a constant depending only on f, $\mathcal{C}$, d, α, ϕ, and b. Here $C \geq 1$ is a constant from Lemma 3.8 depending only on f, $\mathcal{C}$, and d.

Proof. We will first establish (7.26). Consider distinct points $x, y \in S^2$.

If $x, y \in S^2 \setminus \mathfrak{X}_i(X)$, then $(\psi_{i,X}(x) - \psi_{i,X}(y))/d(x,y)^\alpha = 0$.

If $x \in S^2 \setminus \mathfrak{X}_i(X)$ and $y \in \mathfrak{X}_i(X)$, then by (7.19),

$$
|\psi_{i,X}(x) - \psi_{i,X}(y)|/d(x,y)^\alpha = d(y, S^2 \setminus \mathfrak{X}_i(X)) \cdot d(x,y)^{-\alpha} \cdot \left(d(y, \mathfrak{X}'_i(X))^\alpha + d(y, S^2 \setminus \mathfrak{X}_i(X))^\alpha \right)^{-1}
$$

$$
\leq d(\mathfrak{X}'_i(X), S^2 \setminus \mathfrak{X}_i(X))^{-\alpha} \leq 10^\alpha \varepsilon^{-\alpha} C^\alpha \Lambda^{\alpha(m(b)+m_0)} \leq 20\varepsilon^{-\alpha} C \Lambda^{\alpha(m(b)+2m_0)}.
$$

Similarly, if $y \in S^2 \setminus \mathfrak{X}_i(X)$ and $x \in \mathfrak{X}_i(X)$, then $|\psi_{i,X}(x) - \psi_{i,X}(y)|/d(x,y)^\alpha \leq 20\varepsilon^{-\alpha}C\Lambda^{\alpha(m(b)+2m_0)}$.

If $x, y \in \mathfrak{X}_i(X)$, then by (7.18), (7.16), and (7.19),

$$
|\psi_{i,X}(x) - \psi_{i,X}(y)|/d(x,y)^\alpha
$$

$$
\leq \frac{d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha |d(x, \mathfrak{X}'_i(X))^\alpha - d(y, \mathfrak{X}'_i(X))^\alpha|}{d(x,y)^\alpha (d(x, \mathfrak{X}'_i(X))^\alpha + d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha)(d(y, \mathfrak{X}'_i(X))^\alpha + d(y, S^2 \setminus \mathfrak{X}_i(X))^\alpha)}
$$

$$
+ \frac{|d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha - d(y, S^2 \setminus \mathfrak{X}_i(X))^\alpha| d(x, \mathfrak{X}'_i(X))^\alpha}{d(x,y)^\alpha (d(x, \mathfrak{X}'_i(X))^\alpha + d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha)(d(y, \mathfrak{X}'_i(X))^\alpha + d(y, S^2 \setminus \mathfrak{X}_i(X))^\alpha)}
$$

$$
\leq \frac{d(x, S^2 \setminus \mathfrak{X}_i(X))^\alpha d(x,y)^\alpha + d(x,y)^\alpha d(x, \mathfrak{X}'_i(X))^\alpha}{d(x,y)^\alpha d(\mathfrak{X}'_i(X), S^2 \setminus \mathfrak{X}_i(X))^{2\alpha}}
$$

$$
\leq \left(10^{-\alpha}\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha m(b)} + 10^{-\alpha}\varepsilon^\alpha C^{-\alpha}\Lambda^{-\alpha m(b)} \right)\left(10\varepsilon^{-1}C\Lambda^{m(b)+m_0} \right)^{2\alpha} \leq 20\varepsilon^{-\alpha}C\Lambda^{\alpha(m(b)+2m_0)}.
$$

Hence $|\psi_{i,X}|_{\alpha,\,(S^2,d)} \leq 20\varepsilon^{-\alpha}C\Lambda^{\alpha(m(b)+2m_0)}$, establishing (7.26).

In order to establish (7.27), we only need to observe that for each $j \in \{1, 2\}$, and each $x \in \mathrm{inte}(X_{\mathfrak{b},j}^{N_1+M_0}) \cup \mathrm{inte}(X_{\mathfrak{w},j}^{N_1+M_0})$, at most one term in the summations in (7.24) is nonzero. Indeed, we note that for each pair of distinct tiles $X_1, X_2 \in \mathfrak{C}_b$, $\mathfrak{X}_{i_1}(X_1) \cap \mathfrak{X}_{i_2}(X_2) = \emptyset$ for all $i_1, i_2 \in \{1, 2\}$ by (7.17), and $\mathfrak{X}_1(X_1) \cap \mathfrak{X}_2(X_1) = \emptyset$ by (7.21). Hence, by (7.23), at most one term in the summations in (7.24) is nonzero, and (7.27) follows from (7.12).

We now show the continuity of β_J. Note that for each $i \in \{1, 2\}$ and each $X \in \mathfrak{C}_b$, by (7.17), (7.23), and the continuity of $\psi_{i,X}$, we have

$$\psi_{i,X}\big(f^{N_1}\big(\partial X_{\mathfrak{c},j}^{N_1+M_0}\big)\big) = \psi_{i,X}\big(Y_{\mathfrak{c}}^{M_0}\big) = \{0\}$$

for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $j \in \{1, 2\}$. It follows immediately from (7.24) that β_J is continuous.

Finally, for arbitrary $x, y \in S^2$ with $x \neq y$, we will establish $\frac{|\beta_J(x) - \beta_J(y)|}{d(x,y)^\alpha} \leq L_\beta$ by considering the following two cases.

Case 1. $x, y \in X^{N_1+m(b)}$ for some $X^{N_1+m(b)} \in \mathbf{X}^{N_1+m(b)}$. If

$$X^{N_1+m(b)} \not\subseteq \bigcup\big\{X_{\mathfrak{c},j}^{N_1+M_0} : \mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\},\, j \in \{1, 2\}\big\},$$

then $\beta_J(x) - \beta_J(y) = 1 - 1 = 0$. If

$$X^{N_1+m(b)} \subseteq \bigcup\big\{X_{\mathfrak{c},j}^{N_1+M_0} : \mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\},\, j \in \{1, 2\}\big\},$$

then by (7.23),

$$\frac{|\beta_J(x) - \beta_J(y)|}{d(x,y)^\alpha} = \frac{\big(1 - \frac{\eta}{4} \sum_{i \in \{1,2\}} \psi_{i,X_*}\big(f^{N_1}(x)\big)\big) - \big(1 - \frac{\eta}{4} \sum_{i \in \{1,2\}} \psi_{i,X_*}\big(f^{N_1}(y)\big)\big)}{d(x,y)^\alpha}$$

$$\leq \eta\, |\psi_{i,X_*}|_{\alpha,\,(S^2,d)} \big(\mathrm{LIP}_d(f)\big)^{\alpha N_1} \leq L_\beta,$$

where we denote $X_* := f^{N_1}\big(X^{N_1+m(b)}\big)$.

Case 2. $\mathrm{card}\big(\{x, y\} \cap X^{N_1+m(b)}\big) \leq 1$ for all $X^{N_1+m(b)} \in \mathbf{X}^{N_1+m(b)}$. We assume, without loss of generality, that $\beta_J(x) - \beta_J(y) \neq 0$. Then by (7.23) and (7.17), $d\big(f^{N_1}(x), f^{N_1}(y)\big) \geq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)}$. Thus $d(x,y) \geq \frac{\varepsilon}{10} C^{-1} \Lambda^{-m(b)} (\mathrm{LIP}_d(f))^{-N_1}$. Hence by (7.27), $\frac{|\beta_J(x) - \beta_J(y)|}{d(x,y)^\alpha} \leq 10\varepsilon^{-\alpha} C \Lambda^{\alpha m(b)} (\mathrm{LIP}_d(f))^{\alpha N_1} \eta \leq L_\beta$. $\qquad\square$

Definition 7.7. Let f, $\mathcal{C}$, d, α, ϕ satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. Let $a, b \in \mathbb{R}$ satisfy $|b| \geq 1$. Denote $s := a + \mathbf{i}b$. For each subset $J \subseteq \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$, the *Dolgopyat operator* $\mathbb{M}_{J,s,\phi}$ on $C^{0,\alpha}\big(\big(X_\mathfrak{b}^0, d\big), \mathbb{C}\big) \times C^{0,\alpha}\big(\big(X_\mathfrak{w}^0, d\big), \mathbb{C}\big)$ is defined by

$$(7.29) \qquad \mathbb{M}_{J,s,\phi}(u_\mathfrak{b}, u_\mathfrak{w}) = \mathbb{L}_{\underset{a\phi}{\sim}}^{N_1+M_0}\big(u_\mathfrak{b}\beta_J|_{X_\mathfrak{b}^0}, u_\mathfrak{w}\beta_J|_{X_\mathfrak{w}^0}\big)$$

for $u_\mathfrak{b} \in C^{0,\alpha}\big(\big(X_\mathfrak{b}^0, d\big), \mathbb{C}\big)$ and $u_\mathfrak{w} \in C^{0,\alpha}\big(\big(X_\mathfrak{w}^0, d\big), \mathbb{C}\big)$.

Here $\mathfrak{C}_b$ is defined in (7.14), β_J is defined in (7.24), $M_0 \in \mathbb{N}$ is a constant from Definition 7.1, and N_1 is given in (7.11). Note that in (7.29), since $\beta_J \in C^{0,\alpha}(S^2, d)$ (see Lemma 7.6), we have $u_\mathfrak{c}\beta_J|_{X_\mathfrak{c}^0} \in C^{0,\alpha}\big(\big(X_\mathfrak{c}^0, d\big), \mathbb{C}\big)$ for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$.

7.3. Cancellation argument.

Lemma 7.8. *Let f, $\mathcal{C}$, d satisfy the Assumptions. Let $\varphi \in C^{0,\alpha}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\alpha \in (0, 1]$. Then there exists a constant $C_{\mu_\varphi} \geq 1$ depending only on f, d, and φ such that for all integers m, $n \in \mathbb{N}_0$, and tiles $X^n \in \mathbf{X}^n(f, \mathcal{C})$, $X^{m+n} \in \mathbf{X}^{m+n}(f, \mathcal{C})$ satisfying $X^{m+n} \subseteq X^n$, we have*

$$(7.30) \qquad \mu_\varphi(X^n)/\mu_\varphi(X^{m+n}) \leq C_{\mu_\varphi}^2 \exp(m(\|\varphi\|_{C^0(S^2)} + P(f, \varphi))),$$

where μ_φ is the unique equilibrium state for the map f and the potential φ, and $P(f, \varphi)$ denotes the topological pressure for f and φ.

Proof. By [Li18, Theorems 5.16, 1.1, and Corollary 5.18], the unique equilibrium state μ_φ is a *Gibbs state* with respect to f, $\mathcal{C}$, and φ as defined in Definition 5.3 in [Li18]. More precisely, there exist constants $P_{\mu_\varphi} \in \mathbb{R}$ and $C_{\mu_\varphi} \geq 1$ such that for each $n \in \mathbb{N}_0$, each n-tile $X^n \in \mathbf{X}^n$, and each $x \in X^n$, we have $C_{\mu_\varphi}^{-1} \leq \frac{\mu_\varphi(X^n)}{\exp(S_n\varphi(x) - nP_{\mu_\varphi})} \leq C_{\mu_\varphi}$.
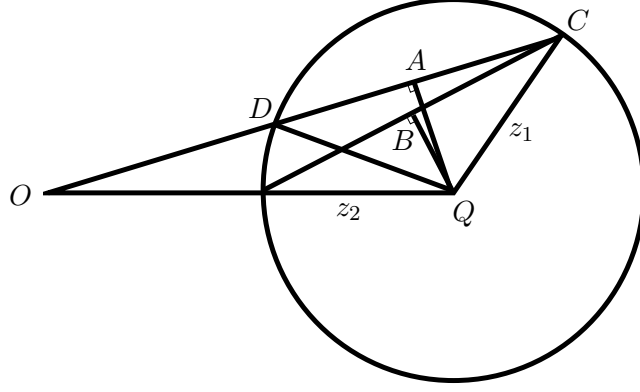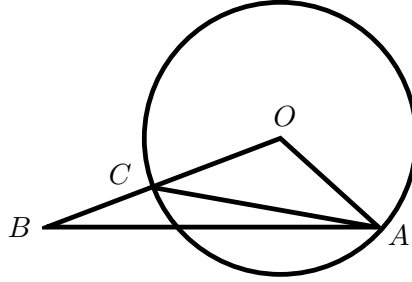
FIGURE 7.1. Proof of (7.32) of Lemma 7.9.

FIGURE 7.2. Proof of (7.33) of Lemma 7.9.

We fix arbitrary integers m, $n \in \mathbb{N}_0$, and tiles $X^n \in \mathbf{X}^n$, $X^{m+n} \in \mathbf{X}^{m+n}$ satisfying $X^{m+n} \subseteq X^n$. Choose an arbitrary point $x \in X^{m+n}$. Then

$$\frac{\mu_\varphi(X^n)}{\mu_\varphi(X^{m+n})} \leq C_{\mu_\varphi}^2 \frac{\exp\left(S_n\varphi(x) - nP_{\mu_\varphi}\right)}{\exp\left(S_{n+m}\varphi(x) - (n+m)P_{\mu_\varphi}\right)} \leq C_{\mu_\varphi}^2 \exp(m(\|\varphi\|_{C^0(S^2)} + P(f,\varphi))).$$

Inequality (7.30) follows immediately from the fact that $P_{\mu_\varphi} = P(f,\varphi)$ (see [Li18, Theorem 5.16 and Proposition 5.17]). $\qquad\square$

Lemma 7.9. *For all z_1, $z_2 \in \mathbb{C} \setminus \{0\}$, the following inequalities hold:*

(7.31) $$|\mathrm{Arg}(z_1 z_2)| \leq |\mathrm{Arg}(z_1)| + |\mathrm{Arg}(z_2)|,$$

(7.32) $$|z_1 + z_2| \leq |z_1| + |z_2| - (\mathrm{Arg}(z_1/z_2))^2 \min\{|z_1|, |z_2|\}/16,$$

(7.33) $$\left|\mathrm{Arg}(z_1/z_2)\right| \leq 2|z_1 - z_2|/\min\{|z_1|, |z_2|\}.$$

Proof. Inequality (7.31) follows immediately from the definition of Arg (see Section 2).

We then verify (7.32). Without loss of generality, we assume that $|z_1| \leq |z_2|$ and $\theta := \mathrm{Arg}\left(\frac{z_1}{z_2}\right) \geq 0$. Using the labeling in Figure 7.1, we let $\overrightarrow{OQ} = z_2$ and $\overrightarrow{QC} = z_1$. Then

$$|z_1 + z_2| = |OA| + |AC| \leq |z_2| + |BC| = |z_2| + |z_1|\cos(\theta/2)$$

$$\leq |z_2| + |z_1|\left(1 - \frac{\theta^2}{8} + \frac{\theta^4}{4!2^4}\right) \leq |z_2| + \left(1 - \frac{\theta^2}{16}\right)|z_1|.$$

Inequality (7.33) follows immediately from the following observation in elementary Euclidean plane geometry. As seen in Figure 7.2, assume $A = z_1$ and $B = z_2$. Then $|z_1 - z_2| = |AB| \geq |AC| \geq \frac{1}{2}|OA|\angle AOC = \frac{1}{2}|z_1||\mathrm{Arg}(z_1/z_2)|$. $\qquad\square$

Lemma 7.10. *Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. Fix $b \in \mathbb{R}$ with $|b| \geq 2s_0 + 1$. Fix $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $h_{\mathfrak{c}} \in K_{A|b|}\left(X_{\mathfrak{c}}^0, d\right)$. For each $m \geq m(b) - M_0$ and each m-tile $X^m \in \mathbf{X}^m(f, \mathcal{C})$ with $X^m \subseteq X_{\mathfrak{c}}^0$, we have*

$$\sup\{h_{\mathfrak{c}}(x) : x \in X^m\} \leq 2\inf\{h_{\mathfrak{c}}(x) : x \in X^m\}.$$

Recall that the cone $K_{A|b|}(X_\mathfrak{c}^0, d)$ is defined in Definition 5.8.

Proof. Fix arbitrary x, $x' \in X^m$. By Definition 5.8, Lemma 3.8 (ii), (7.15), and (7.10),

$$|h_\mathfrak{c}(x) - h_\mathfrak{c}(x')| \le A|b|(h_\mathfrak{c}(x) + h_\mathfrak{c}(x'))d(x, x')^\alpha \le A|b|(h_\mathfrak{c}(x) + h_\mathfrak{c}(x'))(\text{diam}_d(X^m))^\alpha$$

$$\le A|b|(h_\mathfrak{c}(x) + h_\mathfrak{c}(x'))C\Lambda^{\alpha M_0 - \alpha m(b)} \le A|b|(\epsilon_1/|b|)\Lambda^{\alpha M_0}(h_\mathfrak{c}(x) + h_\mathfrak{c}(x')) \le (h_\mathfrak{c}(x) + h_\mathfrak{c}(x'))/4,$$

where $C \ge 1$ is a constant from Lemma 3.8 depending only on f, $\mathcal{C}$, and d. The lemma follows immediately. $\qquad\square$

Lemma 7.11. *Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. Fix $b \in \mathbb{R}$, $m \in \mathbb{N}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, $u_\mathfrak{c} \in C^{0,\alpha}((X_\mathfrak{c}^0, d), \mathbb{C})$, and $h_\mathfrak{c} \in K_{A|b|}(X_\mathfrak{c}^0, d)$ such that $|b| \ge 2s_0 + 1$, $m \ge N_1 + m(b)$, $|u_\mathfrak{c}(y)| \le h_\mathfrak{c}(y)$, and $|u_\mathfrak{c}(y) - u_\mathfrak{c}(y')| \le A|b|(h_\mathfrak{c}(y) + h_\mathfrak{c}(y'))d(y, y')^\alpha$ whenever y, $y' \in X_\mathfrak{c}^0$. Then for each $X^m \in \mathbf{X}^m(f, \mathcal{C})$ with $X^m \subseteq X_\mathfrak{c}^0$, at least one of the following statements holds:*

(1) $|u_\mathfrak{c}(x)| \le \frac{3}{4}h_\mathfrak{c}(x)$ *for all $x \in X^m$.*
(2) $|u_\mathfrak{c}(x)| \ge \frac{1}{4}h_\mathfrak{c}(x)$ *for all $x \in X^m$.*

Proof. Assume that $|u_\mathfrak{c}(x_0)| < \frac{1}{4}h_\mathfrak{c}(x_0)$ for some $x_0 \in X^m$. Then by Lemmas 3.8 (ii), 7.10, and (7.15), for each $x \in X^m$,

$$|u_\mathfrak{c}(x)| < |u_\mathfrak{c}(x) - u_\mathfrak{c}(x_0)| + \frac{1}{4}h_\mathfrak{c}(x_0) \le A|b|(h_\mathfrak{c}(x) + h_\mathfrak{c}(x_0))(\text{diam}_d(X^m))^\alpha + \frac{1}{4}h_\mathfrak{c}(x_0)$$

$$\le \left(2A|b|C\Lambda^{-\alpha N_1 - \alpha m(b)} + \frac{1}{4}\right)\sup\{h_\mathfrak{c}(y) : y \in X^m\} \le \left(4A\epsilon_1\Lambda^{-\alpha N_1} + \frac{1}{2}\right)h_\mathfrak{c}(x) \le \frac{3}{4}h_\mathfrak{c}(x),$$

where $C \ge 1$ is a constant from Lemma 3.8. The last inequality follows from (7.11) and the fact that $\epsilon_1 \in (0, 1)$ (see (7.10)). $\qquad\square$

Lemma 7.12. *Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. Fix arbitrary $s := a + \mathbf{i}b$ with a, $b \in \mathbb{R}$ satisfying $|a - s_0| \le s_0$ and $|b| \ge b_0$. Given arbitrary $h_\mathfrak{b} \in K_{A|b|}(X_\mathfrak{b}^0, d)$, $h_\mathfrak{w} \in K_{A|b|}(X_\mathfrak{w}^0, d)$, $u_\mathfrak{b} \in C^{0,\alpha}((X_\mathfrak{b}^0, d), \mathbb{C})$, and $u_\mathfrak{w} \in C^{0,\alpha}((X_\mathfrak{w}^0, d), \mathbb{C})$ satisfying the property that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, we have $|u_\mathfrak{c}(y)| \le h_\mathfrak{c}(y)$ and $|u_\mathfrak{c}(y) - u_\mathfrak{c}(y')| \le A|b|(h_\mathfrak{c}(y) + h_\mathfrak{c}(y'))d(y, y')^\alpha$ whenever y, $y' \in X_\mathfrak{c}^0$.*

Define the functions $Q_{\mathfrak{c},j} : Y_\mathfrak{c}^{M_0} \to \mathbb{R}$ for $j \in \{1, 2\}$ and $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ by

$$Q_{\mathfrak{c},j}(x) := \frac{\left|\sum_{k \in \{1, 2\}} u_{\varsigma(\mathfrak{c},k)}(\tau_k(x))e^{S_{N_1}\widetilde{-s\phi}(\tau_k(x))}\right|}{-\frac{1}{2}\eta h_{\varsigma(\mathfrak{c},j)}(\tau_j(x))e^{S_{N_1}\widetilde{-a\phi}(\tau_j(x))} + \sum_{k \in \{1, 2\}} h_{\varsigma(\mathfrak{c},k)}(\tau_k(x))e^{S_{N_1}\widetilde{-a\phi}(\tau_k(x))}},$$

for $x \in Y_\mathfrak{c}^{M_0}$, where we write $\tau_k := \left(f^{N_1}\big|_{X_{\mathfrak{c},k}^{N_1+M_0}}\right)^{-1}$ for $k \in \{1, 2\}$, and we set $\varsigma(\mathfrak{c}, j) \in \{\mathfrak{b}, \mathfrak{w}\}$ in such a way that $\tau_j(Y_\mathfrak{c}^{M_0}) \subseteq X_{\varsigma(\mathfrak{c},j)}^0$ for $j \in \{1, 2\}$.

Then for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each $X \in \mathfrak{C}_b$ with $X \subseteq Y_\mathfrak{c}^{M_0}$, we have

$$\min\{\|Q_{\mathfrak{c},j}\|_{C^0(\mathfrak{X}_i(X))} : i, j \in \{1, 2\}\} \le 1.$$

Proof. Fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $X \in \mathfrak{C}_b$ with $X \subseteq Y_\mathfrak{c}^{M_0}$. For typographic reasons, we denote in this proof

(7.34) $u_{i,x} := u_{\varsigma(\mathfrak{c},i)}(\tau_i(x)), \qquad h_{i,x} := h_{\varsigma(\mathfrak{c},i)}(\tau_i(x)), \qquad e_{i,x} := e^{S_{N_1}\widetilde{-s\phi}(\tau_i(x))}$

for $i \in \{1, 2\}$ and $x \in X$.

If $|u_{j,\cdot}| \le \frac{3}{4}h_{j,\cdot}$ on X, for some $j \in \{1, 2\}$, then $\|Q_{\mathfrak{c},j}\|_{C^0(\mathfrak{X}_i(X))} \le 1$ for all $i \in \{1, 2\}$. Thus, by Lemma 7.11, we can assume that

(7.35) $|u_{k,x}| \ge h_{k,x}/4 \qquad$ for all $x \in X$ and $k \in \{1, 2\}$.

We define a function $\Theta : X \to (-\pi, \pi]$ by setting

(7.36) $\Theta(x) := \text{Arg}\left(\frac{u_{1,x}e_{1,x}}{u_{2,x}e_{2,x}}\right) \qquad$ for $x \in X$.

We first claim that for all x, $y \in X$, we have

(7.37) $$\left| \operatorname{Arg}\left(\frac{u_{1,x}/u_{2,x}}{u_{1,y}/u_{2,y}} \right) \right| \leq 16A\epsilon_1 \Lambda^{-\alpha N_1} \leq \pi/16 \qquad \text{and}$$

(7.38) $$|b||-S_{N_1}\phi(\tau_1(x)) + S_{N_1}\phi(\tau_2(x)) + S_{N_1}\phi(\tau_1(y)) - S_{N_1}\phi(\tau_2(y))| \leq \pi/16.$$

Indeed, by (7.31) and (7.33) in Lemma 7.9, (7.34), (7.35), Lemmas 3.8 (ii), 7.10, (7.14), and (7.15),

$$\left| \operatorname{Arg}\left(\frac{u_{1,x}/u_{2,x}}{u_{1,y}/u_{2,y}} \right) \right| \leq \left| \operatorname{Arg}\left(\frac{u_{1,x}}{u_{1,y}} \right) \right| + \left| \operatorname{Arg}\left(\frac{u_{2,x}}{u_{2,y}} \right) \right| \leq \sum_{j \in \{1,2\}} \frac{2|u_{j,x} - u_{j,y}|}{\inf\{|u_{j,z}| : z \in X\}}$$

$$\leq \sum_{j \in \{1,2\}} \frac{8A|b|(h_{j,x} + h_{j,y})}{\inf\{h_{j,z} : z \in X\}} d(\tau_j(x), \tau_j(y))^\alpha$$

$$\leq 16A|b| \sum_{j \in \{1,2\}} \frac{\sup\{h_{j,z} : z \in X\}}{\inf\{h_{j,z} : z \in X\}} C\Lambda^{-\alpha N_1 - \alpha m(b)} \leq 64A|b|(\epsilon_1/|b|)\Lambda^{-\alpha N_1} \leq \pi/16,$$

where $C \geq 1$ is a constant from Lemma 3.8. The last inequality follows from the fact that $N_1 \geq \lceil \frac{1}{\alpha} \log_\Lambda (2^{10}A) \rceil$ (see (7.11)) and the fact that $\epsilon_1 \in (0,1)$ (see (7.10)). We have now verified (7.37). To show (7.38), we note that by Lemma 3.8 (ii), (7.14), (7.15), and (7.10),

$$|b||-S_{N_1}\phi(\tau_1(x)) + S_{N_1}\phi(\tau_2(x)) + S_{N_1}\phi(\tau_1(y)) - S_{N_1}\phi(\tau_2(y))|$$

$$\leq |b|\delta_0^{-1}d(x,y)^\alpha \leq |b|\delta_0^{-1}(\operatorname{diam}_d(X))^\alpha \leq |b|\delta_0^{-1}C^\alpha\Lambda^{-\alpha m(b)} \leq \delta_0^{-1}\epsilon_1 \leq \pi/16.$$

The claim is now verified.

We will choose $i_0 \in \{1, 2\}$, by separate discussions in the following two cases, in such a way that

(7.39) $$|\Theta(x)| \geq 16\eta^{1/2} \qquad \text{for all } x \in \mathfrak{X}_{i_0}(X).$$

Case 1. $|\Theta(y)| \geq \pi/4$ for some $y \in X$. Then by (7.31) in Lemma 7.9, (7.34), (7.36), (7.37), (7.38), and the fact that $\eta \in (0, 2^{-12})$ (see (7.12)), for each $x \in X$,

$$|\Theta(x)| \geq |\Theta(y)| - \left| \operatorname{Arg}\left(\frac{(u_{1,y}e_{1,y})/(u_{2,y}e_{2,y})}{(u_{1,x}e_{1,x})/(u_{2,x}e_{2,x})} \right) \right|$$

$$\geq \frac{\pi}{4} - \left| \operatorname{Arg}\left(\frac{u_{1,y}/u_{2,y}}{u_{1,x}/u_{2,x}} \right) \right| - \left| \operatorname{Arg}\left(\frac{e_{1,y}/e_{2,y}}{e_{1,x}/e_{2,x}} \right) \right| \geq \frac{\pi}{4} - \frac{\pi}{16} - \frac{\pi}{16} \geq \frac{\pi}{8} \geq 16\eta^{1/2}.$$

We can choose $i_0 = 1$ in this case.

Case 2. $|\Theta(z)| < \pi/4$ for all $z \in X$. Then by (7.31) in Lemma 7.9, (7.34), (7.36), (7.37), (7.38), $|b| \geq b_0 \geq 1$ (see (7.8)), (7.25), (7.21), and (7.15), for each $x \in \mathfrak{X}_1(X)$ and each $y \in \mathfrak{X}_2(X)$,

$$|\Theta(x) - \Theta(y)| = \left| \operatorname{Arg}\left(\frac{(u_{1,x}e_{1,x})/(u_{2,x}e_{2,x})}{(u_{1,y}e_{1,y})/(u_{2,y}e_{2,y})} \right) \right| \geq \left| \operatorname{Arg}\left(\frac{e_{1,x}/e_{2,x}}{e_{1,y}/e_{2,y}} \right) \right| - \left| \operatorname{Arg}\left(\frac{u_{2,y}/u_{1,y}}{u_{2,x}/u_{1,x}} \right) \right|$$

$$\geq |b||-S_{N_1}\phi(\tau_1(x)) + S_{N_1}\phi(\tau_2(x)) + S_{N_1}\phi(\tau_1(y)) - S_{N_1}\phi(\tau_2(y))| - 16A\epsilon_1\Lambda^{-\alpha N_1}$$

$$\geq |b|\delta_0 d(x,y)^\alpha - 16A\epsilon_1\Lambda^{-\alpha N_1} \geq |b|\delta_0\left(10^{-1}\varepsilon C^{-1}\Lambda^{-m(b)}\right)^\alpha - 16A\epsilon_1\Lambda^{-\alpha N_1}$$

$$\geq \varepsilon\delta_0(10\Lambda)^{-1}C^{-2}\epsilon_1 - 16A\epsilon_1\Lambda^{-\alpha N_1} \geq \varepsilon\delta_0\epsilon_1/(20\Lambda C^2),$$

where the last inequality follows from the observation that $16A\Lambda^{-\alpha N_1} \leq \frac{\varepsilon\delta_0}{20\Lambda C^2}$ since $N_1 \geq \lceil \frac{1}{\alpha} \log_\Lambda \left(\frac{320A\Lambda C^2}{\varepsilon\delta_0} \right) \rceil$ (see (7.11)).

We now claim that at least one of the following statements holds:

(1) $|\Theta(x)| \geq \frac{\varepsilon\delta_0\epsilon_1}{80\Lambda C^2}$ for all $x \in \mathfrak{X}_1(X)$.

(2) $|\Theta(y)| \geq \frac{\varepsilon\delta_0\epsilon_1}{80\Lambda C^2}$ for all $y \in \mathfrak{X}_2(X)$.

Indeed, assume that statement (1) fails, then there exists $x_0 \in \mathfrak{X}_1(X)$ such that $|\Theta(x_0)| \leq \frac{\varepsilon\delta_0\epsilon_1}{80\Lambda C^2}$. Hence for all $y \in \mathfrak{X}_2(X)$,

$$|\Theta(y)| \geq |\Theta(y) - \Theta(x_0)| - |\Theta(x_0)| \geq \frac{\varepsilon\delta_0\epsilon_1}{20\Lambda C^2} - \frac{\varepsilon\delta_0\epsilon_1}{80\Lambda C^2} \geq \frac{\varepsilon\delta_0\epsilon_1}{80\Lambda C^2}.$$

The claim is now verified.

Thus we can fix $i_0 \in \{1, 2\}$ such that $|\Theta(x)| \geq \frac{\varepsilon \delta_0 \epsilon_1}{80 \Lambda C^2} \geq 16 \eta^{1/2}$ (see (7.12)) for all $x \in \mathfrak{X}_{i_0}(X)$ in this case.

By (7.34), Lemmas 3.15, 7.10, 3.8 (ii), (7.14), and (7.15), for arbitrary x, $y \in \mathfrak{X}_{i_0}(X)$ and $j \in \{1, 2\}$,

$$
\left| \frac{h_{j,x} \exp\left(S_{N_1} \widetilde{-a\phi}(\tau_j(x))\right)}{h_{j,y} \exp\left(S_{N_1} \widetilde{-a\phi}(\tau_j(y))\right)} \right| \leq \left| \frac{h_{j,x}}{h_{j,y}} \right| e^{|S_{N_1} \widetilde{-a\phi}(\tau_j(x)) - S_{N_1} \widetilde{-a\phi}(\tau_j(y))|}
$$

(7.40)
$$
\leq 2 \exp\left(C_0 \big| \widetilde{-a\phi} \big|_{\alpha,\,(S^2,d)} \frac{d(x,y)^\alpha}{1 - \Lambda^{-\alpha}} \right) \leq 2 \exp\left(C_0 \big| \widetilde{-a\phi} \big|_{\alpha,\,(S^2,d)} \frac{C^\alpha \Lambda^{-\alpha m(b)}}{1 - \Lambda^{-\alpha}} \right)
$$

$$
\leq 2 \exp\left(\epsilon_1 |b|^{-1} C_0 \big| \widetilde{-a\phi} \big|_{\alpha,\,(S^2,d)} (1 - \Lambda^{-\alpha})^{-1} \right) \leq 8,
$$

where the last inequality follows from (7.8), (7.13), the condition that $|b| \geq b_0$, and the fact that $\epsilon_1 \in (0, 1)$ (see (7.10)).

We fix $k_0 \in \{1, 2\}$ such that

(7.41) $\inf\{ h_{j,x} |e_{j,x}| : x \in \mathfrak{X}_{i_0}(X), \, j \in \{1, 2\} \} = \inf\{ h_{k_0,x} |e_{k_0,x}| : x \in \mathfrak{X}_{i_0}(X) \}.$

Hence by (7.32) in Lemma 7.9, (7.35), (7.36), (7.34), (7.39), (7.41), (5.14), and (7.40), for each $x \in \mathfrak{X}_{i_0}(X)$, we have

$$
|u_{1,x} e_{1,x} + u_{2,x} e_{2,x}| \leq -\frac{\Theta^2(x)}{16} \min_{k \in \{1, 2\}} \{ |u_{k,x} e_{k,x}| \} + \sum_{j \in \{1, 2\}} |u_{j,x} e_{j,x}|
$$

$$
\leq -\frac{\Theta^2(x)}{64} \min_{k \in \{1, 2\}} \{ h_{k,x} |e_{k,x}| \} + \sum_{j \in \{1, 2\}} h_{j,x} |e_{j,x}|
$$

$$
\leq -4\eta \inf\left\{ h_{k_0,y} e^{S_{N_1} \widetilde{-a\phi}(\tau_{k_0}(y))} : y \in \mathfrak{X}_{i_0}(X) \right\} + \sum_{j \in \{1, 2\}} h_{j,x} e^{S_{N_1} \widetilde{-a\phi}(\tau_j(x))}
$$

$$
\leq -\frac{1}{2} \eta h_{k_0,x} e^{S_{N_1} \widetilde{-a\phi}(\tau_{k_0}(x))} + \sum_{j \in \{1, 2\}} h_{j,x} e^{S_{N_1} \widetilde{-a\phi}(\tau_j(x))}.
$$

Therefore, we conclude that $\|Q_{\mathfrak{c},k_0}\|_{C^0(\mathfrak{X}_{i_0}(X))} \leq 1$. $\square$

Proposition 7.13. *Let f, $\mathcal{C}$, d, α, ϕ, s_0 satisfy the Assumptions. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$ and that ϕ satisfies the α-strong non-integrability condition. We use the notation in this section.*

There exist numbers $a_0 \in (0, s_0)$ and $\rho \in (0, 1)$ such that for all $s := a + \mathbf{i}b$ with a, $b \in \mathbb{R}$ satisfying $|a - s_0| \leq a_0$ and $|b| \geq b_0$, there exists a subset $\mathcal{E}_s \subseteq \mathcal{F}$ of the set $\mathcal{F}$ of all subsets of $\{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$ with a full projection such that the following statements are satisfied:

(i) *The cone $K_{A|b|}(X^0_{\mathfrak{b}}, d) \times K_{A|b|}(X^0_{\mathfrak{w}}, d)$ is invariant under $\mathbb{M}_{J,-s,\phi}$ for all $J \in \mathcal{F}$, i.e.,*

$$
\mathbb{M}_{J,-s,\phi}\left(K_{A|b|}(X^0_{\mathfrak{b}}, d) \times K_{A|b|}(X^0_{\mathfrak{w}}, d) \right) \subseteq K_{A|b|}(X^0_{\mathfrak{b}}, d) \times K_{A|b|}(X^0_{\mathfrak{w}}, d).
$$

(ii) *For all $J \in \mathcal{F}$, $h_{\mathfrak{b}} \in K_{A|b|}(X^0_{\mathfrak{b}}, d)$, and $h_{\mathfrak{w}} \in K_{A|b|}(X^0_{\mathfrak{w}}, d)$, we have*

(7.42)
$$
\sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X^0_{\mathfrak{c}}} |\pi_{\mathfrak{c}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))|^2 \, d\mu_{-s_0\phi} \leq \rho \sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X^0_{\mathfrak{c}}} |h_{\mathfrak{c}}|^2 \, d\mu_{-s_0\phi}.
$$

(iii) *Given arbitrary $h_{\mathfrak{b}} \in K_{A|b|}(X^0_{\mathfrak{b}}, d)$, $h_{\mathfrak{w}} \in K_{A|b|}(X^0_{\mathfrak{w}}, d)$, $u_{\mathfrak{b}} \in C^{0,\alpha}((X^0_{\mathfrak{b}}, d), \mathbb{C})$, and $u_{\mathfrak{w}} \in C^{0,\alpha}((X^0_{\mathfrak{w}}, d), \mathbb{C})$ satisfying the property that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, we have $|u_{\mathfrak{c}}(y)| \leq h_{\mathfrak{c}}(y)$ and $|u_{\mathfrak{c}}(y) - u_{\mathfrak{c}}(y')| \leq A|b|(h_{\mathfrak{c}}(y) + h_{\mathfrak{c}}(y'))d(y, y')^\alpha$ whenever y, $y' \in X^0_{\mathfrak{c}}$. Then the following statement is true:*

There exists $J \in \mathcal{E}_s$ such that

$$(7.43) \qquad \left| \pi_{\mathfrak{c}}\left(\mathbb{L}_{\widetilde{-s\phi}}^{N_1+M_0}(u_{\mathfrak{b}}, u_{\mathfrak{w}}) \right)(x) \right| \leq \pi_{\mathfrak{c}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))(x) \qquad and$$

$$(7.44) \qquad \left| \pi_{\mathfrak{c}}\left(\mathbb{L}_{\widetilde{-s\phi}}^{N_1+M_0}(u_{\mathfrak{b}}, u_{\mathfrak{w}}) \right)(x) - \pi_{\mathfrak{c}}\left(\mathbb{L}_{\widetilde{-s\phi}}^{N_1+M_0}(u_{\mathfrak{b}}, u_{\mathfrak{w}}) \right)(x') \right|$$

$$\leq A|b|(\pi_{\mathfrak{c}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))(x) + \pi_{\mathfrak{c}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))(x'))d(x,x')^{\alpha}$$

for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and all $x, x' \in X_{\mathfrak{c}}^0$.

Proof. For typographical convenience, we write $\iota := N_1 + M_0$ in this proof.

We fix an arbitrary number $s = a + \mathbf{i}b$ with $a, b \in \mathbb{R}$ satisfying $|a - s_0| \leq s_0$ and $|b| \geq b_0$.

(i) Without loss of generality, it suffices to show that for each $J \in \mathcal{F}$,

$$\pi_{\mathfrak{b}}\left(\mathbb{M}_{J,-s,\phi}\left(K_{A|b|}\left(X_{\mathfrak{b}}^0, d \right) \times K_{A|b|}\left(X_{\mathfrak{w}}^0, d \right) \right) \right) \subseteq K_{A|b|}\left(X_{\mathfrak{b}}^0, d \right).$$

Fix $J \in \mathcal{F}$, functions $h_{\mathfrak{b}} \in K_{A|b|}\left(X_{\mathfrak{b}}^0, d \right)$, $h_{\mathfrak{w}} \in K_{A|b|}\left(X_{\mathfrak{w}}^0, d \right)$, and points $x, x' \in X_{\mathfrak{b}}^0$ with $x \neq x'$. For each $X^\iota \in \mathbf{X}_{\mathfrak{b}}^\iota$, denote $y_{X^\iota} := (f^\iota|_{X^\iota})^{-1}(x)$ and $y'_{X^\iota} := (f^\iota|_{X^\iota})^{-1}(x')$.

Then by Definition 7.7, (5.12) in Lemma 5.7, Definition 5.2, and (5.14),

$$|\pi_{\mathfrak{b}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))(x) - \pi_{\mathfrak{b}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))(x')|$$

$$= \left| \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}_{\widetilde{-a\phi},\mathfrak{b},\mathfrak{c}}^{(\iota)}\left(h_{\mathfrak{c}}\beta_J|_{X_{\mathfrak{c}}^0} \right)(x) - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}_{\widetilde{-a\phi},\mathfrak{b},\mathfrak{c}}^{(\iota)}\left(h_{\mathfrak{c}}\beta_J|_{X_{\mathfrak{c}}^0} \right)(x') \right|$$

$$\leq \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} \left| h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} - h_{\mathfrak{c}}(y'_{X^\iota})\beta_J(y'_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota})} \right|$$

$$\leq \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} \left| h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota}) - h_{\mathfrak{c}}(y'_{X^\iota})\beta_J(y'_{X^\iota}) \right| e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota})}$$

$$+ \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota}) \left| e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} - e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota})} \right|$$

$$\leq \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota}) \left| \beta_J(y_{X^\iota}) - \beta_J(y'_{X^\iota}) \right| e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} e^{\left| S_\iota\widetilde{-a\phi}(y'_{X^\iota}) - S_\iota\widetilde{-a\phi}(y_{X^\iota}) \right|}$$

$$+ \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} \left| h_{\mathfrak{c}}(y_{X^\iota}) - h_{\mathfrak{c}}(y'_{X^\iota}) \right| \beta_J(y'_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota})}$$

$$+ \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} \left| 1 - e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota}) - S_\iota\widetilde{-a\phi}(y_{X^\iota})} \right|.$$

By Lemmas 3.15, 7.6, 3.13, and 5.1, the right-hand side of the last inequality is

$$\leq \exp\left(\frac{T_0 C_0 \left(\operatorname{diam}_d(S^2) \right)^{\alpha}}{1 - \Lambda^{-\alpha}} \right) \left(\sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota}) L_{\beta} C_0^{\alpha} \Lambda^{-\iota\alpha} d(x,x')^{\alpha} e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} \right.$$

$$+ \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \sum_{\substack{X^\iota\in\mathbf{X}_{\mathfrak{b}}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} A|b|\left(h_{\mathfrak{c}}(y_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y_{X^\iota})} + h_{\mathfrak{c}}(y'_{X^\iota})e^{S_\iota\widetilde{-a\phi}(y'_{X^\iota})} \right)C_0^{\alpha}\Lambda^{-\alpha\iota}d(x,x')^{\alpha} \right)$$

$$+ C_{10}T_0 d(x,x')^{\alpha} \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}_{\widetilde{-a\phi},\mathfrak{b},\mathfrak{c}}^{(\iota)}\left(h_{\mathfrak{c}}\beta_J|_{X_{\mathfrak{c}}^0} \right)(x),$$

where $C_0 > 1$ is a constant from Lemma 3.13 depending only on f, $\mathcal{C}$, and d; L_β is a constant defined in (7.28) in Lemma 7.6; $T_0 > 0$ is a constant defined in (5.31) giving an upper bound for $\big|\widetilde{-a\phi}\big|_{\alpha,\,(S^2,d)}$ by Lemma 5.11 (c.f. (5.30)); and $C_{10} \coloneqq C_{10}(f, \mathcal{C}, d, \alpha, T_0) > 1$ is a constant defined in (5.2) in Lemma 5.1. Both T_0 and C_{10} depend only on f, $\mathcal{C}$, d, ϕ, and α. Thus by (5.2), (7.27) and (7.28) in Lemma 7.6, Definition 7.7, (7.15), and the calculation above, we get

$$
\frac{|\pi_\mathfrak{b}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(x) - \pi_\mathfrak{b}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(x')|}{A|b|(\pi_\mathfrak{b}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(x) + \pi_\mathfrak{b}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(x'))d(x,x')^\alpha}
$$
$$
\leq \frac{C_{10}}{A|b|(1-\eta)}(L_\beta + A|b|)\Lambda^{-\alpha\iota} + \frac{C_{10}T_0}{A|b|}
$$
$$
\leq \frac{C_{10}}{1-\eta}\left(\frac{40\varepsilon^{-\alpha}}{A|b|}C\Lambda^{2\alpha m_0 + 1}\frac{C|b|}{\epsilon_1}(\mathrm{LIP}_d(f))^{\alpha N_1}\eta + 1\right)\Lambda^{-\alpha(N_1+M_0)} + \frac{C_{10}T_0}{A|b|} \leq 1.
$$

The last inequality follows from the observations that $\frac{C_{10}T_0}{A} \leq \frac{1}{3}$ (see (7.9)), that

$$
40\varepsilon^{-\alpha}C_{10}C^2\eta\Lambda^{-\alpha N_1 - \alpha M_0 + 2\alpha m_0 + 1}(\mathrm{LIP}_d(f))^{\alpha N_1}/(A\epsilon_1(1-\eta)) \leq 1/3
$$

(by (7.12)), and that by (7.11) and (7.12), $\Lambda^{-\alpha(N_1+M_0)} \leq \frac{1}{4C_{10}} \leq \frac{1}{3}\frac{1-\eta}{C_{10}}$.

(ii) Fix $J \in \mathcal{F}$ and two functions $h_\mathfrak{b} \in K_{A|b|}(X_\mathfrak{b}^0, d)$, $h_\mathfrak{w} \in K_{A|b|}(X_\mathfrak{w}^0, d)$.
We first establish that

$$
(7.45) \qquad \big(\pi_\mathfrak{c}\big(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w})\big)(x)\big)^2 \leq \pi_\mathfrak{c}\Big(\mathbb{L}^\iota_{\widetilde{-a\phi}}\big(h_\mathfrak{b}^2, h_\mathfrak{w}^2\big)\Big)(x) \cdot \pi_\mathfrak{c}\Big(\mathbb{L}^\iota_{\widetilde{-a\phi}}\big((\beta_J|_{X_\mathfrak{b}^0})^2, (\beta_J|_{X_\mathfrak{w}^0})^2\big)\Big)(x)
$$

for $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $x \in X_\mathfrak{c}^0$. Indeed, fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $x \in X_\mathfrak{c}^0$. For each $X^\iota \in \mathbf{X}_\mathfrak{c}^\iota$, denote $y_{X^\iota} \coloneqq (f^\iota|_{X^\iota})^{-1}(x)$. Then by Definition 7.7, (5.12) in Lemma 5.7, and the Cauchy–Schwartz inequality, we have

$$
\big(\pi_\mathfrak{c}\big(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w})\big)(x)\big)^2
$$
$$
= \left(\sum_{\mathfrak{c}' \in \{\mathfrak{b},\mathfrak{w}\}} \mathcal{L}^{(\iota)}_{\widetilde{-a\phi},\mathfrak{c},\mathfrak{c}'}\big(h_{\mathfrak{c}'}\beta_J|_{X_{\mathfrak{c}'}^0}\big)(x)\right)^2 = \left(\sum_{\substack{\mathfrak{c}' \in \{\mathfrak{b},\mathfrak{w}\}}} \sum_{\substack{X^\iota \in \mathbf{X}_\mathfrak{c}^\iota \\ X^\iota \subseteq X_{\mathfrak{c}'}^0}} \big(h_{\mathfrak{c}'}\beta_J \exp(S_\iota\widetilde{-a\phi})\big)(y_{X^\iota})\right)^2
$$
$$
\leq \left(\sum_{\substack{\mathfrak{c}' \in \{\mathfrak{b},\mathfrak{w}\}}} \sum_{\substack{X^\iota \in \mathbf{X}_\mathfrak{c}^\iota \\ X^\iota \subseteq X_{\mathfrak{c}'}^0}} \big(h_{\mathfrak{c}'}^2 \exp(S_\iota\widetilde{-a\phi})\big)(y_{X^\iota})\right)\left(\sum_{\substack{\mathfrak{c}' \in \{\mathfrak{b},\mathfrak{w}\}}} \sum_{\substack{X^\iota \in \mathbf{X}_\mathfrak{c}^\iota \\ X^\iota \subseteq X_{\mathfrak{c}'}^0}} \big(\beta_J^2 \exp(S_\iota\widetilde{-a\phi})\big)(y_{X^\iota})\right)
$$
$$
= \pi_\mathfrak{c}\Big(\mathbb{L}^\iota_{\widetilde{-a\phi}}\big(h_\mathfrak{b}^2, h_\mathfrak{w}^2\big)\Big)(x) \cdot \pi_\mathfrak{c}\Big(\mathbb{L}^\iota_{\widetilde{-a\phi}}\big((\beta_J|_{X_\mathfrak{b}^0})^2, (\beta_J|_{X_\mathfrak{w}^0})^2\big)\Big)(x).
$$

We will focus on the case where the potential is $\widetilde{-s_0\phi}$ for now, and only consider the general case at the end of the proof of statement (ii).
Next, we define a set

$$
(7.46) \qquad\qquad W_J \coloneqq \bigcup_{(j,i,X)\in J} f^{M_0}(\mathfrak{X}'_i(X)).
$$

We claim that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each $x \in W_J \cap X_\mathfrak{c}^0$, we have

$$
(7.47) \qquad \pi_\mathfrak{c}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big((\beta_J|_{X_\mathfrak{b}^0})^2, (\beta_J|_{X_\mathfrak{w}^0})^2\big)\Big)(x) \leq 1 - \frac{1}{4}\eta\exp\Big(-\iota\big\|\widetilde{-s_0\phi}\big\|_{C^0(S^2)}\Big).
$$

Indeed, we first fix arbitrary $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $x \in W_J \cap X_{\mathfrak{c}}^0$. Let $X \in \mathfrak{C}_b$ denote the unique $m(b)$-tile in $\mathfrak{C}_b$ with $x \in f^{M_0}(X)$. By (5.12) in Lemma 5.7, Definition 5.2, and (7.24),

$$
\pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(\big(\beta_J|_{X_{\mathfrak{b}}^0}\big)^2, \big(\beta_J|_{X_{\mathfrak{w}}^0}\big)^2\big)\Big)(x)
$$
$$
= \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}^{(\iota)}_{\widetilde{-s_0\phi},\mathfrak{c},\mathfrak{c}'}\big(\big(\beta_J|_{X_{\mathfrak{c}'}^0}\big)^2\big)(x) = \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{\substack{X^{\iota} \in \mathbf{X}_{\mathfrak{c}}^{\iota} \\ X^{\iota} \subseteq X_{\mathfrak{c}'}^0}} \beta_J^2(y_{X^{\iota}}) \exp\big(S_{\iota}\widetilde{-s_0\phi}(y_{X^{\iota}})\big)
$$
$$
\leq \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \mathcal{L}^{(\iota)}_{\widetilde{-s_0\phi},\mathfrak{c},\mathfrak{c}'}\big(\mathbb{1}_{X_{\mathfrak{c}'}^0}\big)(x) - \frac{1}{4}\eta\psi_{i_X,X}\big(f^{N_1}(y_*)\big) \exp\big(S_{\iota}\widetilde{-s_0\phi}(y_*)\big) \leq 1 - \frac{1}{4}\eta \exp\Big(-\iota \big\|\widetilde{-s_0\phi}\big\|_{C^0(S^2)}\Big),
$$

where $i_X, j_X \in \{1, 2\}$ are chosen in such a way that $(j_X, i_X, X) \in J$ (due to the fact that $J \in \mathcal{F}$ has a full projection (see Definition 7.4)), and we denote $y_{X^{\iota}} := (f^{\iota}|_{X^{\iota}})^{-1}(x)$ for $X^{\iota} \in \mathbf{X}_{\mathfrak{c}}^{\iota}$, and write $y_* := y_{X_{\mathfrak{c},j_X}^{N_1+M_0}}$. The last inequality follows from (5.18) in Lemma 5.10, (7.23), and (7.46). The claim is now verified.

Next, we claim that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$,

$$(7.48) \qquad \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \in K_{A|b|}(X_{\mathfrak{c}}^0, d).$$

Indeed, by (5.12) in Lemma 5.7, Lemmas 5.9, and 5.12 (i), for all $x, y \in X_{\mathfrak{c}}^0$,

$$
\Big|\pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(x) - \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(y)\Big|
$$
$$
\leq \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \Big|\mathcal{L}^{(\iota)}_{\widetilde{-s_0\phi},\mathfrak{c},\mathfrak{c}'}\big(h_{\mathfrak{c}'}^2\big)(x) - \mathcal{L}^{(\iota)}_{\widetilde{-s_0\phi},\mathfrak{c},\mathfrak{c}'}\big(h_{\mathfrak{c}'}^2\big)(y)\Big|
$$
$$
\leq A_0 \bigg(\frac{2A|b|}{\Lambda^{\alpha\iota}} + \frac{\big|\widetilde{-s_0\phi}\big|_{\alpha,(S^2,d)}}{1 - \Lambda^{-\alpha}}\bigg) d(x,y)^{\alpha} \sum_{\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}} \sum_{z \in \{x,y\}} \mathcal{L}^{(\iota)}_{\widetilde{-s_0\phi},\mathfrak{c},\mathfrak{c}'}\big(h_{\mathfrak{c}'}^2\big)(z)
$$
$$
\leq A|b| d(x,y)^{\alpha} \sum_{z \in \{x,y\}} \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(z),
$$

where $A_0 = A_0\big(f, \mathcal{C}, d, |\phi|_{\alpha,(S^2,d)}, \alpha\big) > 2$ is a constant from Lemma 5.12 depending only on f, $\mathcal{C}$, d, $|\phi|_{\alpha,(S^2,d)}$, and α; and $C \geq 1$ is a constant from Lemma 3.8 depending only on f, $\mathcal{C}$, and d. The last inequality follows from $\frac{A_0}{\Lambda^{\alpha(N_1+M_0)}} \leq \frac{1}{4}$ (see (7.11)) and $\frac{A_0\big|\widetilde{-s_0\phi}\big|_{\alpha,(S^2,d)}}{1-\Lambda^{-\alpha}} \leq \frac{1}{2}b_0 \leq \frac{1}{2}Ab_0 \leq \frac{1}{2}A|b|$ (see (7.8) and (7.9)). The claim now follows immediately.

We now combine (7.48), Lemmas 7.10, 7.8, (7.46), and $|b| \geq b_0 \geq 2s_0 + 1$ (see (7.8)) to deduce that for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, we have

$$(7.49) \qquad \int_{X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \, \mathrm{d}\mu_{-s_0\phi} \leq \sum_{\substack{X \in \mathfrak{C}_b \\ X \subseteq Y_{\mathfrak{c}}^{M_0}}} \int_{f^{M_0}(X)} \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \, \mathrm{d}\mu_{-s_0\phi}$$
$$
\leq \sum_{\substack{X \in \mathfrak{C}_b \\ X \subseteq Y_{\mathfrak{c}}^{M_0}}} \mu_{-s_0\phi}\big(f^{M_0}(X)\big) \sup_{x \in f^{M_0}(X)} \Big\{\pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(x)\Big\}
$$
$$
\leq \sum_{\substack{X \in \mathfrak{C}_b \\ X \subseteq Y_{\mathfrak{c}}^{M_0}}} \mu_{-s_0\phi}\big(f^{M_0}(X)\big) \cdot 2 \inf_{x \in f^{M_0}(X)} \Big\{\pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(x)\Big\}
$$
$$
\leq C_{18} \sum_{\substack{X \in \mathfrak{C}_b \\ X \subseteq Y_{\mathfrak{c}}^{M_0}}} \mu_{-s_0\phi}\big(f^{M_0}\big(\mathfrak{X}'_{i_{J,X}}(X)\big)\big) \inf_{x \in f^{M_0}\big(\mathfrak{X}'_{i_{J,X}}(X)\big)} \Big\{\pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)(x)\Big\}
$$
$$
\leq C_{18} \sum_{\substack{X \in \mathfrak{C}_b \\ X \subseteq Y_{\mathfrak{c}}^{M_0}}} \int_{f^{M_0}\big(\mathfrak{X}'_{i_{J,X}}(X)\big)} \pi_{\mathfrak{c}}\Big(\mathbb{L}^{\iota}_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \, \mathrm{d}\mu_{-s_0\phi}
$$

$$\leq C_{18} \int_{W_J \cap X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)\, \mathrm{d}\mu_{-s_0\phi},$$

where $i_{J,X} \in \{1, 2\}$ can be set in such a way that either $(1, i_{J,X}, X) \in J$ or $(2, i_{J,X}, X) \in J$ due to the assumption that $J \in \mathcal{F}$ has a full projection, and the constant C_{18} can be chosen as $C_{18} := 2C_{\mu_{-s_0\phi}}^2 \exp\big(2m_0\big(\|-s_0\phi\|_{C^0(S^2)} + P(f, -s_0\phi)\big)\big) > 1$, which depends only on f, $\mathcal{C}$, d, and ϕ. Here the constant $C_{\mu_{-s_0\phi}} \geq 1$ is from Lemma 7.8, depending only on f, d, and ϕ.

We now observe that by (5.12) in Lemma 5.7 and Lemma 5.15,

$$(7.50) \qquad \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)\, \mathrm{d}\mu_{-s_0\phi} = \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} h_{\mathfrak{c}}^2 \, \mathrm{d}\mu_{-s_0\phi}.$$

Combining (7.50), (7.45), Lemma 5.10, (7.27) in Lemma 7.6, (7.47), and (7.49), we get

$$(7.51) \qquad \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} h_{\mathfrak{c}}^2 \, \mathrm{d}\mu_{-s_0\phi} - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \big|\pi_{\mathfrak{c}}\big(\mathbb{M}_{J,-s_0,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)\big|^2 \, \mathrm{d}\mu_{-s_0\phi}$$

$$= \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)\, \mathrm{d}\mu_{-s_0\phi} - \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \big|\pi_{\mathfrak{c}}\big(\mathbb{M}_{J,-s_0,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)\big|^2 \, \mathrm{d}\mu_{-s_0\phi}$$

$$\geq \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \cdot \Big(1 - \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(\big(\beta_J|_{X_{\mathfrak{b}}^0}\big)^2, \big(\beta_J|_{X_{\mathfrak{w}}^0}\big)^2\big)\Big)\Big)\, \mathrm{d}\mu_{-s_0\phi}$$

$$\geq \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{W_J \cap X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big) \cdot \Big(1 - \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(\big(\beta_J|_{X_{\mathfrak{b}}^0}\big)^2, \big(\beta_J|_{X_{\mathfrak{w}}^0}\big)^2\big)\Big)\Big)\, \mathrm{d}\mu_{-s_0\phi}$$

$$\geq \frac{\eta}{4} \exp\Big(-\iota\|\widetilde{-s_0\phi}\|_{C^0(S^2)}\Big) \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{W_J \cap X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)\, \mathrm{d}\mu_{-s_0\phi}$$

$$\geq \frac{\eta}{4C_{18}} \exp\Big(-\iota\|\widetilde{-s_0\phi}\|_{C^0(S^2)}\Big) \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} \pi_{\mathfrak{c}}\Big(\mathbb{L}^\iota_{\widetilde{-s_0\phi}}\big(h_{\mathfrak{b}}^2, h_{\mathfrak{w}}^2\big)\Big)\, \mathrm{d}\mu_{-s_0\phi}$$

$$\geq \frac{\eta}{4C_{18}} \exp\Big(-\iota\|\widetilde{-s_0\phi}\|_{C^0(S^2)}\Big) \sum_{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} h_{\mathfrak{c}}^2 \, \mathrm{d}\mu_{-s_0\phi}.$$

We now consider the general case where the potential is $\widetilde{-s\phi}$. Fix $\mathfrak{c}' \in \{\mathfrak{b}, \mathfrak{w}\}$ and an arbitrary point $x \in X_{\mathfrak{c}'}^0$. For each $X^\iota \in \mathbf{X}_{\mathfrak{c}'}^\iota$, denote $y_{X^\iota} := (f^\iota|_{X^\iota})^{-1}(x)$. Then by Definition 7.7 and (5.12) in Lemma 5.7,

$$\pi_{\mathfrak{c}'}\big(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)(x) = \sum_{\substack{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\ X^\iota\in\mathbf{X}_{\mathfrak{c}'}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota}) \exp\big(S_\iota\widetilde{-a\phi}(y_{X^\iota})\big)$$

$$\leq \sum_{\substack{\mathfrak{c}\in\{\mathfrak{b},\mathfrak{w}\}\ X^\iota\in\mathbf{X}_{\mathfrak{c}'}^\iota \\ X^\iota\subseteq X_{\mathfrak{c}}^0}} h_{\mathfrak{c}}(y_{X^\iota})\beta_J(y_{X^\iota}) \exp\big(S_\iota\widetilde{-s_0\phi}(y_{X^\iota})\big) \exp\big(\big|S_\iota\widetilde{-a\phi}(y_{X^\iota}) - S_\iota\widetilde{-s_0\phi}(y_{X^\iota})\big|\big)$$

$$\leq \pi_{\mathfrak{c}'}\big(\mathbb{M}_{J,-s_0,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)(x)e^{\iota\big(|a-s_0|\|\phi\|_{C^0(S^2)}+|P(f,-a\phi)-P(f,-s_0\phi)|+2\|\log u_{-a\phi}-\log u_{-s_0\phi}\|_{C^0(S^2)}\big)}.$$

Since the function $t \mapsto P(f, t\phi)$ is continuous (see for example, [PU10, Theorem 3.6.1]) and the map $t \mapsto u_{t\phi}$ is continuous on $C^{0,\alpha}(S^2, d)$ equipped with the uniform norm $\|\cdot\|_{C^0(S^2)}$ by Lemma 3.18, we can choose $a_0 \in (0, s_0)$ small enough, depending only on f, $\mathcal{C}$, d, α, and ϕ such that if $s = a + \mathbf{i}b$ with $a, b \in \mathbb{R}$ satisfies $|a - s_0| \leq a_0$ and $|b| \geq 2s_0 + 1$, then

$$\exp\big(\iota\big(|a - s_0|\|\phi\|_{C^0(S^2)} + |P(f, -a\phi) - P(f, -s_0\phi)| + 2\|\log u_{-a\phi} - \log u_{-s_0\phi}\|_{C^0(S^2)}\big)\big)$$

$$\leq \big(1 + (4C_{18})^{-1}\eta \exp\big(-\iota\|\widetilde{-s_0\phi}\|_{C^0(S^2)}\big)\big)^{1/2},$$

and consequently

$$(7.52) \qquad \pi_{\mathfrak{c}'}\big(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)(x) \leq \Big(1 + \frac{\exp\big(-\iota\|\widetilde{-s_0\phi}\|_{C^0(S^2)}\big)}{4C_{18}}\Big)^{1/2} \pi_{\mathfrak{c}'}\big(\mathbb{M}_{J,-s_0,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}})\big)(x).$$

Therefore, if $s = a + \mathbf{i}b$ with $a, b \in \mathbb{R}$ satisfies $|a - s_0| \leq a_0$ and $|b| \geq b_0 \geq 2s_0 + 1$ (see (7.8)), we get from (7.52) and (7.51) that

$$\sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} |\pi_{\mathfrak{c}}(\mathbb{M}_{J,-s,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))|^2 \, d\mu_{-s_0\phi}$$

$$\leq \left(1 + \frac{\eta \exp\left(-\iota \|\widetilde{-s_0\phi}\|_{C^0(S^2)}\right)}{4C_{18}}\right) \sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} |\pi_{\mathfrak{c}}(\mathbb{M}_{J,-s_0,\phi}(h_{\mathfrak{b}}, h_{\mathfrak{w}}))|^2 \, d\mu_{-s_0\phi}$$

$$\leq \left(1 - \frac{\eta^2 \exp\left(-2\iota \|\widetilde{-s_0\phi}\|_{C^0(S^2)}\right)}{16C_{18}^2}\right) \sum_{\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}} \int_{X_{\mathfrak{c}}^0} |h_{\mathfrak{c}}|^2 \, d\mu_{-s_0\phi}.$$

We finish the proof of (ii) by choosing

$$\rho := 1 - 16^{-1} C_{18}^{-2} \eta^2 \exp\left(-2\iota \|\widetilde{-s_0\phi}\|_{C^0(S^2)}\right) \in (0, 1),$$

which depends only on f, $\mathcal{C}$, d, α, and ϕ.

(iii) Given arbitrary $h_{\mathfrak{b}}$, $h_{\mathfrak{w}}$, $u_{\mathfrak{b}}$, and $u_{\mathfrak{w}}$ satisfying the hypotheses in (iii), we construct a subset $J \subseteq \{1, 2\} \times \{1, 2\} \times \mathfrak{C}_b$ as follows: For each $X \in \mathfrak{C}_b$,

    (1) if $\left\|Q_{\mathfrak{c}_X, 1}\right\|_{C^0(\mathfrak{X}_1(X))} \leq 1$, then include $(1, 1, X)$ in J, otherwise

    (2) if $\left\|Q_{\mathfrak{c}_X, 2}\right\|_{C^0(\mathfrak{X}_1(X))} \leq 1$, then include $(2, 1, X)$ in J, otherwise

    (3) if $\left\|Q_{\mathfrak{c}_X, 1}\right\|_{C^0(\mathfrak{X}_2(X))} \leq 1$, then include $(1, 2, X)$ in J, otherwise

    (4) if $\left\|Q_{\mathfrak{c}_X, 2}\right\|_{C^0(\mathfrak{X}_2(X))} \leq 1$, then include $(2, 2, X)$ in J,

where we denote $\mathfrak{c}_X \in \{\mathfrak{b}, \mathfrak{w}\}$ with the property that $X \subseteq Y_{\mathfrak{c}_X}^{M_0}$. Here functions $Q_{\mathfrak{c}, j} \colon Y_{\mathfrak{c}}^{M_0} \to \mathbb{R}$, $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $j \in \{1, 2\}$, are defined in Lemma 7.12.

By Lemma 7.12, at least one of the four cases above occurs for each $X \in \mathfrak{C}_b$. Thus, the set J constructed above has a full projection (c.f. Definition 7.4).

We finally set $\mathcal{E}_s := \bigcup\{J\}$, where the union ranges over all $h_{\mathfrak{b}}$, $h_{\mathfrak{w}}$, $u_{\mathfrak{b}}$, and $u_{\mathfrak{w}}$ satisfying the hypotheses in (iii).

We now fix such $h_{\mathfrak{b}}$, $h_{\mathfrak{w}}$, $u_{\mathfrak{b}}$, $u_{\mathfrak{w}}$, and the corresponding J constructed above. Then for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and each $x \in X_{\mathfrak{c}}^0$, we will establish (7.43) as follows:

    (1) If $x \notin \bigcup_{X \in \mathfrak{C}_b} f^{M_0}(\mathfrak{X}_1(X) \cup \mathfrak{X}_2(X))$, then by (7.23) and (7.24), $\beta_J(y) = 1$ for all $y \in f^{-(N_1 + M_0)}(x)$. Thus (7.43) holds for x by Definition 7.7, (5.12) in Lemma 5.7, and Definition 5.2.

    (2) If $x \in f^{M_0}(\mathfrak{X}_i(X))$ for some $X \in \mathfrak{C}_b$ and $i \in \{1, 2\}$, then one of the following two cases occurs:

        (a) $(1, i, X) \notin J$ and $(2, i, X) \notin J$. Then by (7.24), $\beta_J(y) = 1$ for all $y \in f^{-(N_1 + M_0)}(x)$. Thus (7.43) holds for x by Definition 7.7, (5.12) in Lemma 5.7, and Definition 5.2.

        (b) $(j, i, X) \in J$ for some $j \in \{1, 2\}$. Then by the construction of J, we have $(j', i', X) \in J$ if and only if $(j', i') = (j, i)$. We denote the inverse branches $\tau_k := \left(f^{N_1}\big|_{X_{\mathfrak{c}, k}^{N_1 + M_0}}\right)^{-1}$ for $k \in \{1, 2\}$. Write $z := \left(f^{N_1 + M_0}\big|_{X_{\mathfrak{c}, j}^{N_1 + M_0}}\right)^{-1}(x)$. Then $\beta_J(y) = 1$ for each $y \in f^{-(N_1 + M_0)}(x) \setminus \tau_j(\mathfrak{X}_i(X)) = f^{-(N_1 + M_0)}(x) \setminus \{z\}$. In particular, $\beta_J\left(\tau_{j_*}\left(f^{N_1}(z)\right)\right) = 1$, where $j_* \in \{1, 2\}$ and $j_* \neq j$. By the construction of J, we get $Q_{\mathfrak{c}, j}\left(f^{N_1}(z)\right) \leq 1$, i.e.,

$$\left|\sum_{k \in \{1, 2\}} \left(u_{\varsigma(\mathfrak{c}, k)} e^{S_{N_1} \widetilde{-s\phi}}\right)\left(\tau_k\left(f^{N_1}(z)\right)\right)\right|$$

$$\leq -\frac{1}{2}\eta h_{\varsigma(\mathfrak{c}, j)}(z) e^{S_{N_1} \widetilde{-a\phi}(z)} + \sum_{k \in \{1, 2\}} \left(h_{\varsigma(\mathfrak{c}, k)} e^{S_{N_1} \widetilde{-a\phi}}\right)\left(\tau_k\left(f^{N_1}(z)\right)\right)$$

$$\leq \left(\beta_J h_{\varsigma(\mathfrak{c}, j)} e^{S_{N_1} \widetilde{-a\phi}}\right)(z) + \left(\beta_J h_{\varsigma(\mathfrak{c}, j_*)} e^{S_{N_1} \widetilde{-a\phi}}\right)\left(\tau_{j_*}\left(f^{N_1}(z)\right)\right),$$

        where $\varsigma(\mathfrak{c}, k)$ is defined as in the statement of Lemma 7.12. Hence (7.43) holds for x by Definition 7.7, (5.12) in Lemma 5.7, and Definition 5.2.

We are going to establish (7.44) now. By (5.12) in Lemma 5.7, (5.29) in Lemma 5.12, Definition 5.2, and (7.27), for all $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$ and $x, x' \in X_{\mathfrak{c}}^0$ with $x \neq x'$,

$$\frac{1}{d(x,x')^\alpha}\left|\pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{N_1+M_0}(u_\mathfrak{b}, u_\mathfrak{w})\Big)(x) - \pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{N_1+M_0}(u_\mathfrak{b}, u_\mathfrak{w})\Big)(x')\right|$$

$$\leq \frac{1}{d(x,x')^\alpha} \sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\left|\mathcal{L}_{-s\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(\iota)}(u_{\mathfrak{c}'})(x) - \mathcal{L}_{-s\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(\iota)}(u_{\mathfrak{c}'})(x')\right|$$

$$\leq A_0 \sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\left(\bigg(\frac{A|b|}{\Lambda^{\alpha\iota}}\sum_{z\in\{x,x'\}}\mathcal{L}_{-a\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(\iota)}(h_{\mathfrak{c}'})(z)\bigg) + |b|\mathcal{L}_{-a\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(\iota)}(h_{\mathfrak{c}'})(x)\right)$$

$$\leq \left(\frac{A_0 A}{\Lambda^{\alpha\iota}} + A_0\right)|b| \sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\sum_{z\in\{x,x'\}}\mathcal{L}_{-a\widetilde{\phi},\mathfrak{c},\mathfrak{c}'}^{(\iota)}\big(2h_{\mathfrak{c}'}\beta_J|_{X_{\mathfrak{c}'}^0}\big)(z)$$

$$\leq \left(\frac{2A_0 A}{\Lambda^{\alpha\iota}} + 2A_0\right)|b| \sum_{z\in\{x,x'\}}\pi_\mathfrak{c}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(z) \leq A|b| \sum_{z\in\{x,x'\}}\pi_\mathfrak{c}(\mathbb{M}_{J,-s,\phi}(h_\mathfrak{b}, h_\mathfrak{w}))(z),$$

where the last inequality follows from $\frac{2A_0}{\Lambda^{\alpha\iota}} \leq \frac{1}{2}$ (see (7.11)) and $A \geq 4A_0$ (see (7.9)). $\qquad\square$

Proof of Theorem 6.2. We set $\iota := N_1 + M_0$, where $N_1 \in \mathbb{Z}$ is defined in (7.11) and $M_0 \in \mathbb{N}$ is a constant from Definition 7.1. We take the constants $a_0 \in (0, s_0)$ and $\rho \in (0, 1)$ from Proposition 7.13, and b_0 as defined in (7.8).

Fix arbitrary $s := a + \mathbf{i}b$ with $a, b \in \mathbb{R}$ satisfying $|a - s_0| \leq a_0$ and $|b| \geq b_0$. Fix arbitrary $u_\mathfrak{b} \in C^{0,\alpha}\big((X_\mathfrak{b}^0, d), \mathbb{C}\big)$ and $u_\mathfrak{w} \in C^{0,\alpha}\big((X_\mathfrak{w}^0, d), \mathbb{C}\big)$ satisfying

$$(7.53) \qquad\qquad \|u_\mathfrak{b}\|_{C^{0,\alpha}(X_\mathfrak{b}^0,d)}^{[\Im(s)]} \leq 1 \qquad \text{and} \qquad \|u_\mathfrak{w}\|_{C^{0,\alpha}(X_\mathfrak{w}^0,d)}^{[\Im(s)]} \leq 1.$$

We recall the constant $A \in \mathbb{R}$ defined in (7.9) and the subset $\mathcal{E}_s \subseteq \mathcal{F}$ constructed in Proposition 7.13.

We will construct sequences $\{h_{\mathfrak{b},k}\}_{k=-1}^{+\infty}$ in $K_{A|b|}\big(X_\mathfrak{b}^0, d\big)$, $\{h_{\mathfrak{w},k}\}_{k=-1}^{+\infty}$ in $K_{A|b|}\big(X_\mathfrak{w}^0, d\big)$, $\{u_{\mathfrak{b},k}\}_{k=0}^{+\infty}$ in $C^{0,\alpha}\big((X_\mathfrak{b}^0, d), \mathbb{C}\big)$, $\{u_{\mathfrak{w},k}\}_{k=0}^{+\infty}$ in $C^{0,\alpha}\big((X_\mathfrak{w}^0, d), \mathbb{C}\big)$, and $\{J_k\}_{k=0}^{+\infty}$ in $\mathcal{E}_s$ recursively so that the following properties are satisfied for each $k \in \mathbb{N}_0$, each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, and all $x, x' \in X_\mathfrak{c}^0$:

(1) $u_{\mathfrak{c},k} = \pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{k\iota}(u_\mathfrak{b}, u_\mathfrak{w})\Big)$.

(2) $|u_{\mathfrak{c},k}(x)| \leq h_{\mathfrak{c},k}(x)$ and $|u_{\mathfrak{c},k}(x) - u_{\mathfrak{c},k}(x')| \leq A|b|(h_{\mathfrak{c},k}(x) + h_{\mathfrak{c},k}(x'))d(x,x')^\alpha$.

(3) $\sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\int_{X_{\mathfrak{c}'}^0} h_{\mathfrak{c}',k}^2 \,\mathrm{d}\mu_{-s_0\phi} \leq \rho \sum_{\mathfrak{c}'\in\{\mathfrak{b},\mathfrak{w}\}}\int_{X_{\mathfrak{c}'}^0} h_{\mathfrak{c}',k-1}^2 \,\mathrm{d}\mu_{-s_0\phi}$.

(4) $\pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{\iota}(u_{\mathfrak{b},k}, u_{\mathfrak{w},k})\Big)(x) \leq \pi_\mathfrak{c}(\mathbb{M}_{J_k,-s,\phi}(h_{\mathfrak{b},k}, h_{\mathfrak{w},k}))(x)$ and

$$\left|\pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{\iota}(u_{\mathfrak{b},k}, u_{\mathfrak{w},k})\Big)(x) - \pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{\iota}(u_{\mathfrak{b},k}, u_{\mathfrak{w},k})\Big)(x')\right|$$
$$\leq A|b|\big(\pi_\mathfrak{c}\big(\mathbb{M}_{J_k,-s,\phi}(h_{\mathfrak{b},k}, h_{\mathfrak{w},k})\big)(x) + \pi_\mathfrak{c}\big(\mathbb{M}_{J_k,-s,\phi}(h_{\mathfrak{b},k}, h_{\mathfrak{w},k})\big)(x')\big)d(x,x')^\alpha.$$

We first set $h_{\mathfrak{c},-1} := 1/\rho$, $h_{\mathfrak{c},0} := \|u_\mathfrak{c}\|_{C^{0,\alpha}(X_\mathfrak{c}^0,d)}^{[b]} \in [0,1]$, and $u_{\mathfrak{c},0} := u_\mathfrak{c}$ for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$. Then clearly, Properties (1), (2), and (3) are satisfied for $k = 0$. By Property (2) for $k = 0$, we can choose $J_0 \in \mathcal{E}_s$ according to Proposition 7.13 (iii) such that Property (4) holds for $k = 0$.

We continue our construction recursively as follows. Assume that we have chosen $u_{\mathfrak{b},i} \in C^{0,\alpha}\big((X_\mathfrak{b}^0, d), \mathbb{C}\big)$, $u_{\mathfrak{w},i} \in C^{0,\alpha}\big((X_\mathfrak{w}^0, d), \mathbb{C}\big)$, $h_{\mathfrak{b},i} \in K_{A|b|}\big(X_\mathfrak{b}^0, d\big)$, $h_{\mathfrak{w},i} \in K_{A|b|}\big(X_\mathfrak{w}^0, d\big)$, and $J_i \in \mathcal{E}_s$ for some $i \in \mathbb{N}_0$. Then we define, for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$,

$$u_{\mathfrak{c},i+1} := \pi_\mathfrak{c}\Big(\mathbb{L}_{-s\widetilde{\phi}}^{\iota}(u_{\mathfrak{b},i}, u_{\mathfrak{w},i})\Big) \qquad \text{and} \qquad h_{\mathfrak{c},i+1} := \pi_\mathfrak{c}(\mathbb{M}_{J_i,-s,\phi}(h_{\mathfrak{b},i}, h_{\mathfrak{w},i})).$$

Then for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, by (5.9) we get $u_{\mathfrak{c},i+1} \in C^{0,\alpha}\big((X_\mathfrak{c}^0, d), \mathbb{C}\big)$, and by Proposition 7.13 (i) we have $h_{\mathfrak{c},i+1} \in K_{A|b|}\big(X_\mathfrak{c}^0, d\big)$. Property (1) for $k = i+1$ follows from Property (1) for $k = i$. Property (2) for $k = i+1$ follows from Property (4) for $k = i$. Property (3) for $k = i+1$ follows from Proposition 7.13 (ii). By Property (2) for $k = i+1$ and Proposition 7.13 (iii), we can choose $J_{i+1} \in \mathcal{E}_s$ such that Property (4) for $k = i + 1$ holds. This completes the recursive construction and the verification of Properties (1) through (4) for all $k \in \mathbb{N}_0$.

By (5.12) in Lemma 5.7, Properties (1), (2), (3), and Theorem 3.14 (iii), we have

$$\int_{X_{\mathfrak{c}}^0}\Big|\mathcal{L}^{(n\iota)}_{\widetilde{-s\phi},\mathfrak{c},\mathfrak{b}}(u_{\mathfrak{b}}) + \mathcal{L}^{(n\iota)}_{\widetilde{-s\phi},\mathfrak{c},\mathfrak{w}}(u_{\mathfrak{w}})\Big|^2\,\mathrm{d}\mu_{-s_0\phi} = \int_{X_{\mathfrak{c}}^0}\Big|\pi_{\mathfrak{c}}\Big(\mathbb{L}^{n\iota}_{-s\phi}(u_{\mathfrak{b}},u_{\mathfrak{w}})\Big)\Big|^2\,\mathrm{d}\mu_{-s_0\phi}$$

$$= \int_{X_{\mathfrak{c}}^0}|u_{\mathfrak{c},n}|^2\,\mathrm{d}\mu_{-s_0\phi} \le \int_{X_{\mathfrak{c}}^0}h^2_{\mathfrak{c},n}\,\mathrm{d}\mu_{-s_0\phi} \le \rho^n\left(\int_{X_{\mathfrak{b}}^0}h^2_{\mathfrak{b},0}\,\mathrm{d}\mu_{-s_0\phi} + \int_{X_{\mathfrak{w}}^0}h^2_{\mathfrak{w},0}\,\mathrm{d}\mu_{-s_0\phi}\right) \le \rho^n,$$

for all $\mathfrak{c} \in \{\mathfrak{b},\mathfrak{w}\}$ and $n \in \mathbb{N}$. □

8. Lattès maps and smooth potentials

8.1. Non-local integrability.
We briefly recall the notion of non-local integrability discussed in [LZhe23a, Section 7].

Let $f\colon S^2 \to S^2$ be an expanding Thurston map, d be a visual metric on S^2 for f, and $\mathcal{C} \subseteq S^2$ be a Jordan curve satisfying $f(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$. We define

$$(8.1) \qquad \Sigma^-_{f,\mathcal{C}} := \big\{\{X_{-i}\}_{i\in\mathbb{N}_0} : X_{-i} \in \mathbf{X}^1(f,\mathcal{C}) \text{ and } f\big(X_{-(i+1)}\big) \supseteq X_{-i}, \text{ for } i \in \mathbb{N}_0\big\}.$$

For each $X \in \mathbf{X}^1(f,\mathcal{C})$, since f is injective on X (see Proposition 3.6 (i)), we denote the inverse branch of f restricted on X by $f_X^{-1}\colon f(X) \to X$, i.e., $f_X^{-1} := (f|_X)^{-1}$.

Let $\psi \in C^{0,\alpha}((S^2,d),\mathbb{C})$ be a complex-valued Hölder continuous function with an exponent $\alpha \in (0,1]$. For each $\xi = \{\xi_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f,\mathcal{C}}$, we define the function

$$(8.2) \qquad \Delta^{f,\mathcal{C}}_{\psi,\xi}(x,y) := \sum_{i=0}^{+\infty}\big((\psi \circ f_{\xi_{-i}}^{-1} \circ \cdots \circ f_{\xi_0}^{-1})(x) - (\psi \circ f_{\xi_{-i}}^{-1} \circ \cdots \circ f_{\xi_0}^{-1})(y)\big)$$

for each $(x,y) \in \bigcup_{\substack{X\in\mathbf{X}^1(f,\mathcal{C})\\X\subseteq f(\xi_0)}} X \times X$.

The following lemma is verified in [LZhe23a, Section 7].

Lemma 8.1. *Let f, $\mathcal{C}$, d, ψ, α satisfy the Assumptions in Section 4. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $\xi = \{\xi_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f,\mathcal{C}}$. Then for each $X \in \mathbf{X}^1(f,\mathcal{C})$ with $X \subseteq f(\xi_0)$, we get that $\Delta^{f,\mathcal{C}}_{\psi,\xi}(x,y)$ as a series defined in (8.2) converges absolutely and uniformly in x, $y \in X$, and moreover, for each triple of x, $y, z \in X$, the identity*

$$(8.3) \qquad \Delta^{f,\mathcal{C}}_{\psi,\xi}(x,y) = \Delta^{f,\mathcal{C}}_{\psi,\xi}(z,y) - \Delta^{f,\mathcal{C}}_{\psi,\xi}(z,x)$$

holds with $\big|\Delta^{f,\mathcal{C}}_{\psi,\xi}(x,y)\big| \le C_1 d(x,y)^\alpha$, where $C_1 = C_1(f,\mathcal{C},d,\psi,\alpha)$ is a constant depending on f, $\mathcal{C}$, d, ψ, and α from Lemma 3.15.

Definition 8.2 (Temporal distance). Let f, $\mathcal{C}$, d, ψ, α satisfy the Assumptions in Section 4. We assume, in addition, that $f(\mathcal{C}) \subseteq \mathcal{C}$. For $\xi = \{\xi_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f,\mathcal{C}}$ and $\eta = \{\eta_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f,\mathcal{C}}$ with $f(\xi_0) = f(\eta_0)$, we define the *temporal distance* $\psi^{f,\mathcal{C}}_{\xi,\eta}$ as $\psi^{f,\mathcal{C}}_{\xi,\eta}(x,y) := \Delta^{f,\mathcal{C}}_{\psi,\xi}(x,y) - \Delta^{f,\mathcal{C}}_{\psi,\eta}(x,y)$ for each $(x,y) \in \bigcup_{\substack{X\in\mathbf{X}^1(f,\mathcal{C})\\X\subseteq f(\xi_0)}} X \times X$.

Recall that f^n is an expanding Thurston map with post $f^n =$ post f for each expanding Thurston map $f\colon S^2 \to S^2$ and each $n \in \mathbb{N}$.

Definition 8.3 (Local integrability). Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d a visual metric on S^2 for f. A complex-valued Hölder continuous function $\psi \in C^{0,\alpha}((S^2,d),\mathbb{C})$ is *locally integrable* (with respect to f and d) if for each natural number $n \in \mathbb{N}$, and each Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $f^n(\mathcal{C}) \subseteq \mathcal{C}$ and post $f \subseteq \mathcal{C}$, we have $\big(S_n^f\psi\big)^{f^n,\mathcal{C}}_{\xi,\eta}(x,y) = 0$ for all $\xi = \{\xi_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f^n,\mathcal{C}}$ and $\eta = \{\eta_{-i}\}_{i\in\mathbb{N}_0} \in \Sigma^-_{f^n,\mathcal{C}}$ satisfying $f^n(\xi_0) = f^n(\eta_0)$, and all $(x,y) \in \bigcup_{\substack{X\in\mathbf{X}^1(f^n,\mathcal{C})\\X\subseteq f^n(\xi_0)}} X \times X$.

The function ψ is *non-locally integrable* if it is not locally integrable.

8.2. **Characterizations.** In this section, we show that for Lattès maps, in the class of continuously differentiable real-valued potentials, the weaker condition of non-local integrability implies the (stronger) 1-strong non-integrability for some visual metric d for f. This leads to a characterization of the Prime Orbit Theorems in this context (Theorem A). The proof relies on the geometric properties of various metrics in this context and does not generalize to other rational expanding Thurston maps. However, we are able to show the genericity of the α-strong non-integrability condition in $C^{0,\alpha}(S^2, d)$ in the next section.

In order to carry out the cancellation argument in Section 7, it is crucial to have both the lower bound and the upper bound in (7.25). As seen in the proof of Proposition 7.5, the upper bound in (7.25) is guaranteed automatically by the Hölder continuity of the potential ϕ with the right exponent α. If we could assume in addition that the identity map on S^2 is a bi-Lipschitz equivalence (or more generally, snowflake equivalence) from a visual metric d to the Euclidean metric on S^2, and the temporal distance $\phi_{\xi,\xi'}^{f,\mathcal{C}}$ is nonconstant and continuously differentiable, then we could expect a lower bound with the same exponent as that in the upper bound in (7.25) near the same point.

However, for a rational expanding Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the chordal metric σ (see Remark 3.10 for the definition), which is bi-Lipschitz equivalent to the Euclidean metric away from the infinity, is never a visual metric for f (see [BM17, Lemma 8.12]). In fact, (S^2, d) is snowflake equivalent to $(\widehat{\mathbb{C}}, \sigma)$ if and only if f is topologically conjugate to a Lattès map (see [BM17, Theorem 18.1 (iii)] and Definition 8.4 below).

Recall that we call two metric spaces (X_1, d_1) and (X_2, d_2) are *bi-Lipschitz, snowflake,* or *quasisymmetrically equivalent* if there exists a homeomorphism from (X_1, d_1) to (X_2, d_2) with the corresponding property (see Definition 3.9).

We recall a version of the definition of Lattès maps.

Definition 8.4. Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map on the Riemann sphere $\widehat{\mathbb{C}}$. If f is expanding and the orbifold $\mathcal{O}_f = (S^2, \alpha_f)$ associated to f is parabolic, then it is called a *Lattès map*.

See [BM17, Chapter 3] and [Mi06] for other equivalent definitions and more properties of Lattès maps.

The special phenomenon mentioned above is not common in the study of Prime Orbit Theorems for smooth dynamical systems, as we are endeavoring out of Riemannian settings into general self-similar metric spaces. We content ourselves with the smooth examples of strongly non-integrable potentials for Lattès maps in Proposition 8.6 below.

Remark 8.5. For a Lattès map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the universal orbifold covering map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ of the orbifold $\mathcal{O}_f = (\widehat{\mathbb{C}}, \alpha_f)$ associated to f is holomorphic (see [BM17, Theorem A.26, Definition A.27, and Corollary A.29]). Let d_0 be the Euclidean metric on $\mathbb{C}$. Then the *canonical orbifold metric* ω_f of f is the pushforward of d_0 by Θ, more precisely,

$$\omega_f(p, q) := \inf\{d_0(z, w) : z \in \Theta^{-1}(p),\, w \in \Theta^{-1}(q)\}$$

for $p, q \in \widehat{\mathbb{C}}$ (see Section 2.5 and Appendices A.9 and A.10 in [BM17] for more details on the canonical orbifold metric). Let σ be the chordal metric on $\widehat{\mathbb{C}}$ as recalled in Remark 3.10. By [BM17, Proposition 8.5], ω_f is a visual metric for f. By [BM17, Lemma A.34], $(\widehat{\mathbb{C}}, \omega_f)$ and $(\widehat{\mathbb{C}}, \sigma)$ are bi-Lipschitz equivalent, i.e., there exists a bi-Lipschitz homeomorphism $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ from $(\widehat{\mathbb{C}}, \omega_f)$ to $(\widehat{\mathbb{C}}, \sigma)$. Moreover, by the discussion in [BM17, Appendix A.10], h cannot be the identity map.

Proposition 8.6. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Lattès map, and $d := \omega_f$ be the canonical orbifold metric of f on $\widehat{\mathbb{C}}$ (as recalled in Remark 8.5). Let $\phi\colon \widehat{\mathbb{C}} \to \mathbb{R}$ be a continuously differentiable real-valued function on the Riemann sphere $\widehat{\mathbb{C}}$. Then $\phi \in C^{0,1}(\widehat{\mathbb{C}}, d)$, and the following statements are equivalent:*

    (i) *ϕ is not co-homologous to a constant in $C(\widehat{\mathbb{C}}, \mathbb{C})$.*

    (ii) *ϕ is non-locally integrable with respect to f and d (in the sense of Definition 8.3).*

    (iii) *ϕ satisfies the 1-strong non-integrability condition with respect to f and d (in the sense of Definition 7.1).*

See Definition 3.1 for the notion of co-homologous functions.

Proof. We denote the Euclidean metric on $\mathbb{C}$ by d_0. Let σ be the chordal metric on $\mathbb{C}$ as recalled in Remark 3.10. By [BM17, Proposition 8.5], the canonical orbifold metric $d = \omega_f$ is a visual metric for f. Let $\Lambda > 1$ be the expansion factor of d for f.

Let $\mathcal{O}_f = (S^2, \alpha_f)$ be the orbifold associated to f (see Subsection 7.2 in [LZhe23a]). Since f has no periodic critical points, the ramification function $\alpha_f(z) < +\infty$ for all $z \in \widehat{\mathbb{C}}$ (see Definition 7.4 in [LZhe23a]).

By inequality (A.43) in [BM17, Appendix A.10],

$$(8.4) \qquad \sup\{\sigma(z_1, z_2)/d(z_1, z_2) : z_1,\, z_2 \in \widehat{\mathbb{C}},\, z_1 \neq z_2\} < +\infty.$$

By (8.4) and the assumption that ϕ is continuously differentiable, we get $\phi \in C^{0,1}(\widehat{\mathbb{C}}, \sigma) \subseteq C^{0,1}(\widehat{\mathbb{C}}, d)$.

We establish the equivalence of statements (i) through (iii) as follows.

(i) $\Longleftrightarrow$ (ii): The equivalence follows immediately from Theorem F in [LZhe23a].

(ii) $\Longleftrightarrow$ (iii). The backward implication follows from Proposition 7.3. To show the forward implication, we assume that ϕ is non-locally integrable. We observe from Lemma 3.11, Theorem F in [LZhe23a], and Lemma 7.2 that by replacing f with an iterate of f if necessary, we can assume without loss of generality that there exists a Jordan curve $\mathcal{C} \subseteq S^2$ such that $\operatorname{post} f \subseteq \mathcal{C}$, $f(\mathcal{C}) \subseteq \mathcal{C}$, and that there exist $\xi = \{\xi_{-i}\}_{i \in \mathbb{N}_0} \in \Sigma_{f,\mathcal{C}}^-$ and $\eta = \{\eta_{-i}\}_{i \in \mathbb{N}_0} \in \Sigma_{f,\mathcal{C}}^-$, $X^1 \in \mathbf{X}^1(f, \mathcal{C})$, and $u_0, v_0 \in X^1$ with $X^1 \subseteq f(\xi_0) = f(\eta_0)$, and

$$(8.5) \qquad \phi_{\xi,\eta}^{f,\mathcal{C}}(u_0, v_0) \neq 0.$$

By the continuity of $\phi_{\xi,\eta}^{f,\mathcal{C}}$ (see Lemma 8.1 and Definition 8.2), we can assume that $u_0,\, v_0 \in \operatorname{inte}(X^1)$. Without loss of generality, we can assume that $\infty \notin X^1$. We use the usual coordinate $z = (x, y) \in \mathbb{R}^2$ on X^1. We fix a constant $C_{22} \geq 1$ depending only on f and $\mathcal{C}$ such that

$$(8.6) \qquad C_{22}^{-1}\sigma(z_1, z_2) \leq d_0(z_1, z_2) \leq C_{22}d(z_1, z_2) \qquad \text{for all } z_1,\, z_2 \in X^1.$$

Note that $\alpha_f(z) = 1$ for all $z \in \widehat{\mathbb{C}} \setminus \operatorname{post} f$ (see Definition 7.4 in [LZhe23a]). Recall the notion of *singular conformal metrics* from [BM17, Appendix A.1]. By Proposition A.33 and the discussion proceeding it in [BM17, Appendix A.10], the following statements hold:

(1) The canonical orbifold metric d is a singular conformal metric with a conformal factor ρ that is continuous and positive everywhere except at the points in $\operatorname{supp}(\alpha_f) \subseteq \operatorname{post} f$.

(2) $d(z_1, z_2) = \inf_\gamma \int_\gamma \rho \, d\sigma$, where the infimum is taken over all σ-rectifiable paths γ in $\widehat{\mathbb{C}}$ joining z_1 and z_2.

(3) For each $z \in \widehat{\mathbb{C}} \setminus \operatorname{supp}(\alpha_f)$, there exists a neighborhood $U_z \subseteq \widehat{\mathbb{C}}$ containing z and a constant $C_z \geq 1$ such that $C_z^{-1} \leq \rho(u) \leq C_z$ for all $u \in U_z$.

Choose connected open sets V and U such that $u_0,\, v_0 \in V \subseteq \overline{V} \subseteq U \subseteq \overline{U} \subseteq \operatorname{inte}(X^1)$. By compactness and statement (3) above, there exists a constant $C_{23} \geq 1$ such that

$$(8.7) \qquad C_{23}^{-1} \leq \rho(z) \leq C_{23} \qquad \text{for all } z \in \overline{U}.$$

Thus by (8.6), (8.4), and a simple covering argument using statement (2) above, inequality (8.7), and the fact that $\overline{V} \subseteq U$, there exists a constant $C_{24} \geq 1$ depending only on f, $\mathcal{C}$, d, ϕ, and the choices of U and V such that

$$(8.8) \qquad C_{24}^{-1}d(z_1, z_2) \leq d_0(z_1, z_2) \leq C_{24}d(z_1, z_2) \qquad \text{for all } z_1,\, z_2 \in \overline{V}.$$

We denote, for each $i \in \mathbb{N}$,

$$(8.9) \qquad \tau_i := (f|_{\xi_{1-i}})^{-1} \circ \cdots \circ (f|_{\xi_{-1}})^{-1} \circ (f|_{\xi_0})^{-1} \text{ and } \tau_i' := (f|_{\eta_{1-i}})^{-1} \circ \cdots \circ (f|_{\eta_{-1}})^{-1} \circ (f|_{\eta_0})^{-1}.$$

We define a function $\Phi \colon X^1 \to \mathbb{R}$ by $\Phi(z) := \phi_{\xi,\eta}^{f,\mathcal{C}}(u_0, z)$ for $z \in X^1$ (see Definition 8.2 and Lemma 8.1).

Claim. Φ is continuously differentiable on V.

By Definition 8.2, it suffices to show that the function $\mathcal{D}(\cdot) := \Delta_{\phi,\xi}^{f,\mathcal{C}}(u_0, \cdot)$ is continuously differentiable on V. By Lemma 8.1, the function $\mathcal{D}(z) = \sum_{i=0}^{+\infty}((\phi \circ \tau_i)(u_0) - (\phi \circ \tau_i)(z))$ is the uniform limit of a series of continuous functions on V. Since $V \subseteq \operatorname{inte}(X^1)$, by (8.9) and Proposition 3.6 (i), the function $\phi \circ \tau_i$ is differentiable on V for each $i \in \mathbb{N}$.

We fix an arbitrary integer $i \in \mathbb{N}$. For each pair of distinct points $z_1, z_2 \in \text{inte}(X^1)$, we choose the maximal integer $m \in \mathbb{N}$ with the property that there exist two m-tiles $X_1^m, X_2^m \in \mathbf{X}^m(f, \mathcal{C})$ such that $z_1 \in X_1^m$, $z_2 \in X_2^m$, and $X_1^m \cap X_2^m \neq \emptyset$. Then by Proposition 3.6 (i) and Lemma 3.8 (i) and (ii),

$$\frac{|(\phi \circ \tau_i)(z_1) - (\phi \circ \tau_i)(z_2)|}{d(z_1, z_2)} \leq \frac{\|\phi\|_{C^{0,1}(\widehat{\mathbb{C}}, d)} \, \text{diam}_d(\tau_i(X_1^m \cup X_2^m))}{C^{-1} \Lambda^{-(m+1)}}$$

$$\leq \|\phi\|_{C^{0,1}(\widehat{\mathbb{C}}, d)} \frac{2C\Lambda^{-(m+i)}}{C^{-1} \Lambda^{-(m+1)}} \leq 2C^2 \|\phi\|_{C^{0,1}(\widehat{\mathbb{C}}, d)} \Lambda^{1-i},$$

where $C \geq 1$ is a constant from Lemma 3.8 depending only on f, $\mathcal{C}$, and d. Thus by (8.8),

$$\sup\left\{\left|\frac{\partial}{\partial x}(\phi \circ \tau_i)(z)\right| : z \in V\right\} \leq \sup\left\{\frac{|(\phi \circ \tau_i)(z_1) - (\phi \circ \tau_i)(z_2)|}{d_0(z_1, z_2)} : z_1, z_2 \in V, z_1 \neq z_2\right\}$$

$$\leq C_{24} \sup\left\{\frac{|(\phi \circ \tau_i)(z_1) - (\phi \circ \tau_i)(z_2)|}{d(z_1, z_2)} : z_1, z_2 \in V, z_1 \neq z_2\right\} \leq 2C_{24}C^2 \|\phi\|_{C^{0,1}(\widehat{\mathbb{C}}, d)} \Lambda^{1-i}.$$

Hence $\frac{\partial}{\partial x}\mathcal{D}$ exists and is continuous on V. Similarly, $\frac{\partial}{\partial y}\mathcal{D}$ exists and is continuous on V. Therefore, $\mathcal{D}$ is continuously differentiable on V, establishing the claim.

By the claim, (8.5), and the simple observation that $\phi_{\xi, \eta}^{f, \mathcal{C}}(u_0, u_0) = 0$, there exist numbers $M_0 \in \mathbb{N}$, $\varepsilon \in (0, 1)$, and $C_{25} > 1$, and M_0-tiles $Y_{\mathfrak{b}}^{M_0} \in \mathbf{X}_{\mathfrak{b}}^{M_0}(f, \mathcal{C})$ and $Y_{\mathfrak{w}}^{M_0} \in \mathbf{X}_{\mathfrak{w}}^{M_0}(f, \mathcal{C})$ such that $C_{25} \geq C_{24}$, $Y_{\mathfrak{b}}^{M_0} \cup Y_{\mathfrak{w}}^{M_0} \subseteq V \subseteq \text{inte}(X^1)$, and at least one of the following two inequalities holds:

(a) $\inf\left\{\left|\frac{\partial}{\partial x}\Phi(z)\right| : z \in h^{-1}\left(Y_{\mathfrak{b}}^{M_0} \cup Y_{\mathfrak{w}}^{M_0}\right)\right\} \geq 2C_{25}\varepsilon$,

(b) $\inf\left\{\left|\frac{\partial}{\partial y}\Phi(z)\right| : z \in h^{-1}\left(Y_{\mathfrak{b}}^{M_0} \cup Y_{\mathfrak{w}}^{M_0}\right)\right\} \geq 2C_{25}\varepsilon$.

We assume now that inequality (a) holds and remark that the proof in the other case is similar. Without loss of generality, we can assume that $\varepsilon \in \left(0, (2C_{25}C)^{-2}\right)$.

Then by Lemma 3.8 (v), for each $\mathfrak{c} \in \{\mathfrak{b}, \mathfrak{w}\}$, each integer $M \geq M_0$, and each M-tile $X \in \mathbf{X}^M(f, \mathcal{C})$ with $X \subseteq Y_{\mathfrak{c}}^{M_0}$, there exists a point $u_1(X) = (x_1(X), y_0(X)) \in X$ such that $B_d\left(u_1(X), C^{-1}\Lambda^{-M}\right) \subseteq X$. We choose $x_2(X) \in \mathbb{R}$ such that $|x_1(X) - x_2(X)| = (4C_{25}C)^{-1}\Lambda^{-M}$. Then by (8.8) and $C_{25} \geq C_{24}$, we get

$$u_2(X) := (x_2(X), y_0(X)) \in B_{d_0}\left(u_1(X), (2C_{25}C)^{-1}\Lambda^{-M}\right)$$

$$\tag{8.10} \subseteq B_d\left(u_1(X), (2C)^{-1}\Lambda^{-M}\right) \subseteq B_d\left(u_1(X), C^{-1}\Lambda^{-M}\right) \subseteq X.$$

In particular, the entire horizontal line segment connecting $u_1(X)$ and $u_2(X)$ is contained in $\text{inte}(X)$. By (8.10), Lemma 3.8 (ii), (8.8), and $C_{25} \geq C_{24}$, we get

$$\tag{8.11} \min\left\{d\left(u_1(X), \widehat{\mathbb{C}} \setminus X\right), d\left(u_2(X), \widehat{\mathbb{C}} \setminus X\right), d\left(u_1(X), u_2(X)\right)\right\}$$

$$\geq \min\left\{(2C)^{-1}\Lambda^{-M}, C_{25}^{-1}(4C_{25}C)^{-1}\Lambda^{-M}\right\} \geq \varepsilon \, \text{diam}_d(X).$$

On the other hand, by (8.8), $C_{25} \geq C_{24}$, Definition 8.2, inequality (a) above, and the mean value theorem,

$$\frac{\left|\phi_{\xi, \eta}^{f, \mathcal{C}}(u_1(X), u_2(X))\right|}{d(u_1(X), u_2(X))} \geq \frac{\left|\phi_{\xi, \eta}^{f, \mathcal{C}}(u_1(X), u_2(X))\right|}{C_{25}d_0(u_1(X), u_2(X))} = \frac{|\Phi(u_1(X)) - \Phi(u_2(X))|}{C_{25}|x_1(X) - x_2(X)|} \geq 2\varepsilon.$$

We choose

$$\tag{8.12} N_0 := \left\lceil \log_\Lambda\left(2C^2\varepsilon^{-2} |\phi|_{1, (\widehat{\mathbb{C}}, d)} C_0 / (1 - \Lambda^{-1})\right)\right\rceil.$$

where $C_0 > 1$ is a constant depending only on f, $\mathcal{C}$, and d from Lemma 3.13.

Fix arbitrary $N \geq N_0$. Define $X_{\mathfrak{c}, 1}^{N+M_0} := \tau_N\left(Y_{\mathfrak{c}}^{M_0}\right)$ and $X_{\mathfrak{c}, 2}^{N+M_0} := \tau_N'\left(Y_{\mathfrak{c}}^{M_0}\right)$ (c.f. (8.9)). Note that $\varsigma_1 = \tau_N|_{Y_{\mathfrak{c}}^{M_0}}$ and $\varsigma_2 = \tau_N'|_{Y_{\mathfrak{c}}^{M_0}}$.

Then by Definition 8.2, (8.11), Lemmas 8.1, 3.15, 3.8 (i) and (ii), and Proposition 3.6 (i),

$$\frac{|S_N\phi(\varsigma_1(u_1(X))) - S_N\phi(\varsigma_2(u_1(X))) - S_N\phi(\varsigma_1(u_2(X))) + S_N\phi(\varsigma_2(u_2(X)))|}{d(u_1(X), u_2(X))}$$

$$\geq \frac{\left|\phi_{\xi,\eta}^{f,\mathcal{C}}(u_1(X)), u_2(X))\right|}{d(u_1(X), u_2(X))} - \limsup_{n\to+\infty} \frac{|S_{n-N}\phi(\tau_n(u_1(X))) - S_{n-N}\phi(\tau_n(u_2(X)))|}{\varepsilon \operatorname{diam}_d(X)}$$

$$- \limsup_{n\to+\infty} \frac{|S_{n-N}\phi(\tau_n'(u_1(X))) - S_{n-N}\phi(\tau_n'(u_2(X)))|}{\varepsilon \operatorname{diam}_d(X)}$$

$$\geq 2\varepsilon - \frac{|\phi|_{1,(\widehat{\mathbb{C}},d)}\, C_0}{1 - \Lambda^{-1}} \cdot \frac{d(\tau_N(u_1(X)), \tau_N(u_2(X))) + d(\tau_N'(u_1(X)), \tau_N'(u_2(X)))}{\varepsilon \operatorname{diam}_d(X)}$$

$$\geq 2\varepsilon - \frac{|\phi|_{1,(\widehat{\mathbb{C}},d)}\, C_0}{1 - \Lambda^{-1}} \cdot \frac{\operatorname{diam}_d(\tau_N(X)) + \operatorname{diam}_d(\tau_N'(X))}{\varepsilon \operatorname{diam}_d(X)}$$

$$\geq 2\varepsilon - \frac{|\phi|_{1,(\widehat{\mathbb{C}},d)}\, C_0}{1 - \Lambda^{-1}} \cdot \frac{2C\Lambda^{-(M+N)}}{\varepsilon C^{-1}\Lambda^{-M}} \geq 2\varepsilon - \frac{2C^2\varepsilon^{-1}\,|\phi|_{1,(\widehat{\mathbb{C}},d)}\, C_0}{1 - \Lambda^{-1}}\Lambda^{-N_0} \geq \varepsilon,$$

where the last inequality follows from (8.12).

Therefore, ϕ satisfies the 1-strong non-integrability condition with respect to f and d. $\qquad\square$

Proof of Theorem A. By Proposition 8.6, $\phi \in C^{0,\alpha}(\widehat{\mathbb{C}}, d)$. So the existence and uniqueness of $s_0 > 0$ follows from Corollary 3.20.

The implication (i) $\Longrightarrow$ (iii) follows from Proposition 8.6 and Theorem C. The implication (iii) $\Longrightarrow$ (ii) is trivial. The implication (ii) $\Longrightarrow$ (i) follows immediately by a contradiction argument using [LZhe23a, Theorem F and Proposition 8.1]. $\qquad\square$

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