ON COMPUTABILITY OF EQUILIBRIUM STATES

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Abstract. Equilibrium states are natural dynamical analogs of Gibbs measures in thermodynamic formalism. This paper is devoted to the study of their computability in the sense of Computable Analysis. We show that the unique equilibrium state associated to a pair of a computable, topologically exact, distance-expanding, open transformation $T: X \to X$ and a computable Hölder continuous potential $\varphi: X \to \mathbb{R}$ is always computable. Furthermore, the Hausdorff dimension of the Julia set and the equilibrium state for the geometric potential of a computable hyperbolic rational map are computable. On the other hand, we introduce a mechanism to provide many examples with non-unique equilibrium states for each potential in a dense subset of the space of continuous potentials, which should be of interest independent of Computable Analysis. We also construct some computable dynamical systems whose equilibrium states are all non-computable.

Contents

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1. Introduction

The decidability problem was the third question of Hilbert's program proposed by Hilbert in Paris in 1900. In the 1930s, the *Halting Problem* given by Turing provided a counterexample to this question. Another significant contribution of Turing was the construction of a definite method of computation called *Turing machine*. Turing machines give a standard model of computation of discrete computability and complexity and have become the foundation of modern digital computers.

Nowadays, however, most digital computers are used for computations in continuous settings. The research on computations with real numbers began with Turing's work on the original definition of computable real numbers in 1937 ([Tu37]). The work of Banach and Mazur in 1937 ([BM37]) provided a definition of computability for real objects (such as subsets of $\mathbb{R}^n$ and functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$). The "bit model" has been developed, serving as a foundation of the tradition of Computable Analysis ([Gr55, La55, Ko91, Weih00]).

Computable Analysis on dynamical systems.

In recent decades, there has been dramatic growth in research on the computability and computational complexity of many objects generated by dynamical systems (see for example, [Ya21] and references therein). The sensitivity to initial conditions and the typical instability of many interesting systems implies that what is being observed on the computer screen could be completely unrelated to what was meant to be simulated.

This paper is devoted to the study of computability questions for equilibrium states (see Definition 2.4). It expands and continues the line of inquiry initiated by [BBRY11] of the first-named author and Braverman, Rojas, and Yampolsky. Recall that for a rational map $f(z) = P(z)/Q(z)$ on the Riemann sphere, where $P(z)$ and $Q(z)$ are mutually prime polynomials, with degree $\deg(f) = \max\{\deg(P), \deg(Q)\} \geq 2$, the Julia set $\mathcal{J}_f$ is defined as the locus of the chaotic dynamics of f, namely, the complement of the set where the dynamics of f is stable. Moreover, given a repelling periodic point w, the family of the averages of the Dirac measures supported on the points of $f^{-n}(w)$ converges, as n tends to infinity, in the weak* topology to an f-invariant measure λ (independent of the choice of w) called the Brolin–Lyubich measure.

In [BBRY11], a uniform machine was designed to compute the Brolin–Lyubich measure for any given rational map. Together with the existence of polynomials whose coefficients are all computable, but whose Julia sets are non-computable ([BY09]), the computability of Brolin–Lyubich measures leads to a conflict: heuristically speaking, a measure contains more information than its support, but in Computable Analysis, there exists a computable invariant probability measure whose support is, however, non-computable. Such a conflict can be reconciled by considering these two results as the computability properties of the same physical object from the statistical perspective and the geometric perspective, respectively. In this era of artificial intelligence and data science, the former undoubtedly deserves closer investigations.

Thermodynamic formalism is a powerful method for creating invariant measures with prescribed local behavior under iterations of the dynamical system. This theory, inspired by statistical mechanics, was created by Ruelle, Sinai, and others in the seventies ([Dob68, Si72, Bo75, Wa82]). Since then, the thermodynamic formalism has been applied in many classical contexts (see for example, [Bo75, Ru89, Pr90, KH95, Zi96, MauU03, BS03, Ol03, Yu03, PU10, MayU10]), and has remained at the frontier of researches in dynamical systems.

Among many other applications, thermodynamic formalism has, since its very early days, played a central role in the study of statistical properties of dynamical systems.

The key objects of investigation in thermodynamic formalism are invariant measures called equilibrium states. More precisely, for a continuous transformation $T\colon X \to X$ on a compact metric space (X, ρ), a T-invariant Borel probability measure μ of X, and a real-valued continuous function $\varphi\colon X \to \mathbb{R}$ called potential, one can define the measure-theoretic pressure and the topological pressure to describe the degree of complexity of the given dynamical system from different perspectives. The Variational Principle implies that the supremum of the measure-theoretic pressure equals to the topological pressure, over all invariant Borel T-invariant probability measures. We say that a Borel probability measure μ is an *equilibrium state* of T for φ if it maximizes the measure-theoretic pressure. In particular, if φ is a constant function, then an equilibrium state reduces to a *measure of maximal entropy* (see Subsection 2.2). In particular, the *Brolin–Lyubich measure* of a rational map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the measure of maximal entropy of f.

Statement of main results.

The main theorems of this paper include two positive results on the computability of equilibrium states in uniformly expanding systems, an application to complex dynamics, an application to the classical theory of thermodynamic formalism (independent of the interest from Computable Analysis) on non-uniqueness of equilibrium states, and some counterexamples.

Computability of equilibrium states.

Our first result concerns the computability of equilibrium states. Here is a list of assumptions used in this context.

Assumption A.

(i) $(X,\, \rho,\, \mathcal{S})$ is a computable metric space and X is a recursively compact set (in the sense of Definition 3.12).

(ii) $\varphi\colon X \to \mathbb{R}$ is a real-valued Hölder continuous potential with some exponents v_0 and constants a_0.

(iii) $T\colon X \to X$ is an open, topologically exact, continuous, distance-expanding (in the sense of Definition 2.1) map with respect to the metric ρ with some constants η, λ, and ξ.

(iv) φ and T are both computable (in the sense of Definition 3.9).

By classical results (see for example, Proposition 2.5 and [PU10, Corollary 5.2.14]), the Assumption A (i) through (iii) imply that there exists a unique equilibrium state for the transformation T and the potential φ. The Assumption A (iv) allows us to input the information of the system in the computer. For systems satisfying the above assumptions, we prove the following theorem.

Theorem A. *Let the quintet $(X,\, \rho,\, \mathcal{S},\, \varphi,\, T)$ satisfy the Assumption A in Section 1, and μ be the unique equilibrium state of the transformation T for the potential φ. Then μ is computable.*

For a more precise version of Theorem A, see Theorem 1.1 below. The Assumption A requires that the behavior of the dynamical system has nice properties in the whole space. However, generally speaking, the dynamical behavior may be merely good enough in an invariant subspace rather than in the whole space. For dynamical systems, we prove Theorem B, a "subspace" version of Theorem A. We give a list of assumptions that describes the systems we consider in Theorem B as follows.

Assumption B.

(i) $(X, \rho, \mathcal{S})$ is a computable metric space.

(ii) $(D, \rho, \mathcal{S}')$ is a computable metric space, where $\mathcal{S}'$ is a uniformly computable sequence of points in the computable metric space $(X, \rho, \mathcal{S})$. D is a recursively compact set in the computable metric space $(D, \rho, \mathcal{S}')$.

(iii) $\varphi\colon D \to \mathbb{R}$ is a real-valued Hölder continuous potential with some exponents v_0 and constants a_0.

(iv) $T\colon X \to X$ is a transformation such that D is completely T-invariant and $T|_D\colon D \to D$ is an open, topologically exact, continuous, distance-expanding map with respect to the metric ρ with some constants η, λ, and ξ.

(v) φ and $T|_D$ are both computable in the computable metric space $(D, \rho, \mathcal{S}')$.

The Assumption B (ii) through (iv) give the uniqueness of the equilibrium state of the transformation $T|_D$ for the potential φ. Combined with Theorem 1.1, the Assumption B (ii) through (v) imply that the equilibrium state is computable in the computable metric space $(D, \rho, \mathcal{S}')$. Then the Assumption B (ii) and (iii) allow us to transfer the computability in the computable metric space $(D, \rho, \mathcal{S}')$ to the computability in the computable metric space $(X, \rho, \mathcal{S})$. For systems satisfying the above assumptions, we prove the following theorem.

Theorem B. *Let the septet $(X, D, \rho, \mathcal{S}, \mathcal{S}', \varphi, T)$ satisfy the Assumption B in Section 1, and μ be the unique equilibrium state of the transformation $T|_D$ for the potential φ. Then μ is computable.*

For a more precise version of Theorem B, see Theorem 1.2 below.

As an application, we will apply Theorem 1.2 to study the equilibrium states associated to geometric potentials for hyperbolic rational maps on the Riemann sphere.

We say that a rational map is *hyperbolic* if it is distance-expanding after iterations on the Julia set with respect to the spherical metric. Hyperbolic rational maps are abundant and are expected to be dense among all the rational maps. In the case of quadratic rational maps, see the renowned MLC Conjecture (refer to [Min06, Appendix G] for reference). The studies on equilibrium states as well as the theory of thermodynamic formalism for hyperbolic rational maps are by now well-developed and established. Given $t \in \mathbb{R}$, it is well-known that a hyperbolic rational map f and the geometric potential $t\varphi_f(z) = t\log\bigl(f^{\#}(z)\bigr)$ (see (5.1)) restricted to the Julia set $\mathcal{J}_f$ admit a unique non-atomic equilibrium state μ_f which is supported on $\mathcal{J}_f$. With the above conventions, we state the following theorem.

Theorem C. *There exists an algorithm which, on input oracles of the coefficients of a hyperbolic rational map f of degree $d \geq 2$ and of $t \in \mathbb{R}$, computes the unique equilibrium state of $f|_{\mathcal{J}_f}$ for the potential $t\varphi_f(z)$. Moreover, the Hausdorff dimension of the Julia set $\mathcal{J}_f$ is computable.*

Note that the proof of Theorem C depends on the computability of the hyperbolic Julia sets ([Br04]). Moreover, since the computability of pressure is a consequence of the proof of Theorem 1.1, the pressure function $\mathrm{P}(t) \coloneqq P(f|_{\mathcal{J}_f}, -t\varphi_f)$ is computable. Because $\mathrm{P}(t)$ is strictly decreasing with respect to $t \in \mathbb{R}$, as a byproduct, we can compute its unique zero, that is, by Theorem 9.1.6 and Corollary 9.1.7 in [PU10], the Hausdorff dimension of $\mathcal{J}_f$.

Non-uniqueness of equilibrium states.

Determining which dynamical system admits a unique equilibrium state is a central problem in ergodic theory and a valuable tool for studying statistical properties of a system (for example, detecting the existence of a phase transition).

As indicated by Hofbauer [Hof75], this question was motivated by the studies of *intrinsically ergodic* dynamical systems, i.e., those having a unique measure of maximal entropy, see for examples, [Bu97, Bu05, Bo74, CT12, Hof79, Hof81, Par64, Weis70, Weis73].

For general equilibrium states beyond measures of maximal entropy, most results focus on proving the uniqueness, and there are limited examples of systems and potentials with multiple equilibrium states. In the context of a uniformly hyperbolic (or expanding) map, Bowen [Bo75] proved that an equilibrium state does exist and is unique if the potential is Hölder continuous and the map is topologically transitive. In addition, the theory for finite shifts was developed and used to achieve similar results for smooth dynamics. In contrast, Hofbauer [Hof75] constructed a particular class of continuous but non-Hölder potentials ϕ for the full shift, each of which admits two equilibrium states.

Beyond uniform hyperbolicity, the uniqueness of equilibrium state was studied by many authors, including Bruin, Keller, Li, Rivera-Letelier, Iommi, Doobs, and Todd [BK98, LRL14, IT15, IT10, DT21] for interval maps; Denker and Urbanski [DU91] for rational maps; the third-named author, Das, Przytycki, Tiozzo, Urbański, and Zdunik [Li18, DPTUZ21] for branched covering maps; Buzzi, Sarig, and Yuri [BS03, Yu03], for countable Markov shifts and for piecewise expanding maps in one and higher dimensions. For local diffeomorphisms with some non-uniform expansion, there are results due to Varandas and Viana [VV10], Pinheiro [Pi11], Ramos and Viana [RV17]. All of these results focus on establishing the existence and uniqueness of the equilibrium state for a potential with low oscillation.

On the other hand, much fewer examples that admit multiple equilibrium states have been explored. They include

(i) Makarov and Smirnov [MS00, MS03], and Rivera-Letelier and Przytycki [PRL11] systematically studied the pressure function for rational maps f with geometric potential $\phi_t = -t\log|f^{\#}|$, and constructed examples admitting multiple equilibrium states. For a similar study on (generalized) interval maps, see Rivera-Letelier and Przytycki [PRL14].

(ii) Examples of intermittent maps admitting multiple equilibrium states for the geometric potential were studied in [VV10] via Pesin theory.

Applying Computable Analysis, we establish the following result.

Theorem D. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X a recursively compact set. Assume that $T\colon X \to X$ is a computable transformation of X with $h_{\mathrm{top}}(T) = 0$ whose non-wandering set $\Omega(T)$ contains no computable points. Then for each computable function $\varphi\colon X \to \mathbb{R}$, there exist at least two equilibrium states.*

The non-wandering set $\Omega(T)$ is called in (2.1).

Note that the set of computable functions $\varphi\colon X \to \mathbb{R}$ is dense in the space of real-valued continuous functions on X (equipped with the uniform norm).

For example, let $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the map in Theorem F below. Let the potential $\varphi\colon \mathbb{S}^1 \to \mathbb{R}$ be any rational linear combination of finite compositions of elementary functions (with period 1). Then there exist at least two equilibrium states of T for φ.

More generally, we have the following result.

Theorem E. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X a recursively compact set. Assume that $T\colon X \to X$ is a computable transformation of X satisfying the following conditions:*

(i) *The measure-theoretic entropy function $\mathcal{H}\colon \mathcal{M}_T(X) \to \mathbb{R}$ given by $\mathcal{H}(\mu) \coloneqq h_\mu(T)$, $\mu \in \mathcal{M}_T(X)$, is computable.*

(ii) *There exists no computable point in $\bigcup_{\mu \in \mathcal{M}_T(X)} \operatorname{supp}(\mu)$.*

Then for each computable function $\varphi\colon X \to \mathbb{R}$, there exist at least two equilibrium states of T.

Here $\mathcal{M}_T(X)$ is the space of T-invariant Borel probability measures equipped with the Wasserstein–Kantorovich metric W_ρ (see Section 2.1).

Counterexamples.

In the following theorem, we provide an example of a computable dynamical system with zero topological entropy (hence, for which equilibrium states exist), whose every equilibrium state for every continuous potential is non-computable.

Theorem F. *There exists a computable transformation $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ of the unit circle $\mathbb{S}^1$ such that for each continuous potential $\varphi\colon \mathbb{S}^1 \to \mathbb{R}$, the following statements hold:*

(i) *There exists at least one equilibrium state of T for φ.*

(ii) *There is no computable T-invariant probability measure.*

(iii) *If, in addition, φ is computable, then there exist at least two equilibrium states of T for φ, and all equilibrium states of T for φ are non-computable. Moreover, $P(T, \varphi)$ is computable.*

Recall that the topological entropy reflects a degree of complexity of a dynamical system. We, furthermore, show that there is a dynamical system on the 2-torus with arbitrarily high topological entropy that admits equilibrium states for each Hölder continuous potential, and whose equilibrium states are all non-computable.

Theorem G. *There exists a continuous transformation $\widehat{T}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ with non-zero topological entropy such that for each continuous potential $\widehat{\varphi}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$, the following statements hold:*

(i) *There exists at least one equilibrium state of $\widehat{T}$ for $\widehat{\varphi}$.*

(ii) *There is no computable $\widehat{T}$-invariant probability measure.*

(iii) *If, in addition, $\widehat{\varphi}(x, y) = \varphi_1(x) + \varphi_2(y)$, $x, y \in \mathbb{S}^1$, for some computable functions $\varphi_1, \varphi_2\colon \mathbb{S} \to \mathbb{R}$ with φ_2 being Hölder continuous, then*
   (a) *there exist at least two equilibrium states of $\widehat{T}$ for $\widehat{\varphi}$,*
   (b) *all equilibrium states of $\widehat{T}$ for $\widehat{\varphi}$ are non-computable,*
   (c) *the topological pressure $P\big(\widehat{T}, \widehat{\varphi}\big)$ is computable.*

Strategy of the proof.

We discuss below the strategies applied in the proofs of our main results.

Let us concentrate on the proof of Theorem A first. Here is a precise version of Theorem A.

Theorem 1.1. *There exists an algorithm that satisfies the following property:*

For each quintet $(X, \rho, \mathcal{S}, \varphi, T)$ *satisfying the Assumption A in Section 1, this algorithm outputs a Borel probability measure* $\mu_n \in \mathcal{P}(X)$ *supported on finitely many points of* $\mathcal{S}$ *such that the Wasserstein–Kantorovich distance*

$$W_\rho(\mu_n, \mu) \leq 2^{-n},$$

after inputting the following data in this algorithm:

(i) *an algorithm outputting a net* $G \subseteq \mathcal{S}$ *of* X *with any given precision (in the sense of Definition 3.14),*
(ii) *two algorithms computing functions* φ *and* T *(in the sense of Definition 3.9), respectively,*
(iii) *two rational constants* α_0 *and* v_0 *satisfying that* φ *is Hölder continuous with the exponent* v_0 *and the constant* α_0*,*
(iv) *three rational constants* η*,* λ*, and* ξ *satisfying that* T *is distance-expanding with these constants (in the sense of Definition 2.1),*
(v) *a constant* $n \in \mathbb{N}$*.*

The Wasserstein–Kantorovich metric W_ρ on the set $\mathcal{P}(X)$ of Borel probability measures is defined in Section 2.1.

The structure of our proof of Theorem 1.1 is inspired by the proof of Theorem A in [BBRY11].

However, since we work on equilibrium states, new ideas are needed in our proofs. In particular, in Subsection 4.1 of our paper, we use the technique of cones to estimate the convergence rate of the iterations of a normalized Ruelle operator with explicit expressions for related constants (see Theorem 4.1). There are several such priori bounds in the literature. To our knowledge, a priori upper bound on this rate in the setting of smooth (or more generally Markov) expanding maps with smooth observable was investigated back to Ferrero ([Fer81]), Rychlik ([Ryc89]), and Hunt ([Hun96]). In 1995, Liverani in [Liv95a, Liv95b] reintroduced a technology of cones (used by Birkhoff [Bi57]) and gave a new upper bound of the convergence rate of the iterations of a normalized Perron–Frobenius operator for general (non-Markov) non-smooth hyperbolic systems with discontinuities. In [BJ08], Bandtlow and Jenkinson commented that "This bound seems to be better than Rychlik's (see the calculations in [Fro97]), and the technique has subsequently been used to provide other explicit bounds...". Indeed, the key ingredient of this method is that there exists a convex cone of functions that is 'strictly' inside itself by the Perron–Frobenius operator in the following sense: the image of the cone under the operator is a bounded subspace with respect to the Hilbert projective metric associated to the cone. Moreover, the diameter of the image with respect to such metric gives an estimation for the upper bound of the convergence rate (see Proposition 4.3).

In the setting of smooth expanding map on a compact and connected manifold, Viana [Vi97, Chapter 2] writes a neat and comprehensive estimating for the convergence rate of the iterations of a normalized Perron–Frobenius operator in Hölder norm. It's notable that the local estimations for the range of functions can be extended to the range of the whole manifold because of the connectedness. However, taking into consideration that our setting with totally disconnected set, more ingredients need to be proposed. To get rid of this, we add a new index into the origin cone in [Vi97] to describe the range of functions (see Definition 4.4) in order to estimate the diameter (see Proposition 4.6). Moreover, to make sure that the operator is a strict contraction on these new finer cones, we replace the normalized Ruelle operator by a new operator and combine some ideas from [PU10] in our study (see Proposition 4.5). Finally, we use the convergence rate of the new operator to estimate that of the normalized Ruelle operator.

The estimation for the convergence rate (Theorem 4.1) enables us to establish the computability of the eigenfunction u_φ of the Ruelle operator. Furthermore, in Subsection 4.2, we deduce the computability of the Jacobian of the equilibrium state whose explicit expression is given by

$$J_\varphi(x) = \frac{u_\varphi(T(x))}{u_\varphi(x)} \exp(P(T, \varphi) - \varphi(x)).$$

In Subsection 4.3, we establish the lower-computable openness of some subsets U and W of $\mathcal{P}(X)$ (see Lemma 3.7) by expressing them as unions of uniformly computable sequences of lower-computable open sets. It is worth noting that the lower-computable openness of U depends on the computability of the transformation T, while the lower-computable openness of W depends on the computability of the Jacobian function $J_\varphi(x)$ in addition to the computability of T. In Subsection 4.4, we complete the proof of Theorem 1.1 and apply it to establish Theorem B. Here is a precise version of Theorem B.

Theorem 1.2. *There exists an algorithm that satisfies the following property:*

For each septet $(X, D, \rho, \mathcal{S}, \mathcal{S}', \varphi, T)$ satisfying the Assumption B in Section 1, this algorithm outputs $\mu_n \in \mathcal{P}(X)$ supported on finitely many points of $\mathcal{S}$ such that the Wasserstein–Kantorovich distance

$$W_\rho(\mu_n, \mu) \leq 2^{-n},$$

after inputting the following data in this algorithm:

(i) *an algorithm outputting a net $G \subseteq \mathcal{S}'$ of D with any given precision (in the sense of Definition 3.14),*
(ii) *two algorithms computing functions φ and $T|_D$ (in the sense of Definition 3.9), respectively,*
(iii) *two rational constants α_0 and v_0 satisfying that φ is Hölder continuous with the exponent v_0 and the constant α_0,*
(iv) *three rational constants η, λ, and ξ satisfying that T is distance-expanding with these constants (in the sense of Definition 2.1),*
(v) *a constant $n \in \mathbb{N}$.*

Next, we discuss the strategy of our proof of Theorem C. By Theorem 1.2, it suffices to check that the septet $(\widehat{\mathbb{C}}, \mathcal{J}_f, d_{\widehat{\mathbb{C}}}, \mathbb{Q}^2, \bigcup_{n\in\mathbb{N}} f^{-n}(w), t\varphi_f, f)$ satisfies the Assumption B in Section 1, where $\mathbb{Q}^2$ is defined as $\{a + bi : a, b \in \mathbb{Q}\}$, and w is a repelling periodic point of the rational map f.

The main difficulty in this part is to show that $\mathcal{J}_f$ is recursively compact in the compact metric space $(\mathcal{J}_f, d_{\widehat{\mathbb{C}}}, \bigcup_{n\in\mathbb{N}} f^{-n}(w))$. We use the computability of the hyperbolic Julia sets proved by Braverman [Br04] to establish this statement.

In Section 6, to establish Theorems E and D, we establish Theorem 6.3 and apply it to demonstrate Theorem E, from which we derive Theorem D.

To construct the counterexamples, in Subsection 7.1, we recall a computable homeomorphism $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ satisfying statement (ii) of Theorem F, which is given in [GHR10, Section 4.1]. Since the topological entropy for a homeomorphism of the unit circle is zero, T satisfies statement (i) of Theorem F. Finally, in Subsection 7.2, we construct $\widehat{T} \coloneqq T \times T_d$, where $T_d(x) := dx \pmod 1$. We show that there exists no computable $\widehat{T}$-invariant probability measure. On the other hand, by using results on asymptotic h-expansiveness, we demonstrate that for each continuous potential, there exists at least one equilibrium state for $\widehat{T}$. On the basis of these two ingredients, we establish Theorem G.

All of our analyses for Theorems 1.1 and 1.2 should still work after necessary modifications if we replace the algorithms in the inputs by oracles. For the simplicity of the presentation, we leave the details to the reader.

Structure of the paper.

In Section 2, we introduce our notations and review some basic notions and results in ergodic theory and thermodynamic formalism. Section 3 covers some basics of Computable Analysis. Section 4 is dedicated to the proofs of Theorems 1.1 and 1.2. In Subsection 4.1, we prove that the convergence of the iterations of a normalized Ruelle operator is at an exponential rate with explicit expressions for the related constants. Based on this, we demonstrate the computability of topological pressure $P(T,\varphi)$ and the function u_φ, which leads to the computability of the Jacobian of the equilibrium state in Subsection 4.2. Then, we prove Lemma 4.13 in Subsection 4.3 and establish Theorems 1.1 and 1.2 in Subsection 4.4. In Section 5, we introduce some results on the dynamical and algorithmic aspects of hyperbolic rational maps and prove Theorem C. We establish Theorems D and E by proving and applying Theorem 6.3 in Section 6. In Section 7, we construct some computable systems whose equilibrium states are all non-computable, establishing Theorems F and G.

2. Preliminary on thermodynamic formalism

In this section, we go over some notations, key concepts, and useful results in ergodic theory and thermodynamic formalism.

2.1. **Notations.** Denote $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For each $k \in \mathbb{N}$, we fixed a bijection between $\mathbb{N}^k$ and $\mathbb{N}$ call it the "canonical bijection between $\mathbb{N}^k$ and $\mathbb{N}$". Set $\mathbb{N}^* := \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, and we fixed a bijection between $\mathbb{N}^*$ and $\mathbb{N}$ call it the "canonical bijection between $\mathbb{N}^*$ and $\mathbb{N}$".

For a subset A of a set X, the characteristic function of A is denoted by $\mathbb{1}_A$. Write $\mathbb{1} := \mathbb{1}_X$.

Let (X, ρ) be a compact metric space. Then $C(X)$ and $C^*(X)$ denote the Banach space of continuous functions from X to $\mathbb{R}$ with the norm $\|u\|_\infty := \sup\{|u(x)| : x \in X\}$ and the dual space of $C(X)$, respectively. Given a constant $C > 0$, a function $g \in C(X)$ is *C-Lipschitz* if $|g(x) - g(y)| \leq C\rho(x, y)$ for each pair of $x, y \in X$, and C-Lip(X) denotes the set of real-valued C-Lipschitz continuous functions. For all $x \in X$ and $r > 0$, let $B(x, r)$ denote the ball of radius r centered at x.

Let $T\colon X \to X$ be a continuous transformation. The *non-wandering set* for T is denoted by

$$\Omega(T) := \Big\{x \in X : \bigcup_{n\in\mathbb{N}} T^{-n}(B(x, r)) \cap B(x, r) \neq \emptyset \text{ for each } r > 0\Big\}. \tag{2.1}$$

For all $\varphi \in C(X)$, $n \in \mathbb{N}$, and $x \in X$, denote

$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi \circ T^k(x).$$

Let $\mathcal{P}(X)$ and $\mathcal{M}_T(X)$ denote the set of Borel probability measures and the set of T-invariant Borel probability measures on X, respectively. The *Wasserstein–Kantorovich metric* W_ρ on

$\mathcal{P}(X)$ is defined by

$$W_\rho(\mu,\nu) \coloneqq \sup\biggl\{\biggl|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu\biggr| : f \in 1\text{-Lip}(X)\biggr\}.$$

2.2. **Basic concepts in ergodic theory.** We begin with a brief introduction of distance-expanding transformations and topologically transitive (resp. exact) transformations. See for example, [PU10, Chapters 3 and 4] for more details.

Given constants $\eta > 0$ and $\lambda > 1$, assume that (X, ρ) is a compact metric space and $T\colon X \to X$ is a continuous open transformation satisfying

$$\rho(T(x), T(y)) \geq \lambda\rho(x, y) \quad \text{for each pair of } x,\, y \in X \text{ with } \rho(x, y) \leq 2\eta. \tag{2.2}$$

Then there exists a constant $\xi > 0$ such that $B(T(x), \xi) \subseteq T(B(x, \eta))$ for each $x \in X$. Hence, for each $x \in X$ and each $n \in \mathbb{N}$, one can define the branches of the inverse map $T_x^{-n}\colon B(T^n(x), \xi) \to B(x, \eta)$ of T^n in the sense of

$$T^{-n}(A) = \bigcup_{x \in T^{-n}(y)} T_x^{-n}(A)$$

for all $y \in X$ and $A \subseteq B(y, \xi)$ such that the following inequalities hold (see for example, [PU10, Section 4.1]):

$$\rho(T_x^{-n}(y), T_x^{-n}(z)) \leq \lambda^{-n}\rho(y, z) \text{ for all } n \in \mathbb{N},\, x \in X, \text{ and } y,\, z \in B(T^n(x), \xi). \tag{2.3}$$

Definition 2.1. Let (X, ρ) be a compact metric space. Consider $\eta,\, \xi > 0$, and $\lambda > 1$. A transformation $T\colon X \to X$ is said to be *distance-expanding* with respect to the metric ρ with the constants η, λ, and ξ if (2.2) and (2.3) both hold.

In a topological space X, recall that a continuous transformation $T\colon X \to X$ is *topologically transitive* if, for each pair of non-empty open sets $U,\, V \subseteq X$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Moreover, as a stronger property, we say a continuous transformation $T\colon X \to X$ is *topologically exact* if, for each non-empty open sets $U \subseteq X$, there exists $n \in \mathbb{N}$ such that $T^n(U) = X$.

Under the above conventions, given the constants $\eta,\, \xi > 0$, and $\lambda > 1$, a list of assumptions that are applied in the rest of this section is displayed below:

Assumption C.

(i) The metric space (X, ρ) is compact.
(ii) The transformation $T\colon X \to X$ is continuous, topologically transitive, open, and distance-expanding with respect to the metric ρ with the constants η, λ, and ξ.

Next, we recall some notations and results in ergodic theory. For more details, we refer to [PU10, Chapter 2].

Let X be a topological space. A *measurable partition* $\mathcal{A}$ of X is a cover $\mathcal{A} = \{A_j : j \in J\}$ of X consisting of finite or countably many mutually disjoint Borel sets. Unless otherwise stated, we assume that our partitions are all measurable. For each partition $\mathcal{A}$ of X and $x \in X$, the unique element of $\mathcal{A}$ containing x is denoted by $\mathcal{A}(x)$.

Let $\mathcal{A} = \{A_j : j \in J\}$ and $\mathcal{B} = \{B_k : k \in K\}$ be two covers of X, respectively, where J and K are corresponding index sets. We say that $\mathcal{A}$ is a *refinement* of $\mathcal{B}$ if for each $j \in J$, there exists $k = k(j) \in K$ such that $A_j \subseteq B_k$. The *common refinement* $\mathcal{A} \vee \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is a cover given by $\mathcal{A} \vee \mathcal{B} \coloneqq \{A_j \cap B_k : j \in J,\, k \in K\}$.

For each continuous transformation $T\colon X \to X$, by $T^{-1}(\mathcal{A})$ we denote the partition $\{T^{-1}(A_j) : j \in J\}$, and for each $n \in \mathbb{N}$, write

$$\mathcal{A}_T^n \coloneqq \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{A}) = \mathcal{A} \vee T^{-1}(\mathcal{A}) \vee \cdots \vee T^{-(n-1)}(\mathcal{A}).$$

Definition 2.2. Let (X, ρ) be a compact metric space, $T\colon X \to X$ be a continuous transformation, and $\varphi\colon X \to \mathbb{R}$ be a continuous function (which is often called a *potential*). Denote by $\mathcal{Y}$ the set of the sequences $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of open finite covers of X satisfying that the meshes of the covers $\mathcal{V}_n$ converge to zero as $n \to +\infty$, that is,

$$\mathcal{Y} \coloneqq \bigl\{\{\mathcal{V}_n\}_{n\in\mathbb{N}} : \lim_{n\to+\infty} \operatorname{mesh}(\mathcal{V}_n) = 0\bigr\},$$

where the mesh of a cover $\mathcal{V}$ of X is given by

$$\operatorname{mesh}(\mathcal{V}) \coloneqq \sup\{\operatorname{diam}(V) : V \in \mathcal{V}\}.$$

Define the *topological pressure* as the supremum of the following limits taken over all sequences $\{\mathcal{V}_n\} \in \mathcal{Y}$:

$$\lim_{n\to+\infty} \lim_{m\to+\infty} \frac{1}{n} \log\left(\inf_{\mathcal{V}}\left\{\sum_{U\in\mathcal{V}} \exp(\sup\{S_m\varphi(x) : x \in U\})\right\}\right), \tag{2.4}$$

where V ranges over all covers of X contained (in the sense of inclusion) in $(\mathcal{V}_n)_T^m$. In particular, if the potential φ is identically zero, then the topological pressure $P(T, \varphi)$ is also called the *topological entropy* of the transformation T, and is denoted by $h_{\mathrm{top}}(T)$.

We refer to [PU10, Subsection 3.2] to demonstrate that the topological pressure is well-defined by (2.4).

Let (X, ρ) be a compact metric space. For all Borel probability measure $\mu \in \mathcal{P}(X)$ and continuous transformation T of X, one can define *measure-theoretic entropy* $h_\mu(T)$ of T with respect to μ (see [PU10, Chapter 2.4]). In particular, assume that X, ρ, T satisfy the Assumption C in Subsection 2.2, and μ is a T-invariant Borel probability measure. Recall that a measurable function $J\colon X \to [0, +\infty)$ is a *Jacobian* of T with respect to μ if for each μ-measurable set $A \subseteq X$ on which T is injective, $T(A)$ is still μ-measurable and $\mu(T(A)) = \int_A J \,\mathrm{d}\mu$. By [PU10, Proposition 2.9.5], there exists a unique Jacobian J_μ of T with respect to μ. Then the *measure-theoretic entropy* of T with respect to μ has the following explicit expression $h_\mu(T) = \int \log(J_\mu) \,\mathrm{d}\mu$ (see for example, [PU10, Proposition 2.9.7]).

Under the above notations, one has the following theorem (see for example, [PU10, Theorem 3.4.1]).

Theorem 2.3 (Variational Principle)**.** *Let (X, ρ) be a compact metric space. Then for each continuous transformation $T\colon X \to X$ and each potential $\varphi \in C(X)$, we have*

$$P(T, \varphi) = \sup\left\{h_\mu(T) + \int \varphi \,\mathrm{d}\mu : \mu \in \mathcal{M}_T(X)\right\}.$$

In particular, if $\varphi \equiv 0$, then $h_{\mathrm{top}}(T) = \sup\{h_\mu(T) : \mu \in \mathcal{M}_T(X)\}$.

Definition 2.4. Let (X, ρ) be a compact metric space. Then for each continuous transformation $T\colon X \to X$ and each potential $\varphi \in C(X)$, we say that a measure $\mu \in \mathcal{M}_T(X)$ is an *equilibrium state* of the transformation T for the potential φ if

$$P(T, \varphi) = h_\mu(T) + \int \varphi \,\mathrm{d}\mu.$$

In particular, for $\varphi \equiv 0$, an equilibrium state of T for φ is also called a *measure of maximal entropy* of T.

2.3. **Ruelle operators and Gibbs states.** In this subsection, we review some notions and results for the Ruelle operator, the Gibbs state, and the equilibrium state.

Let X, ρ, T satisfy the Assumption C in Subsection 2.2, and the potential $\varphi\colon X \to \mathbb{R}$ is a continuous function. Recall that the *Ruelle operator* $\mathcal{L}_\varphi$ acting on $C(X)$ is given by

$$\mathcal{L}_\varphi(u)(x) := \sum_{y \in T^{-1}(x)} u(y) \exp(\varphi(y)) \tag{2.5}$$

for each $u \in C(X)$ and each $x \in X$.

By [PU10, Theorem 5.2.8], there exists an eigenmeasure $m \in \mathcal{P}(X)$ and an eigenvalue $c > 0$ of the adjoint operator $\mathcal{L}_\varphi^*$ of $\mathcal{L}_\varphi$, i.e., $\mathcal{L}_\varphi^*(m) = cm$. Moreover, by [PU10, Proposition 5.2.11], the constant c is precisely $\exp(P(T, \varphi))$.

It is convenient to consider the normalized operator $\mathcal{L}_{\overline{\varphi}}$ with $\overline{\varphi} := \varphi - P(T, \varphi)$. By (2.5), we have

$$\mathcal{L}_{\overline{\varphi}}^*(m) = e^{-P(T,\varphi)} \mathcal{L}_\varphi^*(m) = m. \tag{2.6}$$

A measure $\mu \in \mathcal{P}(X)$ is a *Gibbs state* for T and φ if there exist two constants $P \in \mathbb{R}$ and $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu(T_x^{-n}(B(T^n(x), \xi)))}{\exp(S_n\varphi(x) - Pn)} \leq C \tag{2.7}$$

for all $x \in X$ and $n \in \mathbb{N}$.

The following result (see for example, [PU10, Propositions 4.4.3, 5.1.5, 5.2.10, Corollaries 5.2.13, Lemma 5.6.1, and Theorem 5.6.2]) implies that a T-invariant Gibbs state is an ergodic equilibrium state of T for the potential φ.

Proposition 2.5. *Let X, ρ, T satisfy the Assumption C in Subsection 2.2. Then for all $\mu \in \mathcal{M}_T(X)$ and Hölder continuous potential $\varphi\colon X \to \mathbb{R}$, the following statements are equivalent:*

(i) *μ is a Gibbs state for T and φ.*
(ii) *μ is an ergodic equilibrium state of T for φ.*
(iii) *$\exp(-\psi)$ is a Jacobian of T with respect to μ, where*

$$\psi := \varphi - P(T, \varphi) - \log(u_\varphi \circ T) + \log(u_\varphi).$$

In the remainder of this subsection, we quantitatively re-develop some familiar facts about the Ruelle operators from the perspective of Computable Analysis in preparation for more technical studies in Section 4.

Lemma 2.6. *Fix arbitrary constants η, ξ, a_0, $v_0 > 0$, and $\lambda > 1$. Let X, ρ, T satisfy the Assumption C in Subsection 2.2 with the constants η, λ, and ξ, and the potential $\varphi\colon X \to \mathbb{R}$ be Hölder continuous with respect to the metric ρ with the exponent v_0 and the constant a_0. Define $\overline{\varphi}(x) := \varphi(x) - P(T, \varphi)$ for each $x \in X$, $a := \frac{a_0}{\lambda^{v_0} - 1}$, and $\xi_0 := \min\{\xi, 1\}$. Then*

$$\mathcal{L}_{\overline{\varphi}}^n(\mathbb{1})(x) \leq \exp(a\rho(x, y)^{v_0}) \mathcal{L}_{\overline{\varphi}}^n(\mathbb{1})(y) \tag{2.8}$$

for each $n \in \mathbb{N}$ and each pair of x, $y \in X$ with $\rho(x, y) \leq \xi_0$.

Proof. Since T is an open transformation that is distance-expanding with the constants η, λ, and ξ. Then for each pair of x, $y \in X$ with $\rho(x, y) \leq \xi_0$ and each $\overline{x} \in T^{-n}(x)$, one can find a point $y(\overline{x}) := T_{\overline{x}}^{-n}(y) \in T^{-n}(y)$ satisfying that $\rho(T^i(\overline{x}), T^i(y(\overline{x}))) \leq \lambda^{i-n}\rho(x, y) \leq \xi_0$ for each integer $0 \leq i \leq n-1$. Noting that $\overline{\varphi}(x) = \varphi(x) - P(T,\varphi)$ is Hölder continuous with exponent v_0 and constant a_0, we have

$$\begin{aligned} S_n\overline{\varphi}(\overline{x}) - S_n\overline{\varphi}(y(\overline{x})) &= \sum_{i=0}^{n-1}(\overline{\varphi}(T^i(\overline{x})) - \overline{\varphi}(T^i(y(\overline{x})))) \\ &\leq \sum_{i=0}^{n-1} a_0\rho(T^i(\overline{x}), T^i(y(\overline{x})))^{v_0} \\ &\leq a_0\rho(x,y)\cdot \sum_{i=0}^{n-1}\lambda^{(i-n)v_0} \\ &\leq \frac{a_0\rho(x,y)^{v_0}}{\lambda^{v_0}-1} \end{aligned}$$

for each $\overline{x} \in T^{-n}(x)$. Together with (2.5), we can conclude that

$$\begin{aligned} \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x) &= \sum_{\overline{x}\in T^{-n}(x)} \exp(S_n\overline{\varphi}(\overline{x})) \\ &\leq \sum_{\overline{x}\in T^{-n}(x)} \exp(S_n\overline{\varphi}(y(\overline{x})))\cdot\exp\biggl(\frac{a_0\rho(x,y)^{v_0}}{\lambda^{v_0}-1}\biggr) \\ &\leq \exp\biggl(\frac{a_0\rho(x,y)^{v_0}}{\lambda^{v_0}-1}\biggr)\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(y) \end{aligned}$$

for each pair of x, $y \in X$ with $\rho(x,y) \leq \xi_0$. □

Lemma 2.7. *Fix arbitrary constants η, ξ, a_0, $v_0 > 0$, and $\lambda > 1$. Let X, ρ, T satisfy the Assumption C in Subsection 2.2 with the constants η, λ, and ξ, and the potential $\varphi\colon X \to \mathbb{R}$ be Hölder continuous with respect to the metric ρ with exponent v_0 and constant a_0. Define $\overline{\varphi}(x) \coloneqq \varphi(x) - P(T,\varphi)$ for each $x \in X$, $a \coloneqq \frac{a_0}{\lambda^{v_0}-1}$, $D \coloneqq \max_{x\in X}\operatorname{card}\bigl(T^{-1}(x)\bigr)$ and $\xi_0 \coloneqq \min\{\xi, 1\}$. Denote by $G = \{x_1, x_2, \dots, x_l\}$ a ξ_0-net of X. Then for all x, $y \in X$, and $n \in \mathbb{N}$, we have*

$$\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x) \leq C\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(y). \tag{2.9}$$

Here $C \coloneqq D^N \exp(4a + 2N\|\varphi\|_\infty)$, where N is a constant satisfying that

$$\bigcup_{k=0}^{N} T^k(B(x_i,\xi_0)) = X \tag{2.10}$$

for each integer $1 \leq i \leq l$.

See [PU10, Theorem 4.3.12 (2)] for the existence of N.

Proof. By Lemma 2.6, we have

$$\frac{\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x)}{\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(y)} \leq \exp(a\rho(x,y)^{v_0}) \leq \exp(a\xi_0^{v_0}) \leq e^a. \tag{2.11}$$

for each pair of x, $y \in X$ with $\rho(x,y) \leq \xi_0$ and each $n \in \mathbb{N}$.

Now we are ready to demonstrate (2.9). Let x, $y \in X$. Since G is a ξ_0-net, there exist two integers $1 \leq i$, $j \leq l$ satisfying that $\rho(x_i, x) < \xi_0$ and $\rho(x_j, y) < \xi_0$. Hence, by (2.10), there exists an integer $0 \leq m \leq N$ such that $T^m(B(x_i, \xi_0)) \cap B(x_j, \xi_0) \neq \emptyset$. Now let u be a point in $T^{-m}(B(x_j, \xi_0)) \cap B(x_i, \xi_0)$. By (2.5) and (2.11), then for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x_i) &\leq e^a \cdot \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(u) \\
&= e^a \sum_{\overline{u} \in T^{-n}(u)} \exp(S_{n+m}\overline{\varphi}(\overline{u}) - S_m\overline{\varphi}(u)) \\
&\leq \exp\bigl(a - m \inf_{x \in X} \overline{\varphi}(x)\bigr) \sum_{\overline{u} \in T^{-(n+m)}(T^m(u))} \exp(S_{n+m}\overline{\varphi}(\overline{u})) \\
&= \exp\bigl(a - m \inf_{x \in X} \overline{\varphi}(x)\bigr) \sum_{\overline{u} \in T^{-(n+m)}(T^m(u))} \exp(S_n\overline{\varphi}(T^m(\overline{u})) + S_m\overline{\varphi}(\overline{u})) \\
&\leq \exp\bigl(a + m \sup_{x \in X} \overline{\varphi}(x) - m \inf_{x \in X} \overline{\varphi}(x)\bigr) \sum_{\overline{u} \in T^{-(n+m)}(T^m(u))} \exp(S_n\overline{\varphi}(T^m(\overline{u}))) \\
&\leq \exp(a + 2N\|\overline{\varphi}\|_\infty) D^N \sum_{\overline{u} \in T^{-n}(T^m(u))} \exp(S_n\overline{\varphi}(\overline{u})) \\
&= \exp(a + 2N\|\varphi\|_\infty) D^N \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(T^m(u)) \\
&\leq \exp(2a + 2N\|\varphi\|_\infty) D^N \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x_j).
\end{aligned}$$

Therefore, by (2.11), it follows from $\rho(x_i, x) < \xi_0$ and $\rho(x_j, y) < \xi_0$ that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x) &\leq e^a \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x_i) \\
&\leq \exp(3a + 2N\|\varphi\|_\infty) D^N \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(x_j) \\
&\leq \exp(4a + 2N\|\varphi\|_\infty) D^N \mathcal{L}^n_{\overline{\varphi}}(\mathbb{1})(y).
\end{aligned}$$

□

Corollary 2.8. *Fix arbitrary constants η, ξ, a_0, $v_0 > 0$, and $\lambda > 1$. Let X, ρ, T satisfy the Assumption C in Subsection 2.2 with the constants η, λ, and ξ, and the potential $\varphi\colon X \to \mathbb{R}$ be Hölder continuous with respect to the metric ρ with exponent v_0 and constant a_0. Define $\overline{\varphi}(x) := \varphi(x) - P(T, \varphi)$ for each $x \in X$ and $\xi_0 := \min\{\xi, 1\}$. Then the sequence $\bigl\{\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}^j_{\overline{\varphi}}(\mathbb{1})\bigr\}_{n \in \mathbb{N}}$ converges uniformly to a function $u_\varphi \in C(X)$ satisfying*

$$\mathcal{L}_{\overline{\varphi}}(u_\varphi) = u_\varphi, \tag{2.12}$$

$$u_\varphi(x) \leq \exp\biggl(\frac{a_0 \rho(x, y)^{v_0}}{\lambda^{v_0} - 1}\biggr) \cdot u_\varphi(y) \tag{2.13}$$

for each pair of x, $y \in X$ with $\rho(x, y) \leq \xi_0$, and

$$C^{-1} \leq u_\varphi(x) \leq C \tag{2.14}$$

for each $x \in X$, where $C \geq 1$ is a constant from Lemma 2.7. Moreover, if $m \in \mathcal{P}(X)$ satisfies (2.6), then

$$\int u_\varphi \,\mathrm{d}m = 1, \tag{2.15}$$

and $\mu \in \mathcal{P}(X)$ given by

$$\mu(A) := \int_A u_\varphi \,\mathrm{d}m, \qquad \text{for each Borel set } A \subseteq X, \tag{2.16}$$

is the unique T-invariant Gibbs state for T and φ.

Proof. To prove this corollary, we first demonstrate (2.12), (2.13), (2.14), and (2.15) for a subsequential limit of the sequence $\big\{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\overline{\varphi}}^{j}(\mathbb{1})\big\}_{n\in\mathbb{N}}$, then prove the above sequence has a unique subsequential limit, and finally establish that μ is the unique T-invariant Gibbs state for T and φ.

By (2.6), for each $n\in\mathbb{N}$,

$$\int \mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})\,\mathrm{d}m = \int \mathbb{1}\,\mathrm{d}m = 1. \tag{2.17}$$

This implies that $\inf_{x\in X}\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(x)\leq 1\leq \sup_{x\in X}\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(x)$. Then by (2.9), we have

$$C^{-1}\leq \mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(x)\leq C \tag{2.18}$$

for all $n\in\mathbb{N}$ and $x\in X$. Together with Lemma 2.6, we have

$$\big|\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(x)-\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(y)\big| \leq \bigg|\frac{\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(x)}{\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(y)}-1\bigg|\cdot\big|\mathcal{L}_{\overline{\varphi}}^{n}(\mathbb{1})(y)\big| \leq C\bigg(\exp\bigg(\frac{a_0\rho(x,y)^{v_0}}{\lambda^{v_0}-1}\bigg)-1\bigg) \tag{2.19}$$

for all $n\in\mathbb{N}$ and $x,y\in X$ with $\rho(x,y)\leq\xi_0$.

For each $n\in\mathbb{N}$, set $u_n:=\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\overline{\varphi}}^{j}(\mathbb{1})$. By (2.18) and (2.19), $\{u_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of equicontinuous functions on X. By the Arzelà–Ascoli Theorem, there exists a continuous function u_φ and an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $u_{n_i}\to u_\varphi$ uniformly on X as $i\to+\infty$. Thus it follows from the definition of $\{u_n\}_{n\in\mathbb{N}}$, (2.8), (2.18), and (2.17) that u_φ satisfies (2.12), (2.13), (2.14), and (2.15).

Next, we demonstrate that u_φ is the unique subsequential limit of $\{u_n\}_{n\in\mathbb{N}}$. Suppose that v_φ is an arbitrary subsequential limit of $\{u_n\}_{n\in\mathbb{N}}$. Then v_φ is also a continuous function satisfying (2.12), (2.13), (2.14), and (2.15). Set

$$t:=\sup\{s\in\mathbb{R}: u_\varphi(x)-sv_\varphi(x)>0 \text{ for all } x\in X\}.$$

Then there exists $y\in X$ such that $u_\varphi(y)=tv_\varphi(y)$. Since $\mathcal{L}_{\overline{\varphi}}^{n}(u_\varphi-tv_\varphi)(y)=u_\varphi(y)-tv_\varphi(y)=0$ for each $n\in\mathbb{N}$, we obtain that $u_\varphi(x)-tv_\varphi(x)=0$ for each $x\in\bigcup_{i\in\mathbb{N}}T^{-i}(y)$. Since T is topologically transitive, the set $\bigcup_{i\in\mathbb{N}}T^{-i}(y)$ is dense in X. Together with the continuity of $u_\varphi-tv_\varphi$, $u_\varphi(x)-tv_\varphi(x)=0$ for each $x\in X$. Because $\int u_\varphi\,\mathrm{d}m=\int v_\varphi\,\mathrm{d}m=1$, we obtain $t=1$. This implies that $u_\varphi=v_\varphi$. Therefore $u_n\to u_\varphi$ uniformly on X as $n\to+\infty$.

By [PU10, Proposition 5.2.11], m is a Gibbs state for T and φ. According to [PU10, Proposition 5.1.1] and (2.16), μ is also a Gibbs state for T and φ. Recall that for each $m_0\in\mathcal{P}(X)$, a linear operator $\mathcal{L}_{m_0}\colon L^1(m_0)\to L^1(m_0)$ is defined in [PU10, Section 5.2]. Here, for the Gibbs state m, $\mathcal{L}_m$ equals exactly $\mathcal{L}_{\overline{\varphi}}$. Then by [PU10, Proposition 5.2.2], it follows from (2.12) that μ is T-invariant. It follows from [PU10, Corollary 5.2.14] that μ is the unique T-invariant Gibbs state for T and φ. □

3. Preliminary on Computable Analysis

In this section, we recall some notions and results in Computable Analysis. The definitions we adopted in this section are consistent with [Weih00]. Consequently, it is convenient to think of the algorithms or machines mentioned below as Type-2 machines defined in [Weih00, Definition 2.1.1]. For more details, we refer the reader to [BBRY11, Chapter 3], [GHR10, Section 2], and [Weih00].

3.1. **Algorithms and computability over the reals.**

Definition 3.1. Given $k \in \mathbb{N}$, we say that a function $f \colon \mathbb{N}^k \to \mathbb{Z}$ is *computable*, if there exists an algorithm $\mathcal{A}$ which, upon input a sequence of k positive integers $\{x_i\}_{i=1}^k$, outputs the value of $f(x_1, x_2, \dots, x_k)$.

Definition 3.2. A real number x is called *computable* if there are two computable functions $f \colon \mathbb{N} \to \mathbb{Z}$ and $g \colon \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$,

$$\left| \frac{f(n)}{g(n)} - x \right| < 2^{-n}.$$

3.2. **Computable metric spaces.** The above definitions equip the real numbers with a computability structure. This can be extended to virtually any separable metric space. We now give a short introduction.

Definition 3.3. A *computable metric space* is a triple $(X, \rho, \mathcal{S})$, where

(i) (X, ρ) is a separable metric space;
(ii) $\mathcal{S} = \{s_n : n \in \mathbb{N}\}$ is a dense subset of X;
(iii) there exists an algorithm which, on input i, j, $m \in \mathbb{N}$, outputs $y_{i,j,m} \in \mathbb{Q}$ satisfying $|y_{i,j,m} - \rho(s_i, s_j)| < 2^{-m}$.

The points in $\mathcal{S}$ are said to be *ideal*. By the canonical bijection between $\mathbb{N}^3$ and $\mathbb{N}$ (see Subsection 2.1), we fix the enumeration $\{B_i\}_{i \in \mathbb{N}}$ for the set $\{B(s_i, j/k) : i, j, k \in \mathbb{N}\}$ of balls with rational radii centered at points in $\mathcal{S}$. These balls are called the *ideal balls* in $(X, \rho, \mathcal{S})$.

Definition 3.4. Let $(X, \rho, \mathcal{S})$ be a computable metric space. We say that a point $x \in X$ is *computable* if there exists a computable function $f \colon \mathbb{N} \to \mathbb{N}$ such that $\rho(s_{f(n)}, x) < 2^{-n}$ for each $n \in \mathbb{N}$. Moreover, a sequence of points $\{x_i\}_{i \in \mathbb{N}}$ is said to be a *uniformly computable sequence of points* if there exists a computable function $f \colon \mathbb{N}^2 \to \mathbb{N}$ such that $\rho(s_{f(n,m)}, x_m) < 2^{-n}$ for all n, $m \in \mathbb{N}$.

As a remark, a finite sequence of computable points is always a uniformly computable sequence of points. Similarly, for other definitions of computable objects detailed below, we always have that a finite sequence of computable objects is always a uniformly computable sequence.

Definition 3.5. In a computable metric space $(X, \rho, \mathcal{S})$, an open set $U \subseteq X$ is called *lower-computable* if there is a computable function $f \colon \mathbb{N} \to \mathbb{N}$ such that $U = \bigcup_{n \in \mathbb{N}} B_{f(n)}$. A closed set K is said to be *upper-computable* if its complement is a lower-computable open set.

Definition 3.6. In a computable metric space $(X, \rho, \mathcal{S})$, a family $\{U_i\}_{i \in \mathbb{N}}$ of lower-computable open sets is called a *uniformly computable sequence of lower-computable open sets* if there is a computable function $f \colon \mathbb{N}^2 \to \mathbb{N}$ such that $U_i = \bigcup_{n \in \mathbb{N}} B_{f(i,n)}$.

As a consequence of Definitions 3.5 and 3.6, the canonical bijection between $\mathbb{N}^2$ and $\mathbb{N}$ (see Subsection 2.1) gives the following proposition.

Proposition 3.7. *Assume that $\{U_i\}_{i \in \mathbb{N}}$ is a uniformly computable sequence of lower-computable open sets in a computable metric space $(X, \rho, \mathcal{S})$. Then $U = \bigcup_{i \in \mathbb{N}} U_i$ is a lower-computable open set.*

Before the definition of computable functions between computable metric spaces, we recall the definition of an oracle of a point in computable metric space.

Definition 3.8. Given a computable metric space $(X,\, \rho,\, \mathcal{S})$, and a point $x \in X$, we say that a function $\varphi\colon \mathbb{N} \to \mathbb{N}$ is an *oracle* for $x \in X$ if $\rho(s_{\varphi(m)}, x) < 2^{-m}$ for each $m \in \mathbb{N}$.

Definition 3.9. Assume that $(X,\, \rho,\, \mathcal{S})$ and $(X',\, \rho',\, \mathcal{S}')$ are two computable metric spaces. Then a function $f\colon X \to X'$ is *computable* if there exists an algorithm which, for each $x \in X$ and each $n \in \mathbb{N}$, on input $n \in \mathbb{N}$ and an oracle φ for x, outputs $m \in \mathbb{N}$ satisfying that $\rho'(s'_m, f(x)) < 2^{-n}$. Moreover, a sequence $\{f_i\}_{i\in\mathbb{N}}$ of functions $f_i\colon X \to X'$ is called a *uniformly computable sequence of functions* if there exists an algorithm which, for each $x \in X$ and each $n \in \mathbb{N}$, on input $i,\, n \in \mathbb{N}$, and an oracle φ for x, outputs $m \in \mathbb{N}$ satisfying that $\rho'(s'_m, f_i(x)) < 2^{-n}$.

For example, in [Weih00], Examples 4.3.3 and 4.3.13.5 give the computability of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ and the logarithmic function $\log\colon \mathbb{R}^+ \to \mathbb{R}$, respectively. The following proposition is a classical result that describes a topological property of a computable function. For the equivalence of these two definitions, we refer to [Weih00, Chapters 6 and 9].

Proposition 3.10. *Assume that $(X,\, \rho,\, \mathcal{S})$ and $(X',\, \rho',\, \mathcal{S}')$ are two computable metric spaces, and $\{B'_i\}_{i\in\mathbb{N}}$ is the enumeration of ideal balls of $(X',\, \rho',\, \mathcal{S}')$. Then a function $f\colon X \to X'$ is called* computable *if and only if $\left\{f^{-1}(B'_i)\right\}_{i\in\mathbb{N}}$ is a uniformly computable sequence of lower-computable open sets. Here $\{B'_i\}_{i\in\mathbb{N}}$ is the enumeration of ideal balls in $(X',\, \rho',\, \mathcal{S}')$.*

Definition 3.11. Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, and $T\colon X \to X$ a function with $p = p(x) \coloneqq \operatorname{card}\bigl(T^{-1}(x)\bigr) < +\infty$ for each $x \in X$. Then the inverse T^{-1} of T is said to be *computable* if there exists an algorithm $\mathcal{A}$ such that for each $x \in X$ and each $n \in \mathbb{N}$, we can input an oracle φ for x in $\mathcal{A}$ to produce $\{y_i : 1 \leq i \leq p(x)\}$ that satisfies the following property:

If $T^{-1}(x) = \{x_i : 1 \leq i \leq p(x)\}$, then for each $1 \leq i \leq p(x)$, there exists $1 \leq j \leq p(x)$ such that $\rho(x_i, y_j) < 2^{-n}$.

It follows immediately from Definition 3.9 that the computable real-valued functions are closed under finitely many operators from the following list: addition, multiplication, division, scalar multiplication, max, and min (see for example, [Weih00, Corollary 4.3.4]).

At the end of this subsection, we recall the definitions of a recursively compact set and of a recursively precompact metric space introduced in [GHR10, Section 2].

Definition 3.12. In a computable metric space $(X,\, \rho,\, \mathcal{S})$, a set $K \subseteq X$ is said to be *recursively compact* if it is compact and there is an algorithm which, on input $\{i_j\}_{j=1}^p \subseteq \mathbb{N}$ and $\{q_j\}_{j=1}^p \subseteq \mathbb{Q}^+$, halts if and only if $K \subseteq \bigcup_{j=1}^p B(s_{i_j}, q_j)$.

[GHR10, Proposition 1] gives some basic properties of recursively compact sets.

Proposition 3.13. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space. Then the following statements hold:*

- (i) *A point $x \in X$ is computable if and only if the singleton $\{x\}$ is a recursively compact set.*
- (ii) *Assume that $K \subseteq X$ is a recursively compact set, and $U \subseteq X$ is a lower-computable open set. Then $K \setminus U$ is recursively compact.*

Definition 3.14. A computable metric space $(X,\, \rho,\, \mathcal{S})$ is *recursively precompact* if there is an algorithm which, on input $n \in \mathbb{N}$, outputs a subset $\{i_1,\, i_2,\, \dots,\, i_p\}$ of $\mathbb{N}$ satisfying that $\{s_{i_1},\, s_{i_2},\, \dots,\, s_{i_p}\}$ is a 2^{-n}-net of X.

3.3. **Computability of probability measures.** Following [HR09, Proposition 4.1.3], if $(X,\, \rho,\, \mathcal{S})$ is a computable metric space, and X is bounded, then $\bigl(\mathcal{P}(X),\, W_\rho,\, \mathcal{R}_\mathcal{S}\bigr)$ is a computable metric space, where $\mathcal{R}_\mathcal{S} \subseteq \mathcal{P}(X)$ is the set of the probability measures such that each measure is supported on finitely many points of $\mathcal{S}$ and assigns rational values to them.

Definition 3.15. Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, and X be a bounded set. Then a *computable measure* μ is a computable point of $\bigl(\mathcal{P}(X),\, W_\rho,\, \mathcal{R}_\mathcal{S}\bigr)$.

By [GHR10, Lemma 2.12 and Proposition 4], due to the completeness of $\mathcal{P}(X)$ with respect to the metric W_ρ, one can conclude the following result.

Proposition 3.16. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, and X be recursively compact. Then $\mathcal{P}(X)$ is a recursively compact set in $\bigl(\mathcal{P}(X),\, W_\rho,\, \mathcal{R}_\mathcal{S}\bigr)$.*

Proposition 3.17 ([HR09, Corollary 4.3.2])**.** *Assume that $(X,\, \rho,\, \mathcal{S})$ is a computable metric space, and $\{f_i\}_{i\in\mathbb{N}}$ is a uniformly computable sequence of real-valued functions on X. If there is an algorithm which, on input $i \in \mathbb{N}$, outputs M_i such that $\|f_i\|_\infty \leq M_i$, then the sequence of integral operators $\mathcal{I}_i\colon \mathcal{P}(X) \to \mathbb{R}$, $i \in \mathbb{N}$, defined by $\mathcal{I}_i(\mu) \coloneqq \int f_i \,\mathrm{d}\mu$ is a uniformly computable sequence of functions.*

By [BRY12, Proposition 2.13], the bound of a computable function restricted in a recursively compact set can be computed. Hence, we have the following result.

Corollary 3.18. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, and X be recursively compact. Assume that $\{f_i\}_{i\in\mathbb{N}}$ is a uniformly computable sequence of real-valued functions on X. Then the sequence of integral operators $\mathcal{I}_i\colon \mathcal{P}(X) \to \mathbb{R}$, $i \in \mathbb{N}$, defined by $\mathcal{I}_i(\mu) \coloneqq \int f_i \,\mathrm{d}\mu$ is a uniformly computable sequence of functions.*

Finally, we introduce a family of computable functions.

Definition 3.19. Let (X, ρ) be a metric space, and suppose that $r,\, \epsilon > 0$, and $u \in X$. Then the function $g_{u,r,\epsilon}$ given by

$$g_{u,r,\epsilon}(x) \coloneqq \left| 1 - \frac{|\rho(x,u) - r|^+}{\epsilon} \right|^+ \quad \text{for each } x \in X, \tag{3.1}$$

is called a *hat function*. Here $|a|^+ \coloneqq \max\{a,\, 0\}$.

The hat function $g_{u,r,\epsilon}(x)$ is an ϵ^{-1}-Lipschitz function which equals to 1 in the ball $B(u,r)$, to 0 outside $B(u, r+\epsilon)$, and lies strictly between 0 and 1 in $B(u, r+\epsilon) \setminus \overline{B(u,r)}$.

Definition 3.20. In a computable metric space $(X,\, \rho,\, \mathcal{S})$, assume that $\mathcal{F}_0(\mathcal{S})$ is the set of functions of the form $g_{u,\,r,\,1/n}(x)$, where $u \in \mathcal{S}$, $r \in \mathbb{Q}^+$, $n \in \mathbb{N}$, and $\mathcal{E}(\mathcal{S})$ is the smallest set of functions containing $\mathcal{F}_0$ and the constant function 1 closed under max, min, and finite rational linear combinations. Then the elements in $\mathcal{E}(\mathcal{S})$ are called *test functions*.

According to the canonical bijection between N^* and N (see Subsection 2.1), it follows from Definitions 3.19 and 3.20 that there exists an enumeration $\{\varphi_j\}_{j\in\mathbb{N}}$ for $\mathcal{E}(\mathcal{S})$ such that $\{\varphi_j\}_{j\in\mathbb{N}}$ is a uniformly computable sequence of functions. We call such an enumeration a *computable enumeration*.

Assume that X is a compact separable metric space and $\mathcal{S}$ a dense subset of X, then $\mathcal{E}(\mathcal{S})$ is dense in $C(X)$. Thus, using the dominated convergence theorem, we conclude the following proposition.

Proposition 3.21. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, X be recursively compact, and $\{\varphi_j\}_{j\in\mathbb{N}}$ be a computable enumeration of $\mathcal{E}(\mathcal{S})$. Then for each pair of Borel probability measures $\mu,\, \nu \in \mathcal{P}(X)$, $\mu = \nu$ if and only if $\int \varphi_j \,\mathrm{d}\mu = \int \varphi_j \,\mathrm{d}\nu$, for each $j \in \mathbb{N}$.*

4. Computable Analysis on thermodynamic formalism

In this section, we establish Theorems 1.1 and 1.2. In Subsection 4.1, we estimate the rates of the convergences of the iterations of normalized Ruelle operators. In Subsection 4.2, we prove that the function $J_\varphi(x)$ is computable (see (4.19) for the expression of $J_\varphi(x)$). In Subsection 4.3, we demonstrate that two subsets U and W of $\mathcal{P}(X)$ are both lower-computable open sets (see Lemmas 4.12 and 4.13 for their definitions). In Subsection 4.4, we complete the proofs of Theorems 1.1 and 1.2.

4.1. **Ruelle operators and cones.** In this subsection, we develop the technique of cones, combining ideas from [PU10], to prove in Theorem 4.1 that the convergence of iterations of a normalized Ruelle operator is at the exponential rate with explicit expressions for the related constants. The technique of cones first used by Birkhoff [Bi57] in the study of Banach spaces geometry was reintroduced by Liverani [Liv95a, Liv95b] for the analysis of the decay of correlations. For a quick introduction to cones in the context of connected spaces, see [Vi97, Section 2.2]. The lack of connectedness of X becomes a key obstacle in our study of Computable Analysis. Due to the lack of connectedness of X and the precise demand on the computability (in the sense of Computable Analysis) of various constants involved, we further develop the technique of cones in our study.

In this subsection, we define

$$\widehat{K}(\lambda_1, b, b') := 2\log\left(\frac{1+\lambda_1}{1-\lambda_1}\cdot\frac{b^2}{b-b'}\right) \tag{4.1}$$

for $1 \leq b' < b$ and $\lambda_1 < 1$.

Theorem 4.1. *Fix arbitrary constants η, ξ, a_0, $v_0 > 0$, and $\lambda > 1$. Let (X, ρ) be a compact metric space, $T\colon X \to X$ be a continuous, topologically exact, open transformation which is distance-expanding with respect to the metric ρ with the constants η, λ, and ξ, and the potential $\varphi\colon X \to \mathbb{R}$ be Hölder continuous with respect to the metric ρ with exponent v_0 and constant a_0. Define $\overline{\varphi}(x) := \varphi(x) - P(T, \varphi)$ for each $x \in X$. Then there exists $u_\varphi \in C(X)$ such that*

$$\left\|\mathcal{L}_{\overline{\varphi}}^{mk}(\mathbb{1})(x) - u_\varphi(x)\right\|_\infty \leq CeZ(1-\exp(-Z))^{k-1}$$

for each $k \in \mathbb{N}$ with $0 \leq Z(1-\exp(-Z))^{k-1} \leq 1$, where C is from Lemma 2.7, $Z := \widehat{K}\big(\frac{\lambda^{-v_0}+1}{2}, C^2, \frac{Q^m+2C^2}{2C^{-2}Q^m+2}\big)$, and Q, m are from Proposition 4.5.

The proof of Theorem 4.1 will be given at the end of Subsection 4.1.

We begin by introducing some notations in the cone technique. Let E be a vector space over $\mathbb{R}$. A *convex cone* in E is a subset $\mathcal{C} \subseteq E \setminus \{0\}$ satisfying the following properties:

(i) $tv \in \mathcal{C}$ for all $v \in \mathcal{C}$ and $t > 0$.

(ii) $t_1v_1 + t_2v_2 \in \mathcal{C}$ for all v_1, $v_2 \in \mathcal{C}$ and t_1, $t_2 > 0$.

(iii) $\overline{\mathcal{C}} \cap (-\overline{\mathcal{C}}) = \{0\}$.

Here $\overline{\mathcal{C}}$ is the set consisting of $w \in E$ satisfying that there exist $v \in \mathcal{C}$ and $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $w + t_nv \in \mathcal{C}$ for each $n \in \mathbb{N}$ and that t_n converges to 0 as n tends to $+\infty$. Denote $-\overline{\mathcal{C}} := \{-v \in E : v \in \overline{\mathcal{C}}\}$.

Define

$$\alpha(v_1, v_2) := \sup\{t > 0 : v_2 - tv_1 \in \mathcal{C}\} \quad \text{and} \quad \beta(v_1, v_2) := \inf\{s > 0 : sv_1 - v_2 \in \mathcal{C}\} \tag{4.2}$$

for all $v_1,\ v_2 \in \mathcal{C}$, with the conventions $\sup \emptyset = 0$, $\inf \emptyset = +\infty$, and

$$\theta(v_1, v_2) := \log(\beta(v_1, v_2)) - \log(\alpha(v_1, v_2)), \tag{4.3}$$

with $\theta(v_1, v_2) = +\infty$ if $\alpha(v_1, v_2) = 0$ or $\beta(v_1, v_2) = +\infty$. By property (ii) of the definition of a convex cone, we have $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$, $\alpha(v_1, v_2) < +\infty$, and $\beta(v_1, v_2) > 0$ for all $v_1,\ v_2 \in \mathcal{C}$. Thus $\theta(v_1, v_2)$ is well-defined and takes value in $[0, +\infty]$. We call $\theta(\cdot, \cdot)$ the *projective metric* (on $\mathcal{C}$) associated to the convex cone $\mathcal{C}$.

Proposition 4.2 ([Vi97, Proposition 2.2])**.** *Let $\mathcal{C}$ be a convex cone. Then the projective metric θ has the following properties:*

(i) *$\theta(v_1, v_2) = \theta(v_2, v_1)$ for all $v_1,\ v_2 \in \mathcal{C}$,*
(ii) *$\theta(v_1, v_2) + \theta(v_2, v_3) \geq \theta(v_1, v_3)$ for all $v_1,\ v_2,\ v_3 \in \mathcal{C}$,*
(iii) *for all $v_1,\ v_2 \in \mathcal{C}$, $\theta(v_1, v_2) = 0$ if and only if there exists $t > 0$ such that $v_1 = tv_2$.*

Moreover, the following proposition asserts that $L\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a strict contraction with respect to θ_1 and θ_2 if $L(\mathcal{C}_1)$ has finite θ_2-diameter. Here $\theta_i(\cdot, \cdot)$ is the projective metric associated to $\mathcal{C}_i$ for each $i \in \{1,\ 2\}$.

Proposition 4.3 ([Vi97, Proposition 2.3])**.** *Let $E_1,\ E_2$ be two vector spaces, and $\mathcal{C}_i \subseteq E_i$ be a convex cone with the projective metric $\theta_i(\cdot, \cdot)$ for each $i \in \{1,\ 2\}$. Assume that $L\colon E_1 \to E_2$ is a linear operator with $L(\mathcal{C}_1) \subseteq \mathcal{C}_2$, and put $U := \sup\{\theta_2(L(v), L(w)) : v,\ w \in \mathcal{C}_1\}$. If $U < +\infty$, then*

$$\theta_2(L(v), L(w)) \leq \left(1 - e^{-U}\right) \cdot \theta_1(v, w)$$

for each pair of $v,\ w \in \mathcal{C}_1$.

To prove Theorem 4.1, we consider the following family of convex cones.

Definition 4.4. For all $a,\ v > 0$, and $b > 1$, denote by $\mathcal{C}(a, v, b) \subseteq C(X)$ the convex cone consisting of continuous functions $u\colon X \to \mathbb{R}$ satisfying the following properties:

(i) $u(x) > 0$ for each $x \in X$,
(ii) $u(x) \leq \exp(a\rho(x, y)^v) \cdot u(y)$ for all $x,\ y \in X$ with $\rho(x, y) \leq \xi_0 := \min\{\xi,\ 1\}$,
(iii) $u(x) \leq bu(y)$ for each pair of $x,\ y \in X$,

and let $\alpha_{a,v,b}$ and $\beta_{a,v,b}$ be the corresponding functions associated to the convex cone $\mathcal{C}(a, v, b)$ defined in (4.2), and $\theta_{a,v,b}$ be the projective metric associated to the convex cone $\mathcal{C}(a, v, b)$.

Here we compute the expression of $\alpha_{a,v,b}$ for a given pair of $\varphi_1, \varphi_2 \in \mathcal{C}(a, v, b)$. By (4.2), $\alpha_{a,v,b}(\varphi_1, \varphi_2)$ is the supremum of all positive real number t that satisfies the following three properties for all $s,\ m,\ n,\ x,\ y \in X$ with $0 < \rho(x, y) \leq \xi_0$:

(i) $\varphi_2(s) - t\varphi_1(s) > 0$, or equivalently, $t < \frac{\varphi_2(s)}{\varphi_1(s)}$;
(ii) $\frac{\varphi_2(x) - t\varphi_1(x)}{\varphi_2(y) - t\varphi_1(y)} \leq \exp(a\rho(x, y)^v)$, or equivalently,

$$t(\exp(a\rho(x, y)^v)\varphi_1(y) - \varphi_1(x)) \leq \exp(a\rho(x, y)^v)\varphi_2(y) - \varphi_2(x)),$$

or equivalently, $t \leq \frac{e^{a\rho(x,y)^v}\varphi_2(y) - \varphi_2(x)}{e^{a\rho(x,y)^v}\varphi_1(y) - \varphi_1(x)}$;
(iii) $\frac{\varphi_2(n) - t\varphi_1(n)}{\varphi_2(m) - t\varphi_1(m)} \leq b$, or equivalently, $t \leq \frac{b\varphi_2(m) - \varphi_2(n)}{b\varphi_1(m) - \varphi_1(n)}$.

In other words, $\alpha_{a,v,b}(\varphi_1, \varphi_2)$ equals

$$\min\biggl\{\inf_{x,\,y}\biggl\{\frac{e^{a\rho(x,y)^v}\varphi_2(x)-\varphi_2(y)}{e^{a\rho(x,y)^v}\varphi_1(x)-\varphi_1(y)}\biggr\},\ \inf_{m,\,n\in X}\biggl\{\frac{b\varphi_2(m)-\varphi_2(n)}{b\varphi_1(m)-\varphi_1(n)}\biggr\},\ \inf_{s\in X}\biggl\{\frac{\varphi_2(s)}{\varphi_1(s)}\biggr\}\biggr\}, \tag{4.4}$$

where the inferior of the first term is taken over all pairs of x, $y \in X$ with $0 < \rho(x, y) \leq \xi_0$.

We demonstrate that the images of the convex cone under iterations of the Ruelle operator contract uniformly.

Proposition 4.5. *Fix arbitrary η, ξ, a_0, $v_0 > 0$, and $\lambda > 1$. Let (X, ρ) be a compact metric space, $T\colon X \to X$ be a continuous, topologically exact, open transformation which is distance-expanding with respect to the metric ρ with the constants η, λ, and ξ, and the potential $\varphi\colon X \to \mathbb{R}$ be Hölder continuous with respect to the metric ρ with exponent v_0 and constant a_0. Fix constants b, v, λ_1, and a' with $b \geq 1$, $0 < v \leq v_0$, $\lambda^{-v} < \lambda_1 < 1$, and $a' \geq \frac{a_0\lambda^{-v}+2a_0(\lambda^{v_0}-1)^{-1}}{\lambda_1-\lambda^{-v}}$. Define $\overline{\varphi}(x) := \varphi(x) - P(T, \varphi)$ for each $x \in X$, $\xi_0 := \min\{\xi, 1\}$, and $Q := \exp(\inf_{x\in X} \overline{\varphi}(x))$. Suppose that $0 < \epsilon \leq \xi_0$ is a constant with $2\exp(a'\epsilon^v) \leq b$, and that $G' = \{x'_1, x'_2, \dots, x'_{n'}\}$ is an $(\epsilon/2)$-net of X.*

Denote by $L\colon C(X) \to C(X)$ the operator given by

$$L(u)(x) := \frac{\mathcal{L}_{\overline{\varphi}}(uu_\varphi)(x)}{u_\varphi(x)} = \sum_{\overline{x}\in T^{-1}(x)} \frac{u(\overline{x})u_\varphi(\overline{x})}{u_\varphi(x)} \cdot \exp(\overline{\varphi}(\overline{x})) \tag{4.5}$$

for all $u \in C(X)$ and $x \in X$. Then $L(\mathbb{1}) = \mathbb{1}$ and

$$L^m(\mathcal{C}(a', v, b)) \subseteq \mathcal{C}(\lambda_1 a', v, b') \subseteq \mathcal{C}(a', v, b) \tag{4.6}$$

for each $m \in \mathbb{N}$ satisfying that

$$T^m(B(x'_i, \epsilon/2)) = X \qquad \text{for each integer } 1 \leq i \leq n', \tag{4.7}$$

where $b' := \frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2} < b$.

Proof. By (2.12) and (4.5), we have $L(\mathbb{1}) = \mathbb{1}$. Then we turn to prove (4.6). For $\lambda_1 < 1$ and $b' < b$, it follows from Definition 4.4 that $\mathcal{C}(\lambda_1 a', v, b') \subseteq \mathcal{C}(a', v, b)$.

Fix an arbitrary $u \in \mathcal{C}(a', v, b)$ and an integer $m \in \mathbb{N}$ with (4.7). It suffices to demonstrate that $L^m(u)$ satisfies properties (i), (ii), and (iii) in Definition 4.4 for $\mathcal{C}(\lambda_1 a', v, b')$.

Since $u \in \mathcal{C}(a', v, b)$, then $u(x) > 0$ for each $x \in X$. By (4.5), $L(u)(x) > 0$ for each $x \in X$. By induction, we obtain that $L^m(u)(x) > 0$ for each $x \in X$, hence $L^m(u)$ satisfies property (i) in Definition 4.4 for $\mathcal{C}(\lambda_1 a', v, b')$.

Next, we prove that $L(u)$ satisfies property (ii) in Definition 4.4 for $\mathcal{C}(\lambda_1 a', v, b')$, namely,

$$L(u)(x) \leq \exp(a'\lambda_1\rho(x, y)^v) \cdot L(u)(y)$$

for each pair of x, $y \in X$ with $\rho(x, y) \leq \xi_0$. Consider a pair of x, $y \in X$ with $\rho(x, y) \leq \xi_0$. Since T is an open transformation that is distance-expanding with the constants η, λ, and ξ, then for each $\overline{x} \in T^{-1}(x)$, one can find a point $y(\overline{x}) := T^{-1}_{\overline{x}}(y) \in T^{-1}(y)$ satisfying that $\rho(\overline{x}, y(\overline{x})) \leq \lambda^{-1}\rho(x, y) \leq \xi_0$, and hence,

$$u(\overline{x}) \leq \exp(a'\rho(\overline{x}, y(\overline{x}))^v) \cdot u(y(\overline{x})) \leq \exp(a'\lambda^{-v}\rho(x, y)^v) \cdot u(y(\overline{x})). \tag{4.8}$$

Note that $\overline{\varphi}(x) = \varphi(x) - P(T, \varphi)$ is Hölder continuous with exponent v_0 and constant a_0. So for each $\overline{x} \in T^{-1}(x)$,

$$\begin{aligned}\exp(\overline{\varphi}(\overline{x})) &\leq \exp(a_0\rho(\overline{x}, y(\overline{x}))^{v_0}) \cdot \exp(\overline{\varphi}(y(\overline{x}))) \\ &\leq \exp(a_0\rho(\overline{x}, y(\overline{x}))^{v}) \cdot \exp(\overline{\varphi}(y(\overline{x})))\end{aligned} \tag{4.9}$$

$$\leq \exp(a_0\lambda^{-v}\rho(x,y)^v)\cdot\exp(\overline{\varphi}(y(\overline{x}))).$$

Moreover, by (2.13), for each $\overline{x}\in T^{-1}(x)$,

$$\frac{u_\varphi(\overline{x})}{u_\varphi(x)}\leq\exp\biggl(\frac{2a_0\rho(x,y)^{v_0}}{\lambda^{v_0}-1}\biggr)\cdot\frac{u_\varphi(y(\overline{x}))}{u_\varphi(y)}\leq\exp\biggl(\frac{2a_0\rho(x,y)^{v}}{\lambda^{v_0}-1}\biggr)\cdot\frac{u_\varphi(y(\overline{x}))}{u_\varphi(y)}. \tag{4.10}$$

Thus by (4.5), (4.8), (4.9), and (4.10) we demonstrate conclude that

$$\begin{aligned}L(u)(x)&=\sum_{\overline{x}\in T^{-1}(x)}\frac{u(\overline{x})u_\varphi(\overline{x})}{u_\varphi(x)}\cdot\exp(\overline{\varphi}(\overline{x}))\\&\leq\sum_{\overline{x}\in T^{-1}(x)}\frac{u(y(\overline{x}))u_\varphi(y(\overline{x}))}{u_\varphi(y)}\cdot\exp(\overline{\varphi}(y(\overline{x}))+(a'\lambda^{-v}+a_0\lambda^{-v}+2a_0/(\lambda^{v_0}-1))\rho(x,y)^v)\\&\leq\exp(a'\lambda_1\rho(x,y)^v)\cdot L(u)(y)\end{aligned}$$

for each pair of x, $y\in X$ with $\rho(x,y)\leq\xi_0$. By induction, it follows from $\mathcal{C}(\lambda_1a',v,b')\subseteq\mathcal{C}(a',v,b)$ that $L^m(u)$ satisfies property (ii) in Definition 4.4 for $\mathcal{C}(\lambda_1a',v,b')$.

Finally, we establish that $L^m(u)$ satisfies property (iii) in Definition 4.4 for $\mathcal{C}(\lambda_1a',v,b')$, namely,

$$L^m(u)(x)\leq b'L^m(u)(y)\quad\text{for each pair of } x,\ y\in X. \tag{4.11}$$

Without loss of generality, assume that $\inf_{x\in X}u(x)=1$. Let x_* and x^* be two points in X satisfying that $u(x_*)=1$ and $u(x^*)=\sup_{x\in X}u(x)$. Since $u\in\mathcal{C}(a',v,b)$, we have $u(x^*)\leq b$.

First consider the case that $\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}\leq u(x^*)\leq b$. Consider a point $w\in X$. Since G' is an $(\epsilon/2)$-net of X, there exists $1\leq i\leq n'$ with $x_*\in B(x_i',\epsilon/2)$. By (4.7), there exists $\overline{w}_*\in T^{-m}(w)$ such that $\rho(\overline{w}_*,x_i')<\epsilon/2$. Hence, $\rho(\overline{w}_*,x_*)\leq\rho(\overline{w}_*,x_i')+\rho(x_i',x_*)<\epsilon\leq\xi_0$. Additionally, since $u\in\mathcal{C}(a',v,b)$, we have

$$u(\overline{w}_*)\leq\exp(a'\rho(\overline{w}_*,x_*)^v)u(x_*)\leq\exp(a'\epsilon^v). \tag{4.12}$$

Hence, by the definitions of ϵ, Q, (2.14), (4.5), and (4.12), we have

$$\begin{aligned}L^m(u)(w)&=\frac{u(\overline{w}_*)u_\varphi(\overline{w}_*)}{u_\varphi(w)}\cdot e^{S_m\overline{\varphi}(\overline{w}_*)}+\sum_{\overline{w}\in T^{-m}(w)\setminus\{\overline{w}_*\}}\frac{u(\overline{w})u_\varphi(\overline{w})}{u_\varphi(w)}\cdot e^{S_m\overline{\varphi}(\overline{w})}\\&\leq\frac{\exp(a'\epsilon^v)\cdot u_\varphi(\overline{w}_*)}{u_\varphi(w)}\cdot e^{S_m\overline{\varphi}(\overline{w}_*)}+u(x^*)\sum_{\overline{w}\in T^{-m}(w)\setminus\{\overline{w}_*\}}\frac{u_\varphi(\overline{w})}{u_\varphi(w)}\cdot e^{S_m\overline{\varphi}(\overline{w})}\\&\leq u(x^*)-(u(x^*)-\exp(a'\epsilon^v))\cdot C^{-2}\inf_{x\in X}\exp(S_m\overline{\varphi}(x))\\&\leq b-\biggl(\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}-\frac{b}{2}\biggr)C^{-2}Q^m\\&=\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}.\end{aligned}$$

Otherwise we consider the case that $u(x^*)\leq\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}$. Since $L(\mathbb{1})=\mathbb{1}$, we have $\sup_{w\in X}L^m(u)(w)\leq\sup_{w\in X}u(w)\leq\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}$.

To sum up, we always have $\sup_{w\in X}L^m(u)(w)\leq\frac{(C^{-2}Q^m+2)b}{2C^{-2}Q^m+2}$. Moreover, since $L(\mathbb{1})=\mathbb{1}$, it follows that $\inf_{w\in X}L^m(u)(w)\geq u(x_*)=1$. Hence, we complete the proof of (4.11) for each pair of x, $y\in X$. □

Consider now the convex cone $\mathcal{C}_+$ of positive functions on X. Let α_+ and β_+ be the corresponding functions associated to the convex cone $\mathcal{C}_+$ defined in (4.2). We denote by θ_+ the projective metric associated to the convex cone $\mathcal{C}_+$. Now we prove that the $\theta_{a,v,b}$-diameter of $\mathcal{C}(\lambda_1 a, v, b')$ is finite. Recall $\widehat{K}$ from (4.1).

Proposition 4.6. *For all a, $v > 0$, $1 < b' < b$, and $\lambda_1 < 1$, we have*

$$\operatorname{diam}_{\theta_{a,v,b}}(\mathcal{C}(\lambda_1 a, v, b')) \leq \widehat{K}(\lambda_1, b, b'). \tag{4.13}$$

Proof. Consider a pair of φ_1, $\varphi_2 \in \mathcal{C}(\lambda_1 a, v, b') \subseteq \mathcal{C}_+$. By the definition of $\mathcal{C}(\lambda_1 a, v, b')$, we have

$$\varphi_2(y) \leq \varphi_2(x)\exp(a\lambda_1\rho(x,y)^v) \quad \text{and} \quad \varphi_1(y) \geq \varphi_1(x)\exp(-a\lambda_1\rho(x,y)^v)$$

for each pair of x, $y \in X$ with $\rho(x,y) \leq \xi_0 := \min\{\xi, 1\}$. Hence,

$$\begin{aligned}\frac{\exp(a\rho(x,y)^v)\varphi_2(x) - \varphi_2(y)}{\exp(a\rho(x,y)^v)\varphi_1(x) - \varphi_1(y)} &\geq \frac{\varphi_2(x)}{\varphi_1(x)} \cdot \frac{\exp(a\rho(x,y)^v) - \exp(a\lambda_1\rho(x,y)^v)}{\exp(a\rho(x,y)^v) - \exp(-a\lambda_1\rho(x,y)^v)} \\ &\geq \inf_{z>1}\left\{\frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}}\right\} \cdot \frac{\varphi_2(x)}{\varphi_1(x)} \\ &\geq \frac{1-\lambda_1}{1+\lambda_1} \cdot \frac{\varphi_2(x)}{\varphi_1(x)}\end{aligned} \tag{4.14}$$

for each pair of x, $y \in X$ with $\rho(x,y) \leq \xi_0 := \min\{\xi, 1\}$.

By Definition 4.4, we have

$$\varphi_2(y) \leq b'\varphi_2(x) < b\varphi_2(x) \quad \text{and} \quad 0 < \varphi_1(y) < b\varphi_1(x)$$

for each pair of x, $y \in X$. Hence,

$$\frac{b\varphi_2(x) - \varphi_2(y)}{b\varphi_1(x) - \varphi_1(y)} \geq \frac{b-b'}{b} \cdot \frac{\varphi_2(x)}{\varphi_1(x)} \tag{4.15}$$

for each pair of x, $y \in X$.

Consider an arbitrary pair of φ_1, $\varphi_2 \in \mathcal{C}(\lambda_1 a, v, b')$. Noting that, by the definition of α_+, $\alpha_+(\varphi_1,\varphi_2) = \inf_{t\in X}(\varphi_2(t)/\varphi_1(t))$, it follows from (4.4), (4.14), and (4.15) that $\alpha_{a,v,b}(\varphi_1,\varphi_2) \geq \frac{1-\lambda_1}{1+\lambda_1} \cdot \frac{b-b'}{b} \cdot \alpha_+(\varphi_1,\varphi_2)$. Similarly, we can obtain that $\beta_{a,v,b}(\varphi_1,\varphi_2) \leq \frac{1+\lambda_1}{1-\lambda_1} \cdot \frac{b}{b-b'} \cdot \beta_+(\varphi_1,\varphi_2)$. Additionally, by the definitions of $\theta_{a,v,b}$ and θ_+, one can see that

$$\begin{aligned}\theta_{a,v,b}(\varphi_1,\varphi_2) &= \log(\beta_{a,v,b}(\varphi_1,\varphi_2)) - \log(\alpha_{a,v,b}(\varphi_1,\varphi_2)) \\ &\leq \log(\beta_+(\varphi_1,\varphi_2)) - \log(\alpha_+(\varphi_1,\varphi_2)) + 2\log\left(\frac{1+\lambda_1}{1-\lambda_1} \cdot \frac{b}{b-b'}\right) \\ &= \theta_+(\varphi_1,\varphi_2) + 2\log\left(\frac{1+\lambda_1}{1-\lambda_1} \cdot \frac{b^2}{b-b'}\right)\end{aligned}$$

for each pair of φ_1, $\varphi_2 \in \mathcal{C}(\lambda_1 a, v, b')$. This completes the proof of (4.13). □

We are now ready to establish Theorem 4.1.

Proof of Theorem 4.1. Let $\overline{a} := \frac{6a_0\lambda^{v_0} - 2a_0}{(\lambda^{v_0}-1)^2}$ and C is from Lemma 2.7. Then by Proposition 4.5 for $a' = \overline{a}$, $v = v_0$, $b = C^2$, $\lambda_1 = \frac{1+\lambda}{2}$, we have

$$L^m(\mathcal{C}(\overline{a}, v_0, C^2)) \subseteq \mathcal{C}\left(\frac{\lambda^{-v_0}+1}{2}\overline{a}, v_0, \frac{Q^m + 2C^2}{2C^{-2}Q^m + 2}\right).$$

Additionally, it follows from Proposition 4.6 that

$$\begin{aligned}\operatorname{diam}_{\theta_{\overline{a},v,C^2}}(L^m(\mathcal{C}(\overline{a},v_0,C^2))) &\leq \operatorname{diam}_{\theta_{\overline{a},v,C^2}}\biggl(\mathcal{C}\biggl(\frac{\lambda^{-v_0}+1}{2}\overline{a},v_0,\frac{Q^m+2C^2}{2C^{-2}Q^m+2}\biggr)\biggr)\\ &\leq \widehat{K}\biggl(\frac{\lambda^{-v_0}+1}{2},C^2,\frac{Q^m+2C^2}{2C^{-2}Q^m+2}\biggr).\end{aligned} \tag{4.16}$$

By (2.13) and (2.14), one can see that $1/u_\varphi,\ \mathbb{1}\in\mathcal{C}(\overline{a},v_0,C^2)$. Then by Proposition 4.3 for $L^m\colon\mathcal{C}(\overline{a},v_0,C^2)\to\mathcal{C}(\overline{a},v_0,C^2)$ and (4.16), we have

$$\begin{aligned}\theta_{\overline{a},v_0,C^2}\bigl(L^{mk}(1/u_\varphi),\mathbb{1}\bigr) &\leq \theta_{\overline{a},v_0,C^2}\bigl(L^m(1/u_\varphi),\mathbb{1}\bigr)(1-\exp(-Z))^{k-1}\\ &\leq Z(1-\exp(-Z))^{k-1}\end{aligned} \tag{4.17}$$

for each $k\in\mathbb{N}$, where $Z:=\widehat{K}\bigl(\frac{\lambda^{-v_0}+1}{2},C^2,\frac{Q^m+2C^2}{2C^{-2}Q^m+2}\bigr)$. Since $\mathcal{C}(\overline{a},v_0,C^2)\subseteq\mathcal{C}_+$, then $\theta_+(v_1,v_2)\leq\theta_{\overline{a},v_0,C^2}(v_1,v_2)$ for each pair of $v_1,\ v_2\in\mathcal{C}(\overline{a},v_0,C^2)$. Hence,

$$\theta_+\bigl(L^{mk}(1/u_\varphi),\mathbb{1}\bigr)\leq\theta_{\overline{a},v_0,C^2}\bigl(L^{mk}(1/u_\varphi),\mathbb{1}\bigr). \tag{4.18}$$

Then by (4.5), (4.17), and (4.18), one can see that

$$\theta_+\bigl(\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1}),u_\varphi\bigr)=\theta_+\bigl(\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1})/u_\varphi,\mathbb{1}\bigr)=\theta_+\bigl(L^{mk}(1/u_\varphi),\mathbb{1}\bigr)\leq Z(1-\exp(-Z))^{k-1}.$$

Thus one can conclude that

$$\begin{aligned}\bigl\|\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1})(x)-u_\varphi(x)\bigr\|_\infty &\leq \bigl\|\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1})(x)\bigr\|_\infty\cdot\biggl\|\frac{u_\varphi(x)}{\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1})(x)}-1\biggr\|_\infty\\ &\leq C\bigl(\exp\bigl(Z(1-\exp(-Z))^{k-1}\bigr)-1\bigr).\end{aligned}$$

Since for each $x\in[0,1]$, $e^x\leq ex+1$, we have

$$\bigl\|\mathcal{L}^{mk}_{\overline{\varphi}}(\mathbb{1})(x)-u_\varphi(x)\bigr\|_\infty\leq CeZ(1-\exp(-Z))^{k-1}$$

for each $k\in\mathbb{N}$ with $0\leq Z(1-\exp(-Z))^{k-1}\leq 1$. The proof is, therefore, complete. □

4.2. **Proof of the computability of the Jacobian.** We aim to establish the following theorem in this subsection.

Theorem 4.7. *There exists an algorithm that satisfies the following property:*

For each quintet $(X,\rho,\mathcal{S},\varphi,T)$ satisfying the Assumption A in Section 1, this algorithm outputs a rational 2^{-n}-approximation for the value of $J_\varphi(x)$, where the function J_φ is the function defined by

$$J_\varphi(x):=\frac{u_\varphi(T(x))}{u_\varphi(x)}\exp(P(T,\varphi)-\varphi(x)),\qquad x\in X, \tag{4.19}$$

after inputting the following data in this algorithm:

- (i) *two algorithms computing the potential φ and the transformation T, respectively,*
- (ii) *an algorithm outputting a net of X with any given precision (in the sense of Definition 3.14),*
- (iii) *an oracle of a point $x\in X$,*
- (iv) *two rational constants α_0 and v_0 satisfying that φ is Hölder continuous with the exponent v_0 and the constant α_0,*
- (v) *three rational constants η, λ, and ξ satisfying that T is distance-expanding with these constants (in the sense of Definition 2.1),*

(vi) *a constant* $n \in \mathbb{N}$.

The proof will be given at the end of this subsection. We begin with designing an algorithm that computes the function $\mathcal{L}_\varphi(\mathbb{1})$.

Proposition 4.8. *Let* $(X, \rho, \mathcal{S})$ *be a computable metric space,* X *a recursively compact set, and* $T\colon X \to X$ *a computable distance-expanding transformation with respect to the metric* ρ *with the constants* η, λ, *and* ξ. *Then the inverse* T^{-1} *of* T *is computable (in the sense of Definition 3.11).*

Proof. Consider $x \in X$ and an oracle φ for x. By Definition 3.8, $\{x\} = \bigcap_{n\in\mathbb{N}} \big(\overline{B(s_{\varphi(n)}, 2^{-n})}\big)$. Hence $T^{-1}(x) = \bigcap_{n\in\mathbb{N}} T^{-1}\big(\overline{B(s_{\varphi(n)}, 2^{-n})}\big)$. By Definition 3.6, we get that $\big\{\big(\overline{B(s_{\varphi(n)}, 2^{-n})}\big)^c\big\}_{n\in\mathbb{N}}$ is a uniformly computable sequence of lower-computable open sets. Since T is computable, it follows from Proposition 3.10 that the sequence $\big\{T^{-1}\big(\big(\overline{B(s_{\varphi(n)}, 2^{-n})}\big)^c\big)\big\}_{n\in\mathbb{N}}$ is a uniformly computable sequence of lower-computable open sets. By Proposition 3.7, $\big(T^{-1}(x)\big)^c = \bigcup_{n\in\mathbb{N}} T^{-1}\big(\big(\overline{B(s_{\varphi(n)}, 2^{-n})}\big)^c\big)$ is a lower-computable open set. By Proposition 3.13, noting that X is recursively compact, $T^{-1}(x)$ is a recursively compact set.

By [GHR10, Proposition 4], X is recursively precompact, and consequently, there is an algorithm that outputs an η-net of X. Because T is distance-expanding with the constants η, λ, and ξ, there is at most one preimage point of x in each ball with radius η. Hence, each preimage point of x is computable. □

As an immediate consequence of Proposition 4.8 and the computability of the exponential function, one gets the computability of the Ruelle operator in the following sense:

Corollary 4.9. *There exists an algorithm that satisfies the following property:*

For each quintet $(X, \rho, \mathcal{S}, \varphi, T)$ *satisfying the Assumption A in Section 1, this algorithm outputs a rational* 2^{-n}*-approximation for the value of* $\mathcal{L}_\varphi^m(\mathbb{1})(x)$, *after inputting the following data in this algorithm:*

(i) *two algorithms computing the potential* φ *and the transformation* T, *respectively,*
(ii) *an algorithm outputting a net of* X *with any given precision (in the sense of Definition 3.14),*
(iii) *an oracle of a point* $x \in X$,
(iv) *three rational constants* η, λ, *and* ξ *satisfying that* T *is distance-expanding with these constants (in the sense of Definition 2.1),*
(v) *two constants* n, $m \in \mathbb{N}$.

We can now apply Theorem 4.1 to prove the computability of the pressure $P(T, \varphi)$ and the function $u_\varphi(x)$ (see Theorem 4.1 for the definition of the function $u_\varphi(x)$).

Lemma 4.10. *There exists an algorithm that satisfies the following property:*

For each quintet $(X, \rho, \mathcal{S}, \varphi, T)$ *satisfying the Assumption A in Section 1, this algorithm outputs a rational* 2^{-n}*-approximation for the topological pressure* $P(T, \varphi)$, *after inputting the following data in this algorithm:*

(i) *two algorithms computing the potential* φ *and the transformation* T, *respectively,*
(ii) *an algorithm outputting a net of* X *with any given precision (in the sense of Definition 3.14),*

(iii) *two rational constants* α_0 *and* v_0 *satisfying that* φ *is Hölder continuous with the exponent* v_0 *and the constant* α_0*,*

(iv) *three rational constants* η*,* λ*, and* ξ *satisfying that* T *is distance-expanding with these constants (in the sense of Definition 2.1),*

(v) *a constant* $n \in \mathbb{N}$*.*

Proof. We can design the machine following the steps below:

(1) Read in all the data.
(2) Compute a ξ_0'-net G for the space X, where $\xi_0' := \min\{\eta, \xi, 1\}$. Then use (2.10) to compute N for G.
(3) Use [BRY12, Proposition 2.13] to compute $\|\varphi\|_\infty$.
(4) Find $N_1 \in \mathbb{N}$ with $2^{n+1} N \log(\operatorname{card} G)\bigl(\frac{4a_0}{\lambda^{v_0}-1} + 2N\|\varphi\|_\infty\bigr) \leq N_1$.
(5) By Corollary 4.9, we compute and output the value of
$$v_n \approx w_n := \frac{1}{N_1}\log\bigl(\mathcal{L}_\varphi^{N_1}(\mathbb{1})(x_0)\bigr)$$
with precision 2^{-n-1}, where $x_0 \in \mathcal{S}$ is an ideal point.

Let us verify that v_n satisfies $|v_n - P(T,\varphi)| < 2^{-n}$ for each $n \in \mathbb{N}$. To see this, it suffices to check that $|w_n - P(T,\varphi)| < 2^{-n-1}$ for each $n \in \mathbb{N}$. Since $T\colon X \to X$ is distance-expanding with the constants η, λ, and ξ, then T is injective on each balls of radius ξ_0'. Hence, we have $D := \max_{x\in X} \operatorname{card}\bigl(T^{-1}(x)\bigr) \leq \operatorname{card} G$. Additionally, by Lemma 2.7, steps (4) and (5) above, one can conclude that
$$|w_n - P(T,\varphi)| = \bigl|N_1^{-1}\log\bigl(e^{-N_1 P(T,\varphi)}\mathcal{L}_\varphi^{N_1}(\mathbb{1})(x_0)\bigr)\bigr| < (\log C)/N_1 < 2^{-n-1}. \qquad \square$$

Lemma 4.11. *There exists an algorithm that satisfies the following property:*

For each quintet $(X, \rho, \mathcal{S}, \varphi, T)$ *satisfying the Assumption A in Section 1, this algorithm outputs a rational* 2^{-n}*-approximation for the value of* $u_\varphi(x)$*, after inputting the following data in this algorithm:*

(i) *two algorithms computing the potential* φ *and the transformation* T*, respectively,*

(ii) *an algorithm outputting a net of* X *with any given precision (in the sense of Definition 3.14),*

(iii) *two rational constants* α_0 *and* v_0 *satisfying that* φ *is Hölder continuous with the exponent* v_0 *and the constant* α_0*,*

(iv) *three rational constants* η*,* λ*, and* ξ *satisfying that* T *is distance-expanding with these constants (in the sense of Definition 2.1),*

(v) *an oracle of a point* $x \in X$*,*

(vi) *a constant* $n \in \mathbb{N}$*.*

Proof. We can design the algorithm following the steps below:

(1) Read in all the data.
(2) Use Lemma 4.10 to compute the topological pressure $P(T,\varphi)$ and the constant C defined in Lemma 2.7.
(3) Use Proposition 4.5 to compute the constants m and Q.
(4) Use Theorem 4.1 and (4.1) to compute the constant Z.

(5) Find $k \in \mathbb{N}$ satisfying that

$$0 \leq Z(1-\exp(-Z))^{k-1} \leq 1 \quad \text{and} \quad CeZ(1-\exp(-Z))^{k-1} < 2^{-n-1}. \tag{4.20}$$

(6) By Corollary 4.8 and Lemma 4.10, we compute and output

$$s_n \approx t_n := e^{-mkP(T,\varphi)} \mathcal{L}_{\varphi}^{mk}(\mathbb{1})(x).$$

with precision 2^{-n-1}.

By Theorem 4.1 and (4.20), we have $|t_n - u_\varphi(x)| < |t_n - s_n| + |s_n - u_\varphi(x)| < 2^{-n}$. □

We are now ready to establish Theorem 4.7.

Proof of Theorem 4.7. By Corollary 4.9, Lemmas 4.10, and 4.11, it follows from (4.19) that the function J_φ is computable. □

4.3. **Two lemmas.** In this subsection, we consider two subsets of $\mathcal{P}(X)$ closely related to the equilibrium state μ_φ and prove that they are both lower-computable open sets.

Lemma 4.12. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact, and $T\colon X \to X$ is computable. Then the set U of Borel probability measures that are not T-invariant is a lower-computable open set.*

Proof. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathcal{E}(\mathcal{S})$ (see Definition 3.20) in the computable metric space $(X, \rho, \mathcal{S})$.

Claim. $U = \bigcup_{j\in\mathbb{N}} \Phi_j$, where $\Phi_j := \left\{\mu \in \mathcal{P}(X) : \int \varphi_j \,\mathrm{d}\mu - \int \varphi_j \circ T \,\mathrm{d}\mu \neq 0\right\}$ for each $j \in \mathbb{N}$.

Indeed, consider $\mu \in \bigcap_{j\in\mathbb{N}} \Phi_j^c$. By Proposition 3.21, we conclude that $\mu = \mu \circ T^{-1}$, namely, $\mu \notin U$. Hence $U = \bigcup_{j\in\mathbb{N}} \Phi_j$, establishing the claim.

Next, note that $\{\varphi_j\}_{j\in\mathbb{N}}$ is a uniformly computable sequence of functions and T is computable. Then by Corollary 3.18, the family $\{\mathcal{I}_j\}_{j\in\mathbb{N}}$ of integral operators $\mathcal{I}_j\colon \mathcal{P}(X) \to \mathbb{R}$ defined by

$$\mathcal{I}_j(\mu) := \int (\varphi_j - \varphi_j \circ T) \,\mathrm{d}\mu$$

is a uniformly computable sequence of functions. It follows from Proposition 3.10 that $\{\Phi_j\}_{j\in\mathbb{N}}$ is a uniformly computable sequence of lower-computable open sets. Therefore, by Proposition 3.7 and the claim, U is a lower-computable open set. □

Lemma 4.13. *Let the quintet $(X, \rho, \mathcal{S}, \varphi, T)$ satisfy the Assumption A in Section 1. Define the set W so that a Borel probability measure μ is in W if and only if the Jacobians of $T\colon X \to X$ with respect to μ is not*

$$J_\varphi(x) := \frac{u_\varphi(T(x))}{u_\varphi(x)} \exp(P(T,\varphi) - \varphi(x)).$$

Then W is a lower-computable open set.

Proof. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathcal{E}(\mathcal{S})$ (see Definition 3.20) in the computable metric space $(X, \rho, \mathcal{S})$, and $\{x_k : 1 \leq k \leq m\}$ be an η-net for X. Denote by $\{\mathbb{1}_{k,n} : k, n \in \mathbb{N},\ 1 \leq k \leq m\}$ the set of functions given by $\mathbb{1}_{k,n} := g_{x_k, \eta-\frac{1}{n}, \frac{1}{n}}$.

Claim. $W = \bigcup_{k=1}^{m} \bigcup_{j,\, n\in\mathbb{N}} \Phi_j^{k,n}$, where

$$\Phi_j^{k,n} := \left\{\mu \in \mathcal{P}(X) : \int (\varphi_j \cdot \mathbb{1}_{k,n}) \circ T^{-1} \,\mathrm{d}\mu - \int \varphi_j \cdot \mathbb{1}_{k,n} \cdot J_\varphi \,\mathrm{d}\mu \neq 0\right\} \tag{4.21}$$

for all $1 \leq k \leq m$ and $j,\, n \in \mathbb{N}$.

To prove the claim, we fix an integer $1 \leq k \leq m$. Since T is distance-expanding with the constants η, λ, and ξ, hence, is injective on $B(x_k, \eta)$, then the function $(\varphi_j \cdot \mathbb{1}_{k,n}) \circ T^{-1}$ is well-defined in X.

Consider

$$\mu \in \bigcap_{j,\, n \in \mathbb{N}} \big(\Phi_j^{k,n}\big)^c.$$

Let $n \to +\infty$, one can see that

$$\int \varphi_j \cdot \mathbb{1}_{B(x_k,\eta)} \,\mathrm{d}(\mu \circ T) = \int (\varphi_j \cdot \mathbb{1}_{B(x_k,\eta)}) \circ T^{-1} \,\mathrm{d}\mu = \int \varphi_j \cdot \mathbb{1}_{B(x_k,\eta)} \cdot J_\varphi \,\mathrm{d}\mu \tag{4.22}$$

for each $j \in \mathbb{N}$. Then by Proposition 3.21, it follows that $\mu_1 = \mu_2$, where μ_1 and μ_2 are given by

$$\mu_1(A) \coloneqq \frac{\mu(T(A \cap B(x_k,\eta)))}{\mu(T(B(x_k,\eta)))} \quad \text{and} \quad \mu_2(A) \coloneqq \frac{\int_{A \cap B(x_k,\eta)} J_\varphi \,\mathrm{d}\mu}{\int_{B(x_k,\eta)} J_\varphi \,\mathrm{d}\mu}$$

for each Borel set $A \subseteq X$. Thus

$$\mu(T(A \cap B(x_k,\eta))) = \int_{A \cap B(x_k,\eta)} J_\varphi \,\mathrm{d}\mu$$

for each Borel set $A \subseteq X$.

Since $X = \bigcup_{k=1}^m B(x_k, \eta)$, we get $\bigcap_{k=1}^m \bigcap_{j,\, n \in \mathbb{N}} \big(\Phi_j^{k,n}\big)^c \subseteq W^c$. Hence the claim holds.

Now we return to the proof of Lemma 4.13. By Definition 3.19, Theorem 4.7, and Proposition 4.8, $\{(\varphi_j \cdot \mathbb{1}_{k,n}) \circ T^{-1} - \varphi_j \cdot \mathbb{1}_{k,n} \cdot J_\varphi : j,\, n \in \mathbb{N}, \text{ and } 1 \leq k \leq m\}$ is a uniformly computable sequence of functions. Then by Corollary 3.18, the family $\{\mathcal{I}_j^{k,n} : j,\, n \in \mathbb{N}, \text{ and } 1 \leq k \leq m\}$ of integral operators $\mathcal{I}_j^{k,n} \colon \mathcal{P}(X) \to \mathbb{R}$ defined by

$$\mathcal{I}_j^{k,n}(\mu) \coloneqq \int \big((\varphi_j \cdot \mathbb{1}_{k,n}) \circ T^{-1} - \varphi_j \cdot \mathbb{1}_{k,n} \cdot J_\varphi \big) \,\mathrm{d}\mu, \qquad \mu \in \mathcal{P}(X),$$

is a uniformly computable sequence of functions. Thus by Proposition 3.10 and (4.21), it follows that $\big\{\Phi_j^{k,n} : j,\, n \in \mathbb{N}, \text{ and } 1 \leq k \leq m\big\}$ is a uniformly computable sequence of lower-computable open sets. Therefore, by Proposition 3.7 and the claim, W is a lower-computable open set. □

4.4. **Proofs of Theorems 1.1 and 1.2.** We are now ready to complete the proof of Theorem 1.1 inspired by the proof of [BBRY11, Theorem A].

Proof of Theorem 1.1. Assume that X is recursively compact, then by Corollary 3.16, $\mathcal{P}(X)$ is recursively compact. By Corollary 2.8, there exists a unique T-invariant Gibbs state μ_φ for T and φ, hence by Proposition 2.5, an unique equilibrium state μ_φ of T for φ. Together with the definitions of U and W, we have $\{\mu_\varphi\} = \mathcal{P}(X) \setminus (U \cup W)$. By Lemmas 4.12 and 4.13, U and W are both lower-computable open sets. It follows from Proposition 3.13 (ii) that $\{\mu_\varphi\}$ is recursively compact. Therefore by Proposition 3.13 (i), the equilibrium state μ_φ of T for φ is computable. □

Applying Theorem 1.1, we establish Theorem 1.2.

Proof of Theorem 1.2. In the computable metric space $(D, \rho, \mathcal{S}')$, by Theorem 1.1, there exists an algorithm which, on input $n \in \mathbb{N}$, outputs $\{k_i\}_{i=1}^m \subseteq \mathbb{Q}^+$ and $\{x_i\}_{i=1}^m \subseteq \mathcal{S}'$ satisfying that $\sum_{i=1}^m k_i = 1$, and $W_\rho(\mu_\varphi, \mu_n) \leq 2^{-n}$, where $\mu_n = \sum_{i=1}^m k_i \delta_{x_i} \in \mathcal{R}_{\mathcal{S}'}$. Note that $\mathcal{S}'$ is a uniformly computable sequence of points in the computable metric space $(X, \rho, \mathcal{S})$. So for each integer $1 \leq i \leq m$, we can find a point $y_i \in \mathcal{S}$ such that $\rho\bigl(x_i, y_i\bigr) < 2^{-n}$. Define $\widehat{\mu}_n := \sum_{i=1}^m k_i \delta_{y_i} \in \mathcal{R}_{\mathcal{S}}$. Then

$$W_\rho(\mu_\varphi, \widehat{\mu}_n) \leq W_\rho(\mu_\varphi, \mu_n) + W_\rho(\mu_n, \widehat{\mu}_n) < 2^{1-n}.$$

Therefore, μ_φ is a computable point in the computable metric space $(\mathcal{P}(X),\, W_\rho,\, \mathcal{R}_{\mathcal{S}})$. □

5. Hyperbolic rational maps

In this section, we review some notations and results in complex dynamics and apply Theorem 1.2 to prove Theorem C.

Recall that the *spherical metric* $d_{\widehat{\mathbb{C}}}$ on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is given by the length element $\mathrm{d}s(z) = 2|\mathrm{d}z|/(1+|z|^2)$ for each $z \in \widehat{\mathbb{C}}$ is a conformal metric. Define $B_{d_{\widehat{\mathbb{C}}}}(z, r) := \{z' \in \widehat{\mathbb{C}} : d_{\widehat{\mathbb{C}}}(z', z) < r\}$ for each $z \in \widehat{\mathbb{C}}$ and $r > 0$. Moreover, $\bigl(\widehat{\mathbb{C}},\, d_{\widehat{\mathbb{C}}},\, \mathbb{Q}^2\bigr)$ is a computable metric space, where $\mathbb{Q}^2$ is defined by $\{a + bi : a,\, b \in \mathbb{Q}\}$, and $\widehat{\mathbb{C}}$ is recursively compact. The *spherical derivative* $f^{\#}$ of a holomorphic function $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by

$$f^{\#}(z) := \lim_{w \to z} \frac{d_{\widehat{\mathbb{C}}}(f(w), f(z))}{d_{\widehat{\mathbb{C}}}(w, z)} = \frac{1+|z|^2}{1+|f(z)|^2} \cdot |f'(z)| \tag{5.1}$$

for each $z \in \widehat{\mathbb{C}}$. If z or $f(z)$ equals to ∞, then the last expression of (5.1) has to be understood as a suitable limit.

We identify $\widehat{\mathbb{C}}$ with the unit sphere in $\mathbb{R}^3$ via stereographic projection. The *chordal metric* σ on $\widehat{\mathbb{C}}$ is the metric that corresponds to the Euclidean metric in $\mathbb{R}^3$ under this identification. More precisely, we have

$$\sigma(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \quad \text{for each } z,\, w \in \mathbb{C}, \tag{5.2}$$

and $\sigma(\infty, z) = \sigma(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$ for each $z \in \mathbb{C}$.

The first theorem is a classical result (see for example, [Min06, Theorem A.7]).

Theorem 5.1 (Koebe–Bieberbach Quarter Theorem)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$. If f is a univalent function on $B(z_0, r)$, then $B\bigl(f(z_0), \frac{|f'(z_0)|r}{4}\bigr) \subseteq f(B(z_0, r))$.*

The *Fatou set* $\mathcal{F}_f$ of f is defined to be the set of points $z \in \widehat{\mathbb{C}}$ such that there exists an open neighborhood $U(z)$ on which the family $\{f^n|_{U(z)}\}_{n \in \mathbb{N}}$ is equicontinuous with respect to the spherical metric $d_{\widehat{\mathbb{C}}}$. The *Julia set* $\mathcal{J}_f$ is the complement of the Fatou set $\mathcal{F}_f$. Any holomorphic map $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere can be expressed as a rational function, that is, as the quotient $f(z) = p(z)/q(z)$ of two polynomials. Here we may assume that $p(z)$ and $q(z)$ have no common roots. The *degree* d of the rational map f is defined as the maximum of the degrees of p and q.

Proposition 5.2 ([Min06, Corollary 4.13])**.** *If $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map whose degree $d \geq 2$, then for each $z_0 \in \mathcal{J}_f$, the set $\bigcup_{n \in \mathbb{N}} f^{-n}(z_0)$ of all iterated preimages of z_0 is dense in $\mathcal{J}_f$.*

The transformation we consider in this section is a hyperbolic rational map, which is defined as follow (see for example, [Min06, Theorem 19.1]).

Definition 5.3. A rational function f of degree $d \geq 2$ is *hyperbolic* if the closure of the set of post-critical points of f is disjoint from $\mathcal{J}_f$.

For hyperbolic rational maps, we have the following two results.

Lemma 5.4 ([CG93, Lemma 2.1]). *Assume that* $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is a hyperbolic rational map whose degree is bigger than one with* $\infty \notin \mathcal{J}_f$. *Then there exist two constants* $a > 0$ *and* $A > 1$ *which satisfy that* $|(f^n)'(z)| \geq aA^n$ *for each* $n \in \mathbb{N}$ *and* $z \in \mathcal{J}_f$.

Lemma 5.5 ([CG93, Theorem 2.3]). *If* f *is a hyperbolic rational map whose degree is bigger than one, then* $\mathcal{J}_f$ *is of zero area.*

The following lemma is due independently to Braverman [Br04] (in the case of polynomials) and Rettinger [Re04] (in the case of rational maps).

Lemma 5.6. *If* f *is a computable hyperbolic rational map whose degree is bigger than one, then the distance function* $d_{\widehat{\mathbb{C}}}(\mathcal{J}_f, x)$ *is computable.*

Proposition 5.7. *Assume that* $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is a computable hyperbolic rational map of degree* $d \geq 2$, *and* w *is a repelling periodic point of* f. *Then the following statements hold:*

(i) *the set* $\bigcup_{n\in\mathbb{N}} f^{-n}(w)$ *is a uniformly computable sequence of points in the computable metric space* $\big(\widehat{\mathbb{C}},\, d_{\widehat{\mathbb{C}}},\, \mathbb{Q}^2\big)$;

(ii) $\big(\mathcal{J}_f,\, d_{\widehat{\mathbb{C}}},\, \bigcup_{n\in\mathbb{N}} f^{-n}(w)\big)$ *is a computable metric space, and the Julia set* $\mathcal{J}_f$ *is recursively compact in the computable metric space* $\big(\mathcal{J}_f,\, d_{\widehat{\mathbb{C}}},\, \bigcup_{n\in\mathbb{N}} f^{-n}(w)\big)$.

Proof. (i) We use a standard root-finding algorithm (see for example, [BBY10, Appendix A]) to compute the periodic points of f and decide whether these periodic points are repelling until we find a repelling periodic point. Hence, by the same root-finding algorithm, $\bigcup_{n\in\mathbb{N}} f^{-n}(w)$ is a uniformly computable sequence of points.

(ii) By Proposition 5.2 and statement (i), $\big(\mathcal{J}_f,\, d_{\widehat{\mathbb{C}}},\, \bigcup_{n\in\mathbb{N}} f^{-n}(w)\big)$ is a computable metric space. Hence, to demonstrate statement (ii), by Proposition 4 in [GHR10], it suffices to prove that $\big(\mathcal{J}_f,\, d_{\widehat{\mathbb{C}}},\, \bigcup_{n\in\mathbb{N}} f^{-n}(w)\big)$ is recursively precompact.

Fix an integer n. Since $\big(\widehat{\mathbb{C}},\, d_{\widehat{\mathbb{C}}},\, \mathbb{Q}^2\big)$ is recursively precompact, we can compute a 2^{-n-1}-net N''_n of $\widehat{\mathbb{C}}$. By Lemma 5.6, we can compute a 2^{-n-1}-net $N'_n \subseteq N''_n$ of $\mathcal{J}_f$ satisfying that $\mathcal{J}_f \cap B(y, 2^{-n-1}) \neq \emptyset$ for each $y \in N'_n$. By Proposition 5.2 and statement (i), we can compute a point $t(y) \in B(y, 2^{-n-1}) \cap \big(\bigcup_{n\in\mathbb{N}} f^{-n}(w)\big)$ for each $y \in N'_n$. Hence, the set $\{t(y) : y \in N'_n\} \subseteq \mathcal{J}_f$ is a 2^{-n}-net of the Julia set $\mathcal{J}_f$. This completes the proof. $\square$

Proposition 5.8. *Let* $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a hyperbolic rational map of degree* $d \geq 2$ *with* $\infty \notin \mathcal{J}_f$. *Define* $C_f \coloneqq f(\{z \in \widehat{\mathbb{C}} : f'(z) = 0\})$. *Assume that there exist constants* V_1, V_2, V_3, *and* V_4 *satisfying the following properties for each pair* $x, y \in \mathcal{J}_f$:

(i) $B(x, V_1)$ *is disjoint from* C_f;

(ii) $8/V_4 < V_2 \leq |f'(x)| \leq V_3$;

(iii) $d_{\widehat{\mathbb{C}}}(x, y) \geq V_4|x - y|$.

Then the following statements hold:

(i) f *is injective on* $B(x, l)$ *for each* $x \in \mathcal{J}_f$, *where* $l \coloneqq \frac{V_1}{4V_2}$.

(ii) f *is distance-expanding on* $\mathcal{J}_f$ *with respect to the spherical metric with the constants* $\eta \coloneqq V_4 l$, $\lambda \coloneqq V_2V_4/8$, *and* $\xi \coloneqq V_2 l/8$, *where* l *is from statement (i).*

Proof. (i) Given an arbitrary point $x \in \mathcal{J}_f$. By property (i), one can see that there exists no critical value (an image of a critical point under f) in $B(f(x), V_1)$. Hence, we can define a conformal inverse $f_x^{-1}\colon B(f(x), V_1) \to \widehat{\mathbb{C}}$ in the sense of $f(f_x^{-1}(y)) = y$ for each $y \in B(f(x), V_1)$ and $f_x^{-1}(f(x)) = x$. Then by Theorem 5.1, we have $B(x, |(f_x^{-1})'(f(x))|V_1/4) \subseteq f_x^{-1}(B(f(x), V_1))$. By Chain Rule and Property (ii), $|(f_x^{-1})'(f(x))| = 1/|f'(x)| \geq V_3^{-1}$. Thus one can see that $l \coloneqq \frac{V_1}{4V_3} \leq \frac{|(f_x^{-1})'(f(x))|V_1}{4}$. Therefore f is injective on $B(x, l)$.

(ii) Fix an arbitrary pair of x, $y \in \mathcal{J}_f$ with $d_{\widehat{\mathbb{C}}}(x, y) \leq \eta$. First, we demonstrate that $d_{\widehat{\mathbb{C}}}(f(x), f(y)) \geq \lambda d_{\widehat{\mathbb{C}}}(x, y)$. By property (iii), we obtain that $|x - y| \leq \eta/V_4 = l$. Hence, by statement (i), f is injective on $B(x, |x - y|)$. By Theorem 5.1, we have

$$B\biggl(f(x), \frac{|x - y| \cdot |f'(x)|}{4}\biggr) \subseteq f(B(x, |x - y|)).$$

Noting that $f(y) \in \partial f(B(x, |x-y|))$, one can see that $|f(x) - f(y)| \geq \frac{|x-y|\cdot|f'(x)|}{4} \geq \frac{V_2}{4}|x - y|$. Since $\mathcal{J}_f$ is f-invariant, then $f(x)$, $f(y) \in \mathcal{J}_f$. By property (iii), we have

$$d_{\widehat{\mathbb{C}}}(f(x), f(y)) \geq V_4|f(x) - f(y)| \geq \frac{V_2V_4}{4}|x - y| \geq \frac{V_2V_4}{8} d_{\widehat{\mathbb{C}}}(x, y) = \lambda d_{\widehat{\mathbb{C}}}(x, y).$$

Then we verify that $B_{d_{\widehat{\mathbb{C}}}}(f(x), \xi) \subseteq f(B_{d_{\widehat{\mathbb{C}}}}(x, \eta))$. To see this, since $V_4 \leq \frac{d_{\widehat{\mathbb{C}}}(x,y)}{|x-y|} \leq 2$, it suffices to check that $B(f(x), 2\xi) \subseteq f(B(x, l))$. By statement (i), it follows from Theorem 5.1 that

$$B(f(x), 2\xi) = B\biggl(f(x), \frac{V_2 l}{4}\biggr) \subseteq B\biggl(f(x), \frac{|f'(x)|l}{4}\biggr) \subseteq f(B(x, l)).$$

Therefore we complete the proof of statement (ii). □

Proposition 5.9. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a hyperbolic rational map of degree $d \geq 2$ with $\infty \notin \mathcal{J}_f$. Set $\{x_k : 1 \leq k \leq m\} \coloneqq f^{-1}(\infty) \cup \{\infty\}$, and r is a constant satisfying that $\mathcal{J}_f \subseteq K \coloneqq \bigcap_{k=1}^m B_{d_{\widehat{\mathbb{C}}}}(x_k, r)^c$ and $B_{d_{\widehat{\mathbb{C}}}}(x_i, r) \cap B_{d_{\widehat{\mathbb{C}}}}(x_j, r) = \emptyset$ for all $1 \leq i < j \leq m$.*

Assume that constants D_1, D_2, C_1, C_2, $C > 0$ satisfy the following conditions:

1. *$|z| \leq D_1$ and $|f(z)| \leq D_2$ for each $z \in K$;*
2. *$|f'(z)| \leq C_1$ and $|f''(z)| \leq C_2$ for each $z \in K$;*
3. *$|f^{\#}(z)| \geq C$ for each $z \in \mathcal{J}_f$.*

Then the following statements hold:

(i) *$d_K(x, y) \leq \pi d_{\widehat{\mathbb{C}}}(x, y)$ for each x, $y \in K$, where d_K is given by*

$$d_K(x, y) \coloneqq \inf\biggl\{\int_\gamma \mathrm{d}s : \gamma(0) = x,\ \gamma(1) = y,\ \text{and } \gamma\colon [0, 1] \to K \text{ is continuous}\biggr\};$$

(ii) *If $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map, then $|g|$ is $\bigl(A(1 + D_1^2)\pi/2\bigr)$-Lipschitz continuous with respect to $d_{\widehat{\mathbb{C}}}$ on K, where $A \coloneqq \sup_{z\in K}|g'(z)|$;*

(iii) *$\varphi_f(z)$ is (BC^{-1})-Lipschitz continuous on $\mathcal{J}_f$ with respect to $d_{\widehat{\mathbb{C}}}$, where $\varphi_f(z) \coloneqq \log\bigl(f^{\#}(z)\bigr)$ for each $z \in \mathcal{J}_f$, where $B \coloneqq (1+D_1^2)\bigl(C_2(1+D_1^2)+D_2C_1^2(1+D_1^2)+C_1D_1\bigr)\pi$.*

Proof. (i) By the definitions of r and K, one can see that K is path-connected. Hence, d_K is well-defined. Without loss of generality, assume that x, $y \in \partial B_{d_{\widehat{\mathbb{C}}}}(x_i, r)$ for some $1 \leq i \leq m$. Then for each pair of x, $y \in K$, we have

$$d_K(x, y) \leq \pi\sigma(x, y) \leq \pi d_{\widehat{\mathbb{C}}}(x, y).$$

(ii) Fix an arbitrary pair of $x,\, y \in K$. Noting that K is closed, then there exists a continuous function $\gamma\colon [0,1]$ with $\gamma(0) = x$, $\gamma(1) = y$, and $d_K(x,y) = \int_\gamma \mathrm{d}s$. Hence, for each pair of $x,\, y \in K$, we have

$$|g(x)-g(y)| = \left|\int_\gamma g'(z)\,\mathrm{d}z\right| \leq \int_\gamma \frac{A(1+|z|^2)}{2}\,\mathrm{d}s(z) \leq \frac{A(1+D_1^2)d_K(x,y)}{2} \leq \frac{A(1+D_1^2)\pi d_{\widehat{\mathbb{C}}}(x,y)}{2}.$$

(iii) Define $u(x) = 1+|x|^2$, $v(x) \coloneqq \frac{1}{1+|f(x)|^2}$, and $w(x) \coloneqq |f'(x)|$ for each $x \in K$. Then we establish that u, v, and w are all Lipschitz continuous functions, respectively.

By statement (i), conditions (1) and (2), we have

$$|u(x)-u(y)| = (|x|+|y|)\,||x|-|y|| \leq 2D_1|x-y| \leq D_1(1+D_1^2)\pi d_{\widehat{\mathbb{C}}}(x,y), \tag{5.3}$$

$$|v(x)-v(y)| = \frac{||f(x)|^2-|f(y)|^2|}{(1+|f(x)|^2)(1+|f(y)|^2)} \leq 2D_2||f(x)|-|f(y)|| \leq D_2C_1(1+D_1^2)\pi d_{\widehat{\mathbb{C}}}(x,y), \tag{5.4}$$

and

$$|w(x)-w(y)| \leq C_2(1+D_1^2)\pi d_{\widehat{\mathbb{C}}}(x,y) \tag{5.5}$$

for each pair of $x,\, y \in K$. By the definition of K, (5.1) is true for each $z \in K$. Additionally, by (5.3), (5.4), and (5.5), we have

$$\begin{aligned} &\left|f^{\#}(x)-f^{\#}(y)\right| \\ &\quad\leq |u(x)v(x)||w(x)-w(y)| + |u(x)w(y)||v(x)-v(y)| + |v(y)w(y)||u(x)-u(y)| \\ &\quad\leq \big(C_2(1+D_1^2)^2 + D_2C_1^2(1+D_1^2)^2 + C_1D_1(1+D_1^2)\big)\pi d_{\widehat{\mathbb{C}}}(x,y) \\ &\quad= Bd_{\widehat{\mathbb{C}}}(x,y) \end{aligned} \tag{5.6}$$

for each pair of $x,\, y \in K$. Noting that $\log x < x-1$ for each $x > 1$, it follow from condition (3) that

$$|\varphi_f(x)-\varphi_f(y)| = \left|\log\left(\frac{|f^{\#}(x)|}{|f^{\#}(y)|}\right)\right| \leq \frac{|f^{\#}(x)-f^{\#}(y)|}{\min\{|f^{\#}(x)|,\, |f^{\#}(y)|\}} \leq BC^{-1}d_{\widehat{\mathbb{C}}}(x,y)$$

for each pair of $x,\, y \in \mathcal{J}_f$. □

Now we are ready to establish Theorem C.

Proof of Theorem C. Since f is a hyperbolic rational map, it follows from Lemma 5.5 that there exists a computable point u in $\mathcal{J}_f^c$. By Lemma 5.6, we can compute a point $u \in \mathcal{J}_f^c$. Define a new function g by

$$g(z) \coloneqq \big(U \circ f \circ U^{-1}\big)(z) = 1/\big(f\big(u+z^{-1}\big)-u\big)$$

for each $z \in \widehat{\mathbb{C}}$. It follows from the computability of f and u that g is a computable hyperbolic rational map with $\infty \notin \mathcal{J}_g$.

First, we compute constants n, η, λ, and ξ which satisfy that g^n is distance-expanding on $\mathcal{J}_g$ with respect to the spherical metric with η, λ, and ξ. Indeed, by Lemma 5.6 and [BRY12, Proposition 2.13], we can compute a constant $r_0 > 0$ with $|z| \leq r_0$ for each $z \in \mathcal{J}_g$. Then by (5.2), one can conclude that

$$d_{\widehat{\mathbb{C}}}(x,y) \geq \sigma(x,y) \geq \frac{2|x-y|}{1+r_0^2}$$

for each pair of $x,\, y \in \mathcal{J}_g$. By Lemma 5.4, there exist two constants $n \in \mathbb{N}$ and $V_2 > 4(1+r_0^2)$ such that $(g^n)'(z) \geq V_2$ for each $z \in \mathcal{J}_g$. Hence, by [BRY12, Proposition 2.13], we can compute such pair of constants n, V_2. By [BRY12, Proposition 2.13], we can compute two constants V_1, $V_3 > 0$ such that properties (i) and (iii) in Proposition 5.8 is correct when $f = g^n$. Note that, by the definition of Julia sets, $\mathcal{J}_{g^n} = \mathcal{J}_g$. Hence, we can use Proposition 5.8 for the rational map $f = g^n$ and the constants V_1, V_2, V_3, and $V_4 = \frac{2}{1+r_0^2}$ to compute the constants η, λ, and ξ with which g^n is distance-expanding on $\mathcal{J}_g$ with respect to the spherical metric.

Next, we compute the Hölder constant and exponent of φ_{g^n}. Since g^n is computable rational map, we can compute $(g^n)^{-1}(\infty) \cup \{\infty\}$, and we write it as $\{x_k : 1 \leq k \leq m\}$. Then by Lemma 5.6, we can compute $r > 0$ such that $\mathcal{J}_g \subseteq K := \bigcap_{k=1}^m B_{d_{\widehat{\mathbb{C}}}}(x_k, r)^c$ and $B_{d_{\widehat{\mathbb{C}}}}(x_i, r) \cap B_{d_{\widehat{\mathbb{C}}}}(x_j, r) = \emptyset$ for all $1 \leq i < j \leq m$. Noting that, by Proposition 3.13, K is recursively compact, then it follows from [BRY12, Proposition 2.13], we can compute five constants D_1, D_2, C_1, $C_2 > 0$ such that conditions (1) and (2) is correct when $f = g^n$. By statement (ii) in Proposition 5.7, $\mathcal{J}_g$ is a recursively compact set. Hence, by [BRY12, Proposition 2.13], we can compute a constant $C > 0$ such that the condition (3) is correct when $f = g^n$. Therefore we can use Proposition 5.9 for the rational map $f = g^n$ and the constants r, D_1, D_2, C_1, C_2, and C to compute the constants L such that φ_{g^n} is L-Lipschitz on $\mathcal{J}_g$ with respect to the spherical metric.

Finally, we return to the proof of main statement. In deed, by [CG93, Theorem 3.2], $g^n|_{\mathcal{J}_g}$ is topologically exact. Additionally, by Propositions 5.7, 5.8 and 5.9, one can see that the septet $(\widehat{\mathbb{C}}, \mathcal{J}_g, d_{\widehat{\mathbb{C}}}, \mathbb{Q}^2, \bigcup_{n\in\mathbb{N}} g^{-n}(w), t\varphi_{g^n}, g^n)$ satisfies the Assumption B in Section 1, where w is a repelling periodic point of g. Moreover, by the above analysis, we can compute five constants mentioned in (iii) and (iv) in Theorem 1.2 for g^n and $t\varphi_{g^n}$, and statement (ii) in Proposition 5.7 gives the algorithm mentioned in (i) in Theorem 1.2. Therefore, by Theorem 1.2, the equilibrium state of g^n for $t\varphi_{g^n}$ is computable.

By [PU10, Theorem 2.4.6 (a) & Lemma 3.2.7], one can conclude that an equilibrium state of g for $t\varphi_g$ is simultaneously an equilibrium state of g^n for $tS_n\varphi_g$. Moreover, by (5.1), it follows from $\infty \notin \mathcal{J}_g$ that $S_n\varphi_g = \varphi_{g^n}$. Hence, the equilibrium state of g for $t\varphi_g$ is computable. Note that the spherical metric is invariant under the Möbius transformation U. Therefore, the equilibrium state of f for $t\varphi_f$ is also computable. □

6. Non-uniqueness of equilibrium states

In this section, we establish Theorem E and then apply it to prove Theorem D.

We first recall a classical result in Computable Analysis as follows.

Lemma 6.1 ([GHR10, Proposition 7]). *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact, and $\mu \in \mathcal{P}(X)$ is computable. Then there exists at least one computable point in the support of μ.*

Recall the non-wandering set $\Omega(T)$ defined in (2.1).

Lemma 6.2. *Let (X, ρ) be a compact metric space, and $T\colon X \to X$ be a continuous transformation of X. Then* $\operatorname{supp}(\mu) \subseteq \Omega(T)$ *for each $\mu \in \mathcal{M}_T(X)$.*

Proof. Consider $\mu \in \mathcal{M}_T(X)$. By [Wa82, Theorem 5.6 (i)], $\Omega(T)$ is closed. Moreover, it follows from [Wa82, Theorem 6.15 (i)] that $\mu(\Omega(T)) = 1$. Therefore $\operatorname{supp}(\mu) \subseteq \Omega(T)$. □

Theorem 6.3. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X a recursively compact set. Assume that $T\colon X \to X$ is a computable transformation of X satisfying the following conditions:*

(1) *The measure-theoretic entropy function* $\mathcal{H}\colon \mathcal{M}_T(X) \to \mathbb{R}$ *given by* $\mathcal{H}(\mu) := h_\mu(T)$, *for each* $\mu \in \mathcal{M}_T(X)$, *is computable.*

(2) *There exists no computable point in* $\bigcup_{\mu\in\mathcal{M}_T(X)} \operatorname{supp}(\mu)$.

Then the following statements hold:

(i) *Every* T*-invariant Borel probability measure is non-computable.*

(ii) *For each continuous potential* $\varphi\colon X \to \mathbb{R}$, *there exists at least one equilibrium state of* T *for* φ.

(iii) *For each computable potential* $\varphi\colon X \to \mathbb{R}$,

  (a) *the topological pressure* $P(T, \varphi)$ *is computable and*

  (b) *the set of equilibrium states of* T *for* φ *is a recursively compact set.*

Proof. First, we prove statement (i) by contradiction and suppose that μ is a computable T-invariant Borel probability measure. Then by condition (2), $\operatorname{supp}(\mu)$ contains no computable point, which would contradict with Lemma 6.1. Hence, there exists no computable T-invariant Borel probability measure, verifying statement (i).

Moreover, by [PU10, Proposition 3.5.3], for $\varphi \in C(X)$, it follows from condition (1) that there exists at least one equilibrium state of T. Thus statement (ii) holds.

Next, we turn to establish statement (iii)(a). By Corollary 3.18, the integral function $\Phi\colon \mathcal{P}(X) \to \mathbb{R}$ given by $\Phi(\mu) := \int \varphi \,\mathrm{d}\mu$, $\mu \in \mathcal{P}(X)$, is computable. Then by condition (1), the measure-theoretic pressure function $\mathcal{H}(\mu) + \Phi(\mu)$ is computable. By Theorem 2.3, $P(T, \varphi) = \sup\{\mathcal{H}(\mu) + \Phi(\mu) : \mu \in \mathcal{M}_T(X)\}$. Hence, by [BRY12, Proposition 2.13] and Lemma 4.12, $P(T, \varphi)$ is computable.

Finally, we demonstrate statement (iii)(b). Since $P(T, \varphi)$ is computable, the set of equilibrium states for T and φ, the preimage of $P(T, \varphi)$ under $\mathcal{H}(\mu) + \Phi(\mu)$, is recursively compact. □

Now we are ready to prove Theorem E.

Proof of Theorem E. We prove Theorem E by contradiction and suppose that μ is the unique equilibrium state of T for φ. Then by Proposition 3.13 (i), it follows from Theorem 6.3 (iii)(b) that μ is computable, which would contradict with Theorem 6.3 (i). Moreover, by Theorem 6.3 (ii), there exists at least one equilibrium state of T for φ. Therefore, there are at least two equilibrium states of T for φ. □

As an immediate consequence, we establish Theorem D.

Proof of Theorem D. Assume that $T\colon X \to X$ is a computable transformation with zero topological entropy that satisfies that there exists no computable point in $\Omega(T)$. By Theorem E, it suffices to prove that T satisfies the two conditions in Theorem E. For condition (i) in Theorem E, it is easy to see that $\mathcal{H}(\mu)$ is always zero, thus, computable. Next, we demonstrate that T satisfies condition (ii) in Theorem E. By Lemma 6.2, we have $\bigcup_{\mu\in\mathcal{M}_T(X)} \operatorname{supp}(\mu) \subseteq \Omega(T)$. Therefore, as $\Omega(T)$ contains no computable point, T satisfies condition (ii) in Theorem E. □

7. Counterexamples

7.1. **Proof of Theorem F.** In this subsection, we use the computable homeomorphism $T\colon \mathbb{S}^1 \to \mathbb{S}^1$ constructed in [GHR10, Chapter 4], which satisfies the hypotheses in Theorem D and the conditions in Theorem 6.3. Then for each potential $\varphi \in C(\mathbb{S}^1)$, we demonstrate that T satisfies the statements in Theorem F.

Proof of Theorem F. Recall that T constructed in [GHR10, Chapter 4] is a computable homeomorphism of the unit circle $\mathbb{S}^1$ whose non-wandering set $\Omega(T)$ contains no computable points.

By Theorem 6.3, it suffices to verify that T satisfies conditions (1) and (2) in Theorem 6.3. Since T is a homeomorphism, it follows from [Wa82, Theorem 7.14] that the topological entropy of T is zero, which implies that the entropy function $\mathcal{H}$ of T is zero, and thus, computable. So T satisfies condition (1) in Theorem 6.3. By Lemma 6.2, we have $\bigcup_{\mu\in\mathcal{M}_T(X)}\operatorname{supp}(\mu)\subseteq\Omega(T)$. Hence, as $\Omega(T)$ contains no computable point, T satisfies condition (2) in Theorem 6.3. ☐

7.2. **Asymptotic *h*-expansive.** In this subsection, we introduce additional relevant notions and results from ergodic theory. We use them to construct computable systems with arbitrarily high topological entropy whose equilibrium states are all non-computable. We start by recalling the following definition from [Dow11, Remark 6.1.7].

Definition 7.1. A sequence of open covers $\{\xi_i\}_{i\in\mathbb{N}_0}$ of a compact metric space X is a *refining sequence of open covers* of X if the following conditions are satisfied:

(i) ξ_{i+1} is a refinement of ξ_i for each $i\in\mathbb{N}_0$.
(ii) For each open cover η of X, there exists $j\in\mathbb{N}$ such that ξ_i is a refinement of η for each $i\geq j$.

By the Lebesgue Covering Lemma, it is clear that for a compact metric space, refining sequences of open covers always exist.

The topological tail entropy was first introduced by M. Misiurewicz under the name "topological conditional entropy" [Mis73, Mis76]. We use the terminology in [Dow11] (see [Dow11, Remark 6.3.18] for reference).

Definition 7.2. Let (X,d) be a compact metric space and $g\colon X\to X$ a continuous map. The *topological conditional entropy* $h(g|\lambda)$ of g given λ, for some open cover λ, is defined as

$$h(g|\lambda):=\lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{n}H\bigl((\xi_l)_g^n\big|\lambda_g^n\bigr), \tag{7.1}$$

where $\{\xi_l\}_{l\in\mathbb{N}_0}$ is an arbitrary refining sequence of open covers, and for each pair of open covers ξ and η,

$$H(\xi|\eta):=\log\Bigl(\max_{A\in\eta}\Bigl\{\min\Bigl\{\operatorname{card}\xi_A:\xi_A\subseteq\xi,\ A\subseteq\bigcup\xi_A\Bigr\}\Bigr\}\Bigr) \tag{7.2}$$

is the logarithm of the minimal number of sets from ξ sufficient to cover any set in η.

The *topological tail entropy* $h^*(g)$ of g is defined by

$$h^*(g):=\lim_{m\to+\infty}\lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{n}H\bigl((\xi_l)_g^n\big|(\eta_m)_g^n\bigr),$$

where $\{\xi_l\}_{l\in\mathbb{N}_0}$ and $\{\eta_m\}_{m\in\mathbb{N}_0}$ are two arbitrary refining sequences of open covers, and H is as defined in (7.2).

The concept of h-expansiveness was introduced by R. Bowen in [Bo72]. We use the formulation in [Mis76] (see also [Dow11]).

Definition 7.3 (h-expansive)**.** A continuous map $g\colon X\to X$ on a compact metric space (X,d) is called *h-expansive* if there exists a finite open cover λ of X such that $h(g|\lambda)=0$.

Remarks 7.4. As an equivalent definition of h-expansiveness, the map $g\colon X \to X$ is called *h-expansive* if there exists $\epsilon > 0$ such that

$$h_{\mathrm{top}}\big(g|_{\Phi_{\epsilon,g}(x)}\big) = 0$$

for each $x \in X$, where $\Phi_{\epsilon,g}(x) \coloneqq \{y \in X : d(g^n(x), g^n(y)) \leq \epsilon \text{ for each } n \in \mathbb{N}_0\}$.

A weaker property was introduced by M. Misiurewicz in [Mis73] (see also [Mis76, Dow11]).

Definition 7.5 (Asymptotic h-expansive)**.** We say that a continuous map $g\colon X \to X$ on a compact metric space X is *asymptotically h-expansive* if $h^*(g) = 0$.

Remarks 7.6. The topological entropy of g is $h_{\mathrm{top}}(g) = h(g|\{X\})$, where $\{X\}$ is the open cover of X consisting of only one open set X. It also follows from Definition 7.1, for open covers ι and η of X, one can see $h(g|\iota) \leq h(g|\eta)$ if ι is a refinement of η. Hence, we have $h^*(g) \leq h_{\mathrm{top}}(g)$.

We shall say a pair (X, f) is a *cascade* if $f\colon X \to X$ is a continuous transformation of the non-empty compact Hausdorff space X. Under the above notations, we have the following two results.

Proposition 7.7 ([Mis76, Theorem 3.2])**.** *Let $(X_1, f_1), (X_2, f_2)$ be two cascades then*

$$h^*(f_1 \times f_2) = h^*(f_1) + h^*(f_2),$$

where $f_1 \times f_2\colon X_1 \times X_2 \to X_1 \times X_2$ is given by

$$(f_1 \times f_2)(x_1, x_2) \coloneqq (f_1(x_2), f_2(x_2)) \qquad \text{for all } x_1 \in X_1 \text{ and } x_2 \in X_2. \tag{7.3}$$

Proposition 7.8 ([Mis76, Corollary 8.1])**.** *Let (X, f) be a cascade. If f is asymptotically h-expansive, then for each $\varphi \in C(X)$, there exists at least one equilibrium state of the transformation f.*

Now we are ready to prove Proposition 7.9, a precise version of Theorem G. Here we identity $\mathbb{S}$ with $\mathbb{R}/[0,1]$.

Proposition 7.9. *Assume $d \in \mathbb{N}$ and let $T_d\colon \mathbb{S}^1 \to \mathbb{S}^1$ be given by $T_d(x) \coloneqq dx \pmod 1$ for $x \in \mathbb{S}$. Let the transformation $\widehat{T} \coloneqq T \times T_d\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ be defined as in (7.3) and $\widehat{\varphi}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ be a continuous potential. Then statements (i), (ii), and (iii) in Theorem G hold for $\widehat{T}$ and $\widehat{\varphi}$, and $h_{\mathrm{top}}\big(\widehat{T}\big) = \log d$.*

Proof. First, we demonstrate that $\widehat{T}$ satisfies statement (i) in Theorem G. Note that for the transformation T_d and sufficiently small $\epsilon > 0$, the set $\Phi_{\epsilon,T_d}(x) = \{x\}$ for each $x \in \mathbb{S}^1$. Then by Remark 7.4, T_d is h-expansive, and thus $h^*(T_d) = 0$. Noting that $h_{\mathrm{top}}(T) = 0$, it follows from Remark 7.6 that $h^*(T) \leq h_{\mathrm{top}}(T) = 0$, hence, is equal to 0. Hence by Proposition 7.7, $h^*\big(\widehat{T}\big) = h^*(T) + h^*(T_d) = 0$, which implies that $\widehat{T}$ is asymptotically h-expansive. By Proposition 7.8, for each continuous potential $\widehat{\varphi}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$, there exists at least one equilibrium state of $\widehat{T}$.

Next, we argue that there exists no computable $\widehat{T}$-invariant Borel probability measure by contradiction and assume that $\widehat{\mu} \in \mathcal{M}_{\widehat{T}}(\mathbb{S}^1 \times \mathbb{S}^1)$ is computable. Then μ given by $\mu(A) \coloneqq \widehat{\mu}(A \times \mathbb{S}^1)$ for each Borel set $A \subseteq \mathbb{S}^1$ is a computable T-invariant probability measure. This is impossible by statement (ii) in Theorem F. Thus $\widehat{T}$ satisfies statement (ii) in Theorem G.

Finally, we turn to demonstrate that $\widehat{T}$ satisfies statement (iii) in Theorem G. Assume that $\widehat{\varphi}(x, y) = \varphi_1(x) + \varphi_2(y)$, $x, y \in \mathbb{S}^1$, where $\varphi_1, \varphi_2\colon \mathbb{S}^1 \to \mathbb{R}$ are computable functions with φ_2 being Hölder continuous. By [Wa82, Theorem 9.8 (v)], we obtain that $h_{\text{top}}(\widehat{T}) = h_{\text{top}}(T) + h_{\text{top}}(T_d) = \log d$, and $P(\widehat{T}, \widehat{\varphi}) = P(T, \varphi_1) + P(T_d, \varphi_2)$. By Lemma 4.10, $P(T_d, \varphi_2)$ is computable. By statement (iii) in Theorem F, $P(T, \varphi_1)$ is computable. Hence, Theorem G (iii)(c) holds. Moreover, as equilibrium states are invariant, Theorem G (iii)(b) follows immediately from Theorem G (ii).

For the proof of Theorem G (iii)(a), we establish a claim as preparation.

Claim. Let μ_1 be an equilibrium states of T for φ_1, and μ_2 be an equilibrium state of T_d for φ_2. Define μ by $\mu(A \times B) := \mu_1(A)\mu_2(B)$ for each pair of Borel sets $A, B \in \mathbb{S}^1$. Noting that $\{A \times B : A, B \in \mathbb{S}^1\}$ is a topological basis of $\mathbb{S}^1 \times \mathbb{S}^1$, we can extend μ to be a Borel probability measure. Then μ is an equilibrium state of $\widehat{T}$ for $\widehat{\varphi}$.

To see this, we verify that μ satisfies the Variational Principle for $\widehat{T}$ and $\widehat{\varphi}$. By Variational Principle, we have

$$P(T, \varphi_1) = h_{\mu_1}(T) + \int \varphi_1 \, \mathrm{d}\mu_1 \quad \text{and} \quad P(T_d, \varphi_2) = h_{\mu_2}(T_d) + \int \varphi_2 \, \mathrm{d}\mu_2.$$

Additionally, by [Wa82, Theorem 3.2], we obtain that

$$P(\widehat{T}, \widehat{\varphi}) = P(T, \varphi_1) + P(T_d, \varphi_2) = h_\mu(\widehat{T}) + \int \varphi_1 \, \mathrm{d}\mu_1 + \int \varphi_2 \, \mathrm{d}\mu_2 = h_\mu(\widehat{T}) + \int \widehat{\varphi} \, \mathrm{d}\mu.$$

Hence, μ is an equilibrium state of $\widehat{T}$ for $\widehat{\varphi}$, establishing the claim.

Now we turn to the proof of (a). By Theorem F (iii), there exist at least two equilibrium states of T for φ_1. By Theorem A, there exists a unique equilibrium state of T_d for φ_2. Therefore, by the claim, there exist at least two equilibrium states of $\widehat{T}$ for $\widehat{\varphi}$. □

References

[BJ08] Bandtlow, O. and Jenkinson, O. Explicit a priori bounds on the Ruelle eigenvalue sequence Available at `https://webspace.maths.qmul.ac.uk/o.bandtlow/publ/rates12.pdf`.

[Fro97] Froyland, G. Computer-assisted bounds for the rate of decay of correlations, *Comm. Math. Phys.,* 189 (1997), 237—257.

[Hun96] Hunt, B. Estimating invariant measures and Lyapunov exponents *Ergodic Theory Dynam. Systems*, 16 (1996), 735–749

[Fer81] Ferrero, P. Contribution à la théorie des états d'équilibre en mèchanique statistique, Theses, 1981.

[BM37] Banach, S. and Mazur, S., Sur les fonctions caluclables. *Ann. Polon. Math.* 16 (1937).

[BRY12] Binder, I. Rojas, C., and Yampolsky, M., Computable Caratheodory theory. *Adv. Math.* 265 (2012), 280–312.

[BBRY11] Binder, I., Braverman, M., Rojas, C., and Yampolsky, M., Computability of Brolin–Lyubich measure. *Comm. Math. Phys.* 308 (2011), 743–771.

[Bi57] Birkhoff, G., Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* 85 (1957), 219–227.

[BBY10] Bonnot, S., Braverman, M., and Yampolsky, M., Thurston equivalence to a rational map is decidable. *Mosc. Math. J.* 12(4) (2010), 747–763.

[Bo72] Bowen, R., Entropy-expansive maps. *Trans. Amer. Math. Soc.* 164 (1972), 323–331.

[Bo74] Bowen, R., Some systems with unique equilibrium states. *Mathematical systems theory* 8 (1974), 193–202.

[Bo75] Bowen, R., *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.*, Springer, Berlin, 1975.

[Br04] Braverman, M., Computational complexity of Euclidean sets: Hyperbolic Julia sets are Poly-Time computable. Master's thesis, University of Toronto, 2004.

[BY09] Braverman, M. and Yampolsky, M., *Computability of Julia sets*, volume 23 of *Algorithms and Computation in Mathematics*, Springer, Berlin, 2009.

[BK98] Bruin, H. and Keller, G., Equilibrium States for S-Unimodal. *Ergodic Theory Dynam. Systems* 18 (1998), 765–789.

[Bu97] Buzzi, J., Intrinsic ergodicity of smooth interval maps. *Israel J. Math.* 100 (1997), 125–161.

[Bu05] Buzzi, J., Subshifts of quasi-finite type. *Invent. Math.* 159 (2005), 369–406.

[BS03] Buzzi, J. and Sarig, O., Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. *Ergodic Theory Dynam. Systems* 23 (2003), 1383–1400.

[CG93] Carleson, L. and Gamelin, T.W., *Complex dynamics*, Springer, New York, 1993.

[CT12] Climenhaga, V., and Thompson, D., Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors. *Israel J. Math.* 192 (2012), 785–817.

[DPTUZ21] Das, T., Przytycki, F., Tiozzo, G., Urbański, M., and Zdunik, A., Thermodynamic formalism for coarse expanding dynamical systems. *Comm. Math. Phys.* 384 (2021), 165–199.

[DU91] Denker, M. and Urbanski, M. Ergodic theory of equilibrium states for rational maps. *Nonlinearity* 4(1) (1991), 103–134.

[DT21] Dobbs, N. and Todd, M., Free energy and equilibrium states for families of interval maps. Mem. Amer. Math. Soc. 286 (2023), no. 1417.

[Dob68] Dobruschin, P.L., The description of a random field by means of conditional probabilities and conditions of its regularity. *Theory Probab. Appl.* 13(2) (1968), 197–224.

[Dow11] Downarowicz, T., *Entropy in dynamical systems*, Cambridge Univ. Press, Cambridge, 2011.

[GHR10] Galatolo, S., Hoyrup, M., and Rojas, C., Dynamics and abstract computability: computing invariant measures. *Discrete Contin. Dyn. Syst.* 29(1) (2011), 193–212.

[Gr55] Grzegorczyk, A., Computable functionals. *Fund. Math.* 42 (1955), 168-–202.

[Hof75] Hofbauer, F., Examples for the Nonuniqueness of the Equilibrium State. *Trans. Amer. Math. Soc.* 228 (1977), 223–241.

[Hof79] Hofbauer, F., On intrinsic ergodicity of piecewise monotonic transformations with positive entropy. *Israel J. Math.* 34 (1979), 213–237.

[Hof81] Hofbauer, F., On intrinsic ergodicity of piecewise monotonic transformations with positive entropy II. *Israel J. Math.* 38 (1981), 107–115.

[HR09] Hoyrup, M. and Rojas, C., Computability of probability measures and Martin-Löf randomness over metric spaces. *Inform. and Comput.* 207(7) (2009), 830–847.

[IT10] Iommi, G. and Todd, M., Natural equilibrium states for multimodal maps. *Comm. Math. Phys.* 300 (2010), 65–94.

[IT15] Iommi, G. and Todd, M., Recurrence and transience for suspension flows. *Israel J. Math.* 209 (2015), 547–592.

[KH95] Katok, A. and Hasselblatt, B., *Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, Cambridge, 1995.

[Ko91] Ko, K., *Complexity theory of real functions*, Birkhäuser Boston, 1991.

[La55] Lacombe, D., Classes récursivement fermées et fonctions majorantes. *C. R. Acad. Sci. Paris* 240 (1955), 716–718.

[LRL14] Li, H. and Rivera-Letelier, L., Equilibrium states of weakly hyperbolic one-dimensional maps for Hölder potentials. *Comm. Math. Phys.* 328 (2014), 397–419.

[Li18] Li, Z., Equilibrium states for expanding Thurston maps. *Comm. Math. Phys.* 357 (2018), 811–872.

[Liv95a] Liverani, C., Decay of correlations. *Ann. of Math. (2)* 142 (1995), 239–301.

[Liv95b] Liverani, C., Decay of correlations for piecewise expanding maps. *J. Stat. Phys.* 78 (1995), 1111–1129.

[MS00] Makarov, N. and Smirnov, S. On thermodynamics of rational maps. I: negative spectrum. *Comm. Math. Phys.* 211 (2000), 705–743.

[MS03] Makarov, N. and Smirnov, S. On thermodynamics of rational maps. II: non-recurrent maps. *J. Lond. Math. Soc.* 67(2) (2003), 417–432.

[MauU03] Mauldin, D. and Urbański, M., *Graph directed Markov systems: geometry and dynamics of limit sets*, Cambridge Univ. Press, Cambridge, 2003.

[MayU10] Mayer, V. and Urbański, M., Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order. *Mem. Amer. Math. Soc.* 203 (2010), no. 954.

[Min06] Milnor, J., *Dynamics in one complex variable*, 3rd ed., Princeton Univ. Press, Princeton, 2006.

[Mis73] Misiurewicz, M., Diffeomorphisms without any measure with maximal entropy. *Bull. Acad. Pol. Sci.* 21 (1973), 903–910.

[Mis76] Misiurewicz, M., Topological conditional entropy. *Studia Math.* 55 (1976), 175–200.

[Ol03] Oliveira, K., Equilibrium states for non-uniformly expanding maps. *Ergodic Theory Dynam. Systems* 23 (2003), 1891–1905.

[Par64] Parry, W., Intrinsic Markov chains. *Trans. Amer. Math. Soc.* 112 (1964), 55–66.

[Pi11] Pinheiro, V., Expanding measures. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28 (2011), 889–939.

[Pr90] Przytycki, F., On the Perron–Frobenius–Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions. *Bol. Soc. Brasil. Mat.* 20(2) (1990), 95–125.

[PRL11] Przytycki, F. and Rivera-Letelier. L., Nice inducing schemes and the thermodynamics of rational maps. *Comm. Math. Phys.* 301 (2011), 661–707.

[PRL14] Przytycki, F. and Rivera-Letelier, L., Geometric pressure for multimodal maps of the interval. *Mem. Amer. Math. Soc.* 259 (2019), no. 1246.

[PU10] Przytycki, F. and Urbański, M., *Conformal fractals: ergodic theory methods*, Cambridge Univ. Press, Cambridge, 2010.

[RV17] Ramos, V. and Viana, M., Equilibrium states for hyperbolic potentials. *Nonlinearity* 30 (2017), 825–847.

[Re04] Rettinger, R., A fast algorithm for Julia sets of hyperbolic rational functions. *Electron. Notes Theor. Comput. Sci.* 120 (2005), 145–157.

[Ru89] Ruelle, D., The thermodynamical formalism for expanding maps. *Comm. Math. Phys.* 125 (1989), 239–262.

[Ryc89] Rychlik, M. Regularity of the metric entropy for expanding maps, *Trans. Amer. Math. Soc.* 315 (1989), 833–847.

[Si72] Sinai, Ya., Gibbs measures in ergodic theory. *Russian Math. Surveys* 27(4) (1972), 21–69.

[Tu37] Turing, A.M., On computable numbers, with an application to the Entscheidungsproblem. *Proc. Lond. Math. Soc. (3)* s2-42 (1937), 230–265.

[VV10] Varandas, P. and Viana, M., Existence, uniqueness and stability of equilibrium states for non-uniformly expanding maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27 (2010), 555–593.

[Vi97] Viana, M., *Stochastic dynamics of deterministic systems*, IMPA, 1997.

[Wa82] Walters, P., *An introduction to ergodic theory*, Springer, New York, 1982.

[Weih00] Weihrauch, K., *Computable Analysis: An Introduction*, Springer, Berlin, 2000.

[Weis70] Weiss, B., Intrinsically ergodic systems. *Bull. Amer. Math. Soc.* 76(6) (1970), 1266–1269.

[Weis73] Weiss, B., Subshifts of finite type and sofic systems. *Monatsh. Math.* 77 (1973), 462–474.

[Ya21] Yampolsky, M., Towards understanding the theoretical challenges of numerical modeling of dynamical systems. *New Zealand J. Math.* 52 (2021), 453–467.

[Yu03] Yuri, M., Thermodynamic formalism for countable to one Markov systems. *Trans. Amer. Math. Soc.* 335 (2003), 2949–2971.

[Zi96] Zinsmeister, M., *Formalisme thermodynamique et systèmes dynamiques holomorphes*, Panoramas et Synthèses 4, Société Mathématique de France, Paris, 1996.

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