NON-CONFORMING FINITE ELEMENT AND ARTIFICIAL BOUNDARY IN MULTI-ATOMIC YOUNG MEASURE APPROXIMATION FOR MICROMAGNETICS

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ABSTRACT. In this paper, a nonconforming finite element method coupled with an artificial boundary technique is developed in a multi-atomic Young measure approximation to solve the two-dimensional variational problem for the magnetization field in micromagnetics, which has an anisotropic potential energy and a non-convex constraint and thus can develop microstructures. Compared with the conforming finite element approach, which turns out to be unstable in the sense that spurious numerical oscillations can occur in the discrete macroscopic magnetization field, the stability and convergence of the nonconforming finite element method can be established. It is also proved that, for the uniaxial energy density, two-atomic young measure is sufficient to approximate the macroscopic magnetization field. The efficiency of the method is illustrated by some numerical examples.

1. INTRODUCTION

A static magnetization field $\mathbf{m}$ of ferromagnetic materials in $\Omega \subset R^n$ is characterized by the problem of minimizing the total Gibbs free energy [4]

$$E(\mathbf{m}) = \int_\Omega \varphi(\mathbf{m})dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2 dx, \qquad n = 2, 3, \tag{1.1}$$

where $\varphi(\mathbf{m})$ is the magnetocrystalline-anisotropic energy density which depends on the materials' property, $\mathbf{H}(x)$ is the applied static magnetic field, and $u_{\mathbf{m}}$

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is the potential of the stray field energy which is related to $\mathbf{m}$ by the following Maxwell's equation

$$\operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \ \text{ in } \ R^n; \quad \text{ and } u_{\mathbf{m}} \to 0 \text{ as } |x| \to \infty \ . \tag{1.2}$$

By the classical micromagnetics theory, $|\mathbf{m}|$ depends only on the temperature T, and $|\mathbf{m}| \neq 0$ when $T < T_c$ the Curie temperature. For simplicity and without loss of generality, we may assume that the material is put at some fixed temperature below T_c and the system is scaled such that $|\mathbf{m}| = 1$.

Thus, we are led to the following variational problem, with unknowns $\mathbf{m} \in \mathcal{A} := \{\mathbf{m} : \mathbf{m} \in (L^2(\Omega))^n \text{ and } |\mathbf{m}| = 1, \text{ a.e. } x \in \Omega\}$ and $u_{\mathbf{m}} \in H^1(R^n)$,

$$(\mathbf{P}) \begin{cases} \min\limits_{\mathbf{m}\in\mathcal{A}} E(\mathbf{m}) \\ s.t. \\ \operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty \ , \end{cases}$$

where $E(\mathbf{m})$ is defined by (1.1) with $\varphi \in C(S^{n-1})$ and $\mathbf{H} \in (L^2(\Omega))^n$ being given. This is a minimizing problem in which both the energy functional and the admissible set of functions are nonconvex. The minimizing sequences of such a problem would generally generate finer and finer oscillations which do not have weak limits in the admissible set [15]. Some relaxations of the problem have been developed. There are mainly two kinds of relaxations. One is the convex-hull relaxation [6, 7, 10, 11], which is a purely macroscopic model and is of the form

$$(\mathbf{RP1}) \begin{cases} \min\limits_{\mathbf{m}\in\mathcal{A}^{**}} E^{**}(\mathbf{m}) \\ s.t. \\ div(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where $\mathcal{A}^{**} = \{\mathbf{m} : \mathbf{m} \in (L^2(\Omega))^n \text{ and } |\mathbf{m}| \leq 1, \text{ a.e. } x \in \Omega\}$ and

$$E^{**}(\mathbf{m}) = \int_\Omega \hat{\varphi}^{**}(\mathbf{m})dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2 dx, \quad n = 2, 3. \tag{1.3}$$

Here $\hat{\varphi}^{**} = \min\{f : f \leq \hat{\varphi} \ \text{ and } \ f \text{ is convex}\}$ is the convex hull of $\hat{\varphi} : R^n \to R^1 \cup \{\infty\}$, which is defined by

$$\hat{\varphi}(\mathbf{m}) = \begin{cases} \varphi(\mathbf{m}), & |\mathbf{m}| = 1; \\ +\infty, & \text{otherwise.} \end{cases}$$

The other is the so-called Young measure relaxation [9, 16, 20, 22, 23, 24], which is a mesoscopic model and can provide some information on the underlying microstructure of the problem. Denote $\nu = \{\nu_x\}_{x\in\Omega}$ a family of weakly measurable probability measures supported on S^{n-1}, *i.e.*, the mapping $x \to \int_{S^{n-1}} u(A)\nu_x(dA)$ is measurable for any given $u \in C(S^{n-1})$ and $\int_{S^{n-1}} \nu_x(dA) = 1$, $\forall x \in \Omega$. Let

$$\mathcal{A}^\mu = \{\nu = \{\nu_x\}_{x\in\Omega} : \text{supp}\nu_x \subset S^{n-1}, a.e.\ x \in \Omega\}, \tag{1.4}$$

and define the relaxed energy functional by

$$E^\mu(\nu) = \int_\Omega \int_{S^{n-1}} \varphi(A)\nu_x(dA)dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2 dx. \tag{1.5}$$

Then, the Young measure relaxation of (**P**) is given by

$$(\mathbf{RP2}) \begin{cases} \min\limits_{\nu\in\mathcal{A}^\mu} E^\mu(\nu) & \\ s.t. & \\ \mathbf{m}(x) = \int_{S^{n-1}} A\nu_x(dA), & \text{a.e. } x \in \Omega, \\ div(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty. \end{cases}$$

It is easily seen that, by the Maxwell's equation, the stray-field energy $\int_{R^n} |\nabla u_{\mathbf{m}}|^2\,dx$ can be replaced by an equivalent finite integral $\int_\Omega \mathbf{m}\cdot\nabla u_{\mathbf{m}}\,dx$. Sometimes, it is convenient to rewrite the relaxed energy functionals in the following equivalent form

$$E^{**}(\mathbf{m}) = \int_\Omega \hat{\varphi}^{**}(\mathbf{m})\,dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}dx + \frac{1}{2}\int_\Omega \mathbf{m}\cdot\nabla u_{\mathbf{m}}\,dx, \tag{1.6}$$

$$E^\mu(\nu) = \int_\Omega \int_{S^{n-1}} \varphi(A)\nu_x(dA)dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}dx + \frac{1}{2}\int_\Omega \mathbf{m}\cdot\nabla u_{\mathbf{m}}\,dx. \tag{1.7}$$

In this paper, we restrict ourselves to the two-dimensional case ($n = 2$), and consider the numerical approximation of the problem (**RP2**), though many results are readily applied to the three-dimensional case ($n = 3$) as well.

Numerical methods and analysis have been studied in recent years by many researchers, see for example [6, 7] for the convex hull relaxation (**RP1**), and [16, 20] for the Young measure relaxation (**RP2**), among many others. The main idea of the latter is to reduce the original problem to the so-called active

multi-atomic young measure optimization problem, in which the key step is to identify "active" atoms. Kruzik's strategy [16], which could trace back to [5] where the authors considered the Young measure approximation for general non-convex variational problems, is to use the optimality conditions derived from a Weierstrass maximum principle to single out the active atoms from all of the mesh points. Exploiting the idea of the mesh transformation method, Li put the determination of the supports and volume fractions of a limited number of atoms into the optimization process [20]. To fully discretize the problem, these methods need to couple with a finite element approximation to the Maxwell's equation. Numerical examples showed that a direct application of a conforming finite element approximation may develop numerical oscillations [6, 20].

In this paper, we develop a numerical method for (**RP2**), which uses a multi-atomic Young measure approximation for the anisotropic energy density and a nonconforming finite element method coupled with an artificial boundary method for the approximation of the Maxwell's equation. Motivated by the results of mixed finite element methods and also by the method used by Carstensen in the numerical approximation of (**RP1**) [6], we choose the piecewise constant element for $\mathbf{m}$ and the Crouzeix-Raviart element for $u_{\mathbf{m}}$. For such a choice, we are able to prove the existence, convergence and stability of the numerical solutions to the full discrete problem. We are also able to prove, by further exploiting the relationship between the discrete problem of (**RP1**) and that of (**RP2**), that in the uniaxial case two atoms are sufficient in the numerical approximation to produce the macroscopic magnetization field. In the present paper, the proof for the stability of the method will be concentrated on the uniaxial case, which is regarded as a good platform for the numerical analysis of the relaxed models in micromagnetics [6, 7, 11], while the stability for the general case is more delicate and will be treated in a separate paper. However, we notice here that, making use of the Young measure relaxation instead of the convex-hull relaxation, our method is ready to be applied to more general cases, and with some careful handling and subtle analysis, the corresponding stability result, which is much more complicated than the uniaxial case because of possible non-uniqueness of the macroscopic magnetization field, can also be established.

The rest of the paper is organized as follows. In section 2, some analytical results on the multi-atomic Young measure approximation are presented.

In section 3, we will construct and analyze the nonconforming finite element approximation combined with an artificial boundary method for the Maxwell's equation. In section 4, we will analyze the existence, uniqueness and convergence of the numerical solutions of the full discrete problem, and prove the stability of the method for the uniaxial case. In section 5, we discuss the implementation of the algorithm and give some numerical examples.

2. Multi-atomic Young measure approximation

We start with the finite element approximation of the Young measure. It is worth noticing, as we will see below, that the multi-atomic young measure approximation can also be viewed as a finite element approximation.

Let $\Gamma_h^1(\Omega)$ be a regular triangulation of Ω with mesh size h and $\Gamma_{h'}^2(S^{n-1})$ be a regular triangulation of S^{n-1} with mesh size h', which is a finite element partition of S^{n-1} using curved simplex elements on S^{n-1}. Denote $\Gamma_{\mathbf{h}} = \Gamma_h^1(\Omega) \times \Gamma_{h'}^2(S^{n-1})$ the triangulation introduced by $\Gamma_h^1(\Omega)$ and $\Gamma_{h'}^2(S^{n-1})$ on $\Omega \times S^{n-1}$ with mesh size $\mathbf{h} = (h, h')$.

Define a projector $P_h^1 : L^1(\Omega; C(S^{n-1})) \to L^1(\Omega; C(S^{n-1}))$ by

$$[P_h^1 f](x, A) = \frac{1}{K} \int_K f(y, A) dy, \quad \text{if } x \in K \in \Gamma_h^1,$$

and define an operator $P_{h'}^2$ by

$$[P_{h'}^2 f](x, A) = \sum_{i=1}^{N_{h'}} f(x, A_i) v_i(A),$$

where A_i are the nodes of the triangulation $\Gamma_{h'}^2$, and v_i are element wise affine basis functions derived from barycentric coordinates of the finite elements in $\Gamma_{h'}^2$ (see [16, 20]), which satisfy $v_i(A_i) = 1$, $v_i(A_j) = 0$, if $i \neq j$, for all $1 \leq i, j \leq N_{h'}$, and $\sum_{i=1}^{N_{h'}} v_i(A) = 1$ for all $A \in S^{n-1}$. It is easily seen that $P_{\mathbf{h}} = P_{(h,h')} = P_h^1 P_{h'}^2 = P_{h'}^2 P_h^1$ defines a projector which provides an Ω-element-wise constant and S^{n-1}-element-wise affine approximation. If we define its adjoint operator $P_{\mathbf{h}}^* : \mathcal{A}^\mu \to \mathcal{A}^\mu$ by

$$< P_{\mathbf{h}}^* \nu, f > = < \nu, P_{\mathbf{h}} f >, \quad \forall \nu \in \mathcal{A}^\mu,$$

where $\mathcal{A}^\mu$ is the set of Young measures defined by (1.4) and

$$< \nu, f >= \int_\Omega \int_{S^{n-1}} f(x, A)\nu_x(dA)dx,$$

and denote $P^*_{\mathbf{h}}\mathcal{A}^\mu \subset \mathcal{A}^\mu$ by $\mathcal{A}^\mu_{\mathbf{h}}$, then $\mathcal{A}^\mu_{\mathbf{h}}$ is a set of the form [23]

$$\mathcal{A}^\mu_{\mathbf{h}} = \left\{ \nu^{\mathbf{h}}_x : \nu^{\mathbf{h}}|_K = \sum_{i=1}^{N_{h'}} \lambda_{K,i}\delta_{A_i},\ \lambda_{K,i} \geq 0, \sum_{i=1}^{N_{h'}} \lambda_{K,i} = 1,\ \forall K \in \Gamma^1_h \right\},$$

where δ_{A_i} is the Dirac measure supported at $A_i \in S^{n-1}$. We have [23]

$$\lim_{\mathbf{h}\to 0} \|P_{\mathbf{h}}f - f\|_{L^1(\Omega;C(S^{n-1}))} = 0, \quad \forall f \in L^1(\Omega; C(S^{n-1})),$$

and

$$\begin{aligned} &| < \nu - P^*_{\mathbf{h}}\nu, f > | = | < \nu, f - P_{\mathbf{h}}f > | \\ &\leq \|\nu\|_{L^1(\Omega;C(S^{n-1}))^*}\|f - P_{\mathbf{h}}f\|_{L^1(\Omega;C(S^{n-1}))} \to 0, \quad \text{as } \mathbf{h} \to 0, \end{aligned} \tag{2.1}$$

i.e. L^1-weak*-$\lim_{\mathbf{h}\to 0} P^*_{\mathbf{h}}\nu = \nu$ in $L^1(\Omega; C(S^{n-1}))^*$(see [16],[23]).

The finite element Young measure version of (**RP2**) is

$$(\mathbf{FERP}):\ \text{problem } (\mathbf{RP2}) \text{ with } \mathcal{A}^\mu \text{ replaced by } \mathcal{A}^\mu_{\mathbf{h}}. \tag{2.2}$$

Since the Young measure solution is typically supported at a very few atoms (the so-called active atoms), the finite element Young measure method can be terribly inefficient. Currently, there are two ways to improve the efficiency. One is to use the Weierstrass maximum principle to single out the active atoms [16]. Another is to regard the support of the atoms as a set of variables in the minimization process [20]. Here, we take the latter approach.

For a given integer $k \geq 1$, define

$$\begin{aligned} \mathcal{A}^\mu_{h,k} = \Bigg\{ &\nu^{h,k} = \{\nu^{h,k}|_K\}_{K\in\Gamma^1_h} : \nu^{h,k}|_K = \sum_{i=1}^{k} \lambda_{K,i}\delta_{A_{K,i}},\ A_{K,i} \in S^{n-1}, \\ &\lambda_{K,i} \geq 0,\ \sum_{i=1}^{k} \lambda_{K,i} = 1,\ \forall 1 \leq i \leq k \ \text{ and } \ \forall K \in \Gamma^1_h \Bigg\}. \end{aligned} \tag{2.3}$$

Then, (**FERP**) with variable nodes leads to the semi-discrete problem

$$(\mathbf{SDRP}):\ \text{problem } (\mathbf{RP2}) \text{ with } \mathcal{A}^\mu \text{ replaced by } \mathcal{A}^\mu_{h,k}. \tag{2.4}$$

Notice that we use only k atoms in $\mathcal{A}^{\mu}_{h,k}$, that is why we call it the multi-atomic Young measure method.

It is shown in [20] that the problem (**SDRP**) has a minimizer and we have

Theorem 2.1. [20] *Let $\varphi \in C(S^{n-1})$ and $H \in (L^2(\Omega))^n$, then*

$$\lim_{h\to 0, k\to\infty} \inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} E^{\mu}(\nu^{h,k}) = \inf_{\nu\in\mathcal{A}^{\mu}} E^{\mu}(\nu). \tag{2.5}$$

Especially, if the problem admits a Young measure solution supported at k_0 atoms, then we have

$$\lim_{h\to 0} \inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} \mathrm{E}^{\mu}(\nu^{h,k}) = \inf_{\nu\in\mathcal{A}^{\mu}} \mathrm{E}^{\mu}(\nu), \quad \forall k \geq k_0. \tag{2.6}$$

In general, it is known that $\inf\{k_0\} \leq n+1$ [16]. In the following we show that $\inf\{k_0\} = 2$ in the uniaxial case. The result implies that two atoms are sufficient to attain the minimum in the uniaxial case. In fact, we have

Lemma 2.1. *In the uniaxial case, that is when $\varphi(\mathbf{m}) = c_1 m_1^2 + c_2(1-m_2^2)^2$ with $c_1, c_2 > 0$ and $\mathbf{m} = (m_1, m_2)^T$, the following relation holds*

$$\inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} \mathrm{E}^{\mu}(\nu^{h,k}) = \inf_{\nu^{h,2}\in\mathcal{A}^{\mu}_{h,2}} \mathrm{E}^{\mu}(\nu^{h,2}), \quad \forall k \geq 2. \tag{2.7}$$

Proof. First, we claim that

$$\inf_{\mathbf{m}_h\in\mathcal{A}^{**}_h} E^{**}(\mathbf{m}_h) \leq \inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} E^{\mu}(\nu^{h,k}), \quad \forall k \geq 2, \tag{2.8}$$

where $\mathcal{A}^{**}_h = \{\mathbf{m}_h : \mathbf{m}_h|_K \text{ is constant } \forall K \in \Gamma^1_h(\Omega)\}$, E^{**} is defined by (1.3) with $\hat{\varphi}^{**}(\mathbf{m}) = c_1 m_1^2 + c_2 m_1^4$ and $u_{\mathbf{m}_h}$being the solution to the Maxwell's equation (1.2) with respect to $\mathbf{m}_h$. In fact, for any $\nu^{h,k} \in \mathcal{A}^{\mu}_{h,k}$ and $\nu^{h,k}|_K = \sum_{i=1}^k \lambda_{K,i}\delta_{A_{K,i}}$, define $\mathbf{m}_h \in \mathcal{A}^{**}_h$ by $\mathbf{m}_h|_K \equiv \sum_{i=1}^k \lambda_{K,i} A_{K,i}$, then, it follows from the convexity of $\hat{\varphi}^{**}$ and $\hat{\varphi}^{**} \leq \varphi$ that

$$E^{**}(\mathbf{m}_h) - E^{\mu}(\nu^{h,k}) = \sum_{K\in\Gamma^1_h(\Omega)} (\hat{\varphi}^{**}(\mathbf{m}_h|_K) - \sum_{i=1}^k \lambda_{K,i}\varphi(A_{K,i}))|K| \leq 0.$$

Next, we claim that

$$\inf_{\nu^{h,2}\in\mathcal{A}^{\mu}_{h,2}} E^{\mu}(\nu^{h,2}) \le \inf_{\mathbf{m}_h\in\mathcal{A}^{**}_h} E^{**}(\mathbf{m}_h). \tag{2.9}$$

In fact, for a given $\mathbf{m}_h \in \mathcal{A}^{**}_h$, define (see also [6, 10]) ,

$$\nu^{h,2}(\mathbf{m}_h) = \lambda(\mathbf{m}_h)\delta_{A_1(\mathbf{m}_h)} + (1-\lambda(\mathbf{m}_h))\delta_{A_2(\mathbf{m}_h)},$$

where $A_1(\mathbf{m}_h) = m_{h,1}\mathbf{i} + \sqrt{(1-m^2_{h,1})}\ \mathbf{j}$, $A_2(\mathbf{m}_h) = m_{h,1}\mathbf{i} - \sqrt{(1-m^2_{h,1})}\ \mathbf{j}$, $\mathbf{i}$, $\mathbf{j}$ are the unit vectors along the axis of coordinates, and $\lambda(\mathbf{m}_h) = \frac{1}{2} + \frac{m_{h,2}}{2\sqrt{1-m^2_{h,1}}}$, then, a direct calculation yields

$$E^{\mu}(\nu^{h,2}(\mathbf{m}_h)) - E^{**}(\mathbf{m}_h) = \int_{\Omega}(\sum_{i=1}^{2}\lambda_i(\mathbf{m}_h)\varphi(A_i(\mathbf{m}_h)) - \hat{\varphi}^{**}(A(\mathbf{m}_h)))dx = 0.$$

It follows from (2.8) and (2.9), and the fact that, for $k \ge 2$, $\mathcal{A}^{\mu}_{h,2} \subseteq \mathcal{A}^{\mu}_{h,k}$, that

$$\inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} E^{\mu}(\nu^{h,k}) = \inf_{\nu^{h,2}\in\mathcal{A}^{\mu}_{h,2}} E^{\mu}(\nu^{h,2}) = \inf_{\mathbf{m}_h\in\mathcal{A}^{**}_h} E^{**}(\mathbf{m}_h), \quad \forall k \ge 2. \tag{2.10}$$

This completes the proof. □

Theorem 2.2. *In the uniaxial case, we have*

$$\lim_{h\to 0}\inf_{\nu^{h,2}\in\mathcal{A}^{\mu}_{h,2}} \mathrm{E}^{\mu}(\nu^{h,2}) = \inf_{\nu\in\mathcal{A}^{\mu}} \mathrm{E}^{\mu}(\nu) \tag{2.11}$$

Proof. The conclusion follows directly from Theorem 2.1 and Lemma 2.1. □

3. A nonconforming finite element method coupled with an artificial boundary technique for the Maxwell's equation

We consider to solve numerically the Maxwell's equation in R^n

$$\mathrm{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0, \qquad \text{in } H^{-1}(\mathbf{R}^n); \tag{3.1}$$

$$u \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty. \tag{3.2}$$

First, we apply an artificial boundary method to reduce the problem to a boundary value problem of the Maxwell's equation on a bounded domain.

Let $\Omega_i = B(0;\, R) \equiv \{x : |x| < R\}$ with R sufficiently large so that $\bar{\Omega} \subset \Omega_i$, and let $\mathbf{n}$ be the unit outward normal of $\partial\Omega_i = \partial B(0;\, R) \equiv \{x : |x| = R\}$, and let $\Omega_e = R^n \setminus \bar{\Omega}_i$.

Consider the problem defined on the exterior domain Ω_e

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega_e, \\ u = u(R,\theta), & \text{on } \partial\Omega_i, \\ u \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{3.3}$$

The solution of this problem can be written as a Fourier expansion

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \tag{3.4}$$

with the coefficients $a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\cos(n\phi)d\phi$, $b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\phi)\sin(n\phi)d\phi$, and we have

$$\frac{\partial u(R,\theta)}{\partial r} = \sum_{n=1}^{\infty} -\frac{n}{R}(a_n \cos n\theta + b_n \sin n\theta),$$

substituting a_n, b_n into the above equation, we get

$$\frac{\partial u(R,\theta)}{\partial r} = -\sum_{n=1}^{\infty} \frac{n}{\pi R}\int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)\, d\phi.$$

This allows us to define an operator $L : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_i)$(See [12])

$$Lu(R,\theta) = -\sum_{n=1}^{\infty} \frac{n}{\pi R}\int_0^{2\pi} u(R,\phi)\cos n(\theta-\phi)\, d\phi. \tag{3.5}$$

Now, we can reduce the problem (3.1), (3.2) to the following equivalent problem

$$\operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0, \qquad \text{in } \Omega_i; \tag{3.6}$$

$$\frac{\partial u}{\partial \mathbf{n}} = Lu, \qquad \text{on } \partial\Omega_i; \tag{3.7}$$

$$\int_0^{2\pi} u(R,\phi)\, d\phi = 0. \tag{3.8}$$

Define

$$V = \{v \in H^1(\Omega_i) : \int_0^{2\pi} v(R,\phi)\, d\phi = 0\}, \tag{3.9}$$

$$a(u,v)=\int_{\Omega_i}\nabla u\nabla v\,dx\ ,\ \ f(v)=\int_{\Omega_i}\mathbf{m}\nabla v\chi_\Omega dx, \tag{3.10}$$

$$b(u,v)=\sum_{n=1}^{\infty}\frac{n}{\pi R}\int_0^{2\pi}\int_0^{2\pi}u(R,\theta)v(R,\phi)\cos n(\theta-\phi)\,d\phi\,d\theta, \tag{3.11}$$

then (3.6)-(3.8) has the weak formulation

$$\begin{cases}\text{Find } u\in V,\ \text{such that}\\ a(u,v)+b(u,v)=f(v), \qquad \forall v\in V.\end{cases} \tag{3.12}$$

Since the bilinear form $a(u,v)+b(u,v)$ is symmetric, continuous and V-elliptic on $V\times V$, by the Lax-Milgram theorem, we have

Theorem 3.1. *Problem* (3.12) *has a unique solution.*

In a numerical implementation, the exact artificial boundary condition term $b(u,v)$ has to be replaced by a truncated approximate artificial boundary condition term. We are thus led to the following approximation problem

$$\begin{cases}\text{Find } u\in V,\ \text{such that}\\ a(u,v)+b_N(u,v)=f(v), \qquad \forall v\in V.\end{cases} \tag{3.13}$$

where $a(u,v)$, $f(v)$ are defined by (3.10) and

$$b_N(u,v)=\sum_{n=1}^{N}\frac{n}{\pi R}\int_0^{2\pi}\int_0^{2\pi}u(R,\theta)v(R,\phi)\cos n(\theta-\phi)\,d\phi\,d\theta. \tag{3.14}$$

Similarly, we have

Theorem 3.2. *Problem* (3.13) *has a unique solution.*

We also have the following convergence theorem.

Theorem 3.3. *Let* u, $u_N\in H^1(\Omega_i)$ *be the solution of* (3.12) *and* (3.13) *respectively. Suppose there exist* $R_0<R$ *and an integer* $k\geq 1$ *such that* $\bar{\Omega}\subset B(0,\,R_0)$ *and* $u|_{\partial B(0,R_0)}\in H^{k-\frac{1}{2}}(\partial B(0,R_0))$. *Then, we have*

$$|u-u_N|_{1,\Omega_i}\leq\frac{C}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1}|u|_{k-\frac{1}{2},\partial B(0,R_0)}, \tag{3.15}$$

where C *is a constant independent of* k, R_0 *and* N.

We need the following lemma.

Lemma 3.1. [14] *Under the conditions of theorem 3.3 , we have*

$$\sup_{\substack{v\in V\\ |v|_{1,\Omega_i}=1}} |b(u,v)-b_N(u,v)| \le \frac{C}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1} |u|_{k-\frac{1}{2},\partial B(0,R_0)}. \tag{3.16}$$

Proof of theorem 3.3. It follows from (3.12), (3.13) and (3.16) that

$$\begin{aligned} |u-u_N|^2_{1,\Omega_i} &\le a(u-u_N,u-u_N)+b_N(u-u_N,u-u_N)\\ &= b_N(u,u-u_N)-b(u,u-u_N)\\ &\le \frac{C}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{N+1} |u|_{k-\frac{1}{2},\partial B(0,R_0)}|u-u_N|_{1,\Omega_i}. \end{aligned}$$

□

Remark 3.1. In application, we can always choose R and R_0 such that u is sufficiently smooth near $\partial B(0,R_0)$. This indicates that a small N would be sufficient to achieve a good approximation.

Next, we discretize the problem (3.13) by a nonconforming finite element method. Let $\Gamma_h^{(i)}$ be a regular triangulation of Ω_i, which coincides with $\Gamma_h^1(\Omega)$ on Ω. Furthermore, we suppose that the triangulation satisfies

$$|a_{ij}-b_{ij}| \le C\,h_K^2, \quad \forall K \text{ on } \Gamma_e,$$

for some constant C, where a_{ij} is the midpoint of the arc $\widehat{a_ia_j}$, and b_{ij} is the midpoint of the section $\overline{a_ia_j}$ (Figure 1), and h_K is the diameter of element K, which guarantees that the geometric non-conforming error is of higher order (Section 4.3 in [8]).

Let V_h be the Crouzeix-Raviart element space, *i.e.*

$$\begin{aligned} V_h = \big\{ & v\in L^2(\Omega_i):\ \textstyle\int_0^{2\pi} v(R,\phi)\,d\phi = 0,\ v|_K \in P_1(K), \forall K\in\Gamma_h^{(i)},\ \text{and}\\ & v\ \text{ is continuous at the midpoints of all interior element edges}\big\}. \end{aligned} \tag{3.17}$$

The corresponding finite element problem is given as

$$\begin{cases} \text{Find } u_h\in V_h,\ \text{such that}\\ a_h(u_h,v_h)+b_N(u_h,v_h)=f_h(v_h), \qquad \forall v_h\in V_h. \end{cases} \tag{3.18}$$

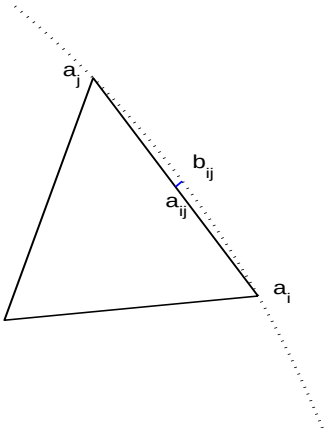

FIGURE 1. The curved element near Γ_e

where $a_h(u_h, v_h) = \sum_{K\in\Gamma_h^{(i)}} \int_K \nabla u_h \nabla v_h dx$, $f_h(v_h) = \sum_{K\in\Gamma_h^{(i)}} \int_K \mathbf{m}\chi_\Omega \cdot \nabla v_h dx$, and $b_N(\cdot,\cdot)$ is defined as

$$\begin{aligned} b_N(u_h, v_h) &= \sum_{n=1}^{N} \frac{n}{\pi R} \int_0^{2\pi} \int_0^{2\pi} \cos n(\theta-\phi) u_h(R,\theta) v_h(R,\phi) d\phi d\theta \\ &= \sum_{n=1}^{N} \frac{n}{\pi R} \left(\int_0^{2\pi} u_h(R,\theta) \cos n\theta d\theta \int_0^{2\pi} v_h(R,\phi) \cos n\phi d\phi \right. \\ &\quad \left. + \int_0^{2\pi} u_h(R,\theta) \sin n\theta d\theta \int_0^{2\pi} v_h(R,\phi) \sin n\phi d\phi \right). \end{aligned} \tag{3.19}$$

In the computation of (3.19), the integrals are calculated by numerical quadrature formulas. For example, by trapezoidal rule we get

$$\int_0^{2\pi} u_h(R,\theta)\cos n\theta d\theta \doteq \sum_{\widehat{a_i a_j}\subset\Gamma_e} |\theta_i - \theta_j| \frac{u_h|_{\overline{a_i a_j}}(a_i)\cos(n\theta_i) + u_h|_{\overline{a_i a_j}}(a_j)\cos(n\theta_j)}{2},$$

where θ_i and θ_j are the polar angles corresponding to a_i and a_j, respectively. Notice that the error introduced by numerical quadrature is of higher order, for simplicity, we would ignore its effect in the following analysis.

With a standard argument, we have

Theorem 3.4. *Problem* (3.18) *has a unique solution.*

We proceed to prove the convergence of the finite element solution. For $v \in V + V_h$, define $\|v\|_h = (a_h(v,v) + b_N(v,v))^{\frac{1}{2}}$, $|v|_{1,h} = (\sum_{K\in\Gamma_h^{(i)}} \int_K |\nabla v|^2 dx)^{\frac{1}{2}}$, and $|v|_{2,h} = (\sum_{K\in\Gamma_h^{(i)}} \int_K |\partial^2 v|^2 dx)^{\frac{1}{2}}$. We have the following error estimate.

Theorem 3.5. *Let $u \in H^1(\Omega_i)$ be the solution of Problem (3.13), $u_h \in V_h$ be the solution of Problem (3.18). Suppose that $u \in H^1(\Omega_i) \cap C(\bar{\Omega}_i)$, $u|_K \in H^2(K)$, $\mathbf{m}|_K \in (H^1(K))^2$, for all $K \in \Gamma_h^{(i)}$, then we have*

$$\|u - u_h\|_h \le C\Big(\sum_{\substack{K\in\Gamma_h^{(i)} \\ K \bigcap \partial\Omega_i = \emptyset}} h_K^2 |u|_{2,K}^2 + \sum_{\substack{K\in\Gamma_h^{(i)} \\ K \bigcap \partial\Omega_i \neq \emptyset}} (1 + N^2 h_K) h_K^2 |u|_{2,K}^2 \Big)^{\frac{1}{2}}. \quad (3.20)$$

Proof. By the second Strang lemma [8], we have, for some constant C, that

$$\|u - u_h\|_h \le C \left(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) + b_N(u, w_h) - f_h(w_h)|}{\|w_h\|_h} \right). \quad (3.21)$$

For the first term on the right hand side of (3.21), we have

$$\begin{aligned} \inf_{v_h \in V_h} \|u - v_h\|_h &\le \|u - \Pi_h u\|_h \\ &\le \big(a_h(u - \Pi_h u, u - \Pi_h u) + b_N(u - \Pi_h u, u - \Pi_h u) \big)^{\frac{1}{2}}, \end{aligned} \quad (3.22)$$

where Π_h is the interpolation operator from $H^1(\Omega)$ to V_h defined by

$$\Pi_h u|_K(a) = u(a), \quad \forall a \in \{\text{the midpoints on the edges of } K\}.$$

The standard Sobolev interpolation theory gives [8]

$$a_h(u - \Pi_h u, u - \Pi_h u) \le C \sum_{K\in\Gamma_h^{(i)}} h_K^2 |u|_{2,K}^2 \quad (3.23)$$

and

$$
\begin{aligned}
& b_N(u-\Pi_h u, u-\Pi_h u) \\
= & \sum_{n=1}^{N} \frac{n}{\pi R}\left(\left(\int_0^{2\pi}(u-\Pi_h u)\cos n\theta d\theta\right)^2+\left(\int_0^{2\pi}(u-\Pi_h u)\sin n\theta d\theta\right)^2\right) \\
\leq & \frac{2}{R}\sum_{k=1}^{N} n\int_0^{2\pi}(u-\Pi_h u)^2 d\theta = \frac{N(N+1)}{R^2}\sum_{E\subset\partial\Omega_i}\int_E (u-\Pi_h u)^2 ds \\
\leq & C\,\frac{N(N+1)}{R^2}\sum_{\substack{K\in\Gamma_h^{(i)}\\ K\bigcap\partial\Omega_i\neq\emptyset}} h_K|u-\Pi_h u|^2_{1,K} \leq \frac{CN(N+1)}{R^2}\sum_{\substack{K\in\Gamma_h^{(i)}\\ K\bigcap\partial\Omega_i\neq\emptyset}} h_K^3|u|^2_{2,K},
\end{aligned}
$$

where E denotes the edges in $\Gamma_h^{(i)}$, or

$$
b_N(u-\Pi_h u, u-\Pi_h u) \leq C\,N^2 \sum_{\substack{K\in\Gamma_h^{(i)}\\ K\bigcap\partial\Omega_i\neq\emptyset}} h_K^3|u|^2_{2,K}. \tag{3.24}
$$

For the second term on the right hand side of (3.21), by the standard error estimate techniques for the Crouzeix-Raviart nonconforming element[3], we have

$$
\begin{aligned}
& |a_h(u,w_h)+b_N(u,w_h)-f_h(w_h)| = \Big|\sum_{K\in\Gamma_h^{(i)}}\sum_{\substack{E\subset\partial K\\ E\not\subset\partial\Omega_i}}\int_E w_h\Big(\frac{\partial u}{\partial \mathbf{n}}+\mathbf{m}\chi_\Omega\cdot\mathbf{n}\Big)\mathrm{d}s\Big| \\
& \leq C\sum_{K\in\Gamma_h^{(i)}} h_K|w_h|_{1,K}(|u|_{2,K}+|\mathrm{div}(\mathbf{m}\chi_\Omega)|_K) \\
& \leq C\|w_h\|_h\Big(\sum_{K\in\Gamma_h^{(i)}} h_K^2|u|^2_{2,K}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.25}
$$

Now, the conclusion of the theorem follows as a consequence of (3.21)-(3.25). □

Corollary 3.1. *In the general case, when* $\mathbf{m}\in(L^2(\Omega))^n$ *and* $u_{\mathbf{m}}$ *is only in* $H^1(\Omega_i)$*, we have*

$$
\lim_{h\to 0}\|u_{\mathbf{m}}-u_{\mathbf{m}}^h\|_h = 0, \quad \textit{uniformly for } \mathbf{m}\in(L^2(\Omega))^n. \tag{3.26}
$$

Proof. For a given $\varepsilon > 0$, define $\mathbf{m}_l = \mathbf{m}\chi_\Omega * \psi_{1/l}$, where $\psi_{1/l}$ is the regularizer (or mollifier) and "$*$" is the convolution operator, then $\mathbf{m}_l \in H^1(\Omega_i)$ and for l sufficiently large, we have $\|\mathbf{m}_l - \mathbf{m}\|_0 < \varepsilon/3$ (*c.f.* [25]). Thus, as a consequence of theorem 3.5, we have

$$\begin{aligned} \|u^h_{\mathbf{m}} - u_{\mathbf{m}}\|_h &\leq \|u^h_{\mathbf{m}} - u^h_{\mathbf{m}_l}\|_h + \|u^h_{\mathbf{m}_l} - u_{\mathbf{m}_l}\|_h + \|u_{\mathbf{m}_l} - u_{\mathbf{m}}\|_h \\ &\leq 2\,\|\mathbf{m}_l - \mathbf{m}\|_0 + \|u^h_{\mathbf{m}_l} - u_{\mathbf{m}_l}\|_h \leq 2\,\varepsilon/3 + Ch\,|u_{\mathbf{m}_l}|_2. \end{aligned}$$

This implies (3.26). □

4. Numerical analysis of the full discrete problem

Let $\mathcal{A}^\mu_{h,k}$ be the k-atomic discrete Young measure space defined by (2.3). For $\nu^{h,k} \in \mathcal{A}^\mu_{h,k}$, let $\mathbf{m}_{h,k} = \int_{S^{n-1}} A\nu^{h,k}(dA)$, and let $u^h_{\mathbf{m}_{h,k}} \in V_h$ be the finite element solution of the problem (3.18). Denote $\mathbf{H}_K = \frac{1}{|K|}\int_K H(x)dx$, and define

$$\begin{aligned} E^\mu_h(\nu^{h,k}) &= \sum_{K\in\Gamma^1_h(\Omega)} \int_K \int_{S^{n-1}} \varphi(A)\nu^{h,k}|_K(dA)dx - \sum_{K\in\Gamma^1_h(\Omega)} \mathbf{H}_k \cdot \mathbf{m}_{h,k}|_K|K| \\ &\quad + \frac{1}{2}\sum_{K\in\Gamma^{(i)}_h} \int_K \mathbf{m}_{h,k}\chi_\Omega \cdot \nabla u^h_{\mathbf{m}_{h,k}}\,dx, \end{aligned} \tag{4.1}$$

then the full discrete relaxation problem is given as

$$(\mathbf{FDRP}): \min_{\nu^{h,k}\in\mathcal{A}^\mu_{h,k}} E^\mu_h(\nu^{h,k}). \tag{4.2}$$

In other words, (**FDRP**) is obtained from (**SDRP**) by replacing the stray-field energy $u_{\mathbf{m}_{h,k}}$ by its finite element solution $u^h_{\mathbf{m}_{h,k}}$ (see also (1.7)).

Theorem 4.1. *Let $\varphi \in C^0(S^{n-1})$ and $\mathbf{H} \in (L^2(\Omega))^n$, then the problem* (**FDRP**) *admits a solution for any given $h > 0$ and $k \geq 1$.*

Proof. The theorem follows from the compactness of both function spaces $\mathcal{A}^\mu_{h,k}$ and V_h, and the continuity of $E^\mu_h(\cdot)$ in $\mathcal{A}^\mu_{h,k}$. □

Theorem 4.2. *Let $\varphi \in C^0(S^{n-1})$ and $\mathbf{H} \in (L^2(\Omega))^n$, then we have*

$$\lim_{h\to 0, k\to\infty} \inf_{\nu^{h,k}\in\mathcal{A}^\mu_{h,k}} \mathrm{E}^\mu_h(\nu^{h,k}) = \inf_{\nu\in\mathcal{A}^\mu} \mathrm{E}^\mu(\nu). \tag{4.3}$$

Proof. Denote $\tilde{\nu}$ the minimizer of $E^\mu(\cdot)$ in $\mathcal{A}^\mu$, let $\tilde{\nu}^{h,h'} = P^*_{(h,h')}\tilde{\nu}$ be the interpolation of $\tilde{\nu}$ in $\mathcal{A}^\mu_{(h,h')} = \mathcal{A}^\mu_{h,k}$, here we assume that the triangulation of S^{n-1} has k nodes and satisfies $\lim_{k\to\infty} h' = 0$. Let $\tilde{\nu}^{h,k}$ be the minimizer of $E^\mu_h(\cdot)$ in $\mathcal{A}^\mu_{h,k}$. Let $\tilde{\mathbf{m}} = \int_{S^{n-1}} A\tilde{\nu}(dA)$, $\tilde{\mathbf{m}}_{h,h'} = \int_{S^{n-1}} A\tilde{\nu}^{h,h'}(dA)$, and $\tilde{\mathbf{m}}_{h,k} = \int_{S^{n-1}} A\tilde{\nu}^{h,k}(dA)$. Then we have

$$\begin{aligned}
&\inf_{\nu^{h,k}\in A^\mu_{h,k}} E^\mu_h(\nu^{h,k}) - \inf_{\nu\in A^\mu} E^\mu(\nu) = \inf_{\nu^{h,k}\in A^\mu_{h,k}} E^\mu_h(\nu^{h,k}) - E(\tilde{\nu}) \\
\leq\ & |E^\mu_h(\tilde{\nu}^{h,h'}) - E^\mu(\tilde{\nu}^{h,h'})| + |E^\mu(\tilde{\nu}^{h,h'}) - E^\mu(\tilde{\nu})| = I_1 + I_2, \qquad (4.4)
\end{aligned}$$

and

$$\inf_{\nu\in\mathcal{A}^\mu} E^\mu(\nu) - \inf_{\nu^{h,k}\in\mathcal{A}^\mu_{h,k}} E^\mu_h(\nu^{h,k}) \leq |E^\mu(\tilde{\nu}^{h,k}) - E^\mu_h(\tilde{\nu}^{h,k})| = I_3. \qquad (4.5)$$

The L^1-weak*-continuity of $P^*_{(h,h')}$ (see (2.1)) gives $\lim_{h\to 0,k\to\infty} I_2 = 0$. It follows from the L^1-weak*-continuity of $P^*_{(h,h')}$, Corollary 3.1 and

$$\begin{aligned}
I_1 &= \frac{1}{2}|\sum_{K\in\Gamma^{(i)}_h}\int_K \mathbf{m}_{h,k}\chi_\Omega\cdot(\nabla u^h_{\tilde{\mathbf{m}}_{h,h'}} - \nabla u_{\tilde{\mathbf{m}}_{h,h'}})\,dx| \\
&\leq C\,\|u^h_{\tilde{\mathbf{m}}_{h,h'}} - u_{\tilde{\mathbf{m}}_{h,h'}}\|_{1,h}, \qquad (4.6)
\end{aligned}$$

$$\begin{aligned}
I_3 &= |\sum_{K\in\Gamma^{(i)}_h}\int_K \mathbf{m}_{h,k}\chi_\Omega\cdot(\nabla u^h_{\tilde{\mathbf{m}}_{h,k}} - \nabla u_{\tilde{\mathbf{m}}_{h,k}})\,dx| \\
&\leq C\|u_{\tilde{\mathbf{m}}_{h,k}} - u^h_{\tilde{\mathbf{m}}_{h,k}}\|_{1,h}, \qquad (4.7)
\end{aligned}$$

that $\lim_{h\to 0,k\to\infty} I_i = 0$, for $i = 1, 3$. This completes the proof. □

As a consequence of Lemma 2.1, Corollary 3.1 and Theorem 4.2, we have

Corollary 4.1. *For the uniaxial case, where $\varphi(\mathbf{m}) = c_1 m_1^2 + c_2(1-m_2^2)^2$, we have*

$$\lim_{h\to 0}\inf_{\nu^{h,2}\in\mathcal{A}^\mu_{h,2}} \mathrm{E}^\mu_h(\nu^{h,2}) = \inf_{\nu\in\mathcal{A}^\mu} \mathrm{E}^\mu(\nu).$$

We have the following uniqueness result for the discrete potential of the stray field energy, which can be useful in determining whether a given number of atoms is sufficient to obtain the minimum of (**FDRP**).

Lemma 4.1. *Suppose that, for some $k_0 > 1$,*

$$\inf_{\nu_{h,k}\in\mathcal{A}^{\mu}_{h,k}} E^{\mu}_h(\nu_{h,k}) = \inf_{\nu_{h,k_0}\in\mathcal{A}^{\mu}_{h,k_0}} E^{\mu}_h(\nu_{h,k_0}), \quad \forall k \geq k_0. \tag{4.8}$$

Then, the potential of the stray field energy $u^h_{\mathbf{m}_h}$ is uniquely determined by the minimizers of (**FDRP**) *with $k \geq k_0$.*

Proof. Suppose $k \geq k_0$ and $\nu^{(1)}_{h,k}$, $\nu^{(2)}_{h,k}$ are two minimizers of E^{μ}_h in $\mathcal{A}^{\mu}_{h,k}$ with $\nu^{(1)}_{h,k}|_K = \sum_{i=1}^k \lambda^{(1)}_{K,i}\delta_{A^{(1)}_{K,i}}$, $\nu^{(2)}_{h,k}|_K = \sum_{i=1}^k \lambda^{(2)}_{K,i}\delta_{A^{(2)}_{K,i}}$. Let u^h_1 and u^h_2 be the solutions of (3.18) corresponding to the macroscopic magnetization fields $\mathbf{m}^{(1)}_h$ and $\mathbf{m}^{(2)}_h$ which are defined by $\mathbf{m}^{(1)}_h|_K = \sum_{i=1}^k \lambda^{(1)}_{K,i}A^{(1)}_{K,i}$, $\mathbf{m}^{(2)}_h|_K = \sum_{i=1}^k \lambda^{(2)}_{K,i}A^{(2)}_{K,i}$ respectively. We only need to show that $u^h_1 = u^h_2$. In fact, for $0 < \xi < 1$, define $\nu^{(\xi)}_{h,2k} \in \mathcal{A}^{\mu}_{h,2k}$ by $\nu^{(\xi)}_{h,2k}|_K = \sum_{i=1}^k (\xi\lambda^{(1)}_{K,i}\delta_{A^{(1)}_{K,i}} + (1-\xi)\lambda^{(2)}_{K,i}\delta_{A^{(2)}_{K,i}})$, then $u^h_\xi = \xi u^h_1 + (1-\xi)u^h_2$ is the solution of (3.18) corresponding to the macroscopic magnetization field $\mathbf{m}^{(\xi)}_h = \xi\mathbf{m}^{(1)}_h + (1-\xi)\mathbf{m}^{(2)}_h$. If $u^h_1 \neq u^h_2$, then (3.18) and the strict convexity of $a_h(\cdot,\cdot) + b_N(\cdot,\cdot)$ would lead to

$$\begin{aligned}
&\sum_{K\in\Gamma^{(i)}_h}\int_K \nabla u^h_\xi \cdot \mathbf{m}^{(\xi)}_h \chi_\Omega\, dx = a_h(u^h_\xi, u^h_\xi) + b_N(u^h_\xi, u^h_\xi)\\
<\ & \xi(a_h(u^h_1,u^h_1) + b_N(u^h_1,u^h_1)) + (1-\xi)(a_h(u^h_2,u^h_2) + b_N(u^h_2,u^h_2))\\
=\ & \xi\sum_{K\in\Gamma^{(i)}_h}\int_K \nabla u^h_1\cdot\mathbf{m}^{(1)}_h\chi_\Omega\,dx + (1-\xi)\sum_{K\in\Gamma^{(i)}_h}\int_K \nabla u^h_2\cdot\mathbf{m}^{(2)}_h\chi_\Omega\,dx
\end{aligned}$$

and as a consequence we would have

$$E^{\mu}_h(\nu^{(\xi)}_{h,2k}) < \xi E^{\mu}_h(\nu^{(1)}_{h,k}) + (1-\xi)E^{\mu}_h(\nu^{(2)}_{h,k}),$$

which contradicts the assumption (4.8). □

Now we are in the situation to prove the stability of the full discrete problem for the uniaxial case. In fact, by establishing a relationship between the full discrete problem of (**RP2**) and that of (**RP1**), we can show that, with the uniaxial energy density, the full discrete problem (**FDRP**) has a unique solution, that is the discrete macroscopic magnetization field $\mathbf{m}_h$ is unique. This

is the most that we can expect, since the Young measure solution is in general not unique, which corresponds to the fact that in physics the microstructure is not unique.

The full discrete problem (**FDRP1**) of the convex-hull relaxation problem (**RP1**) is to minimize the energy functional (see (1.6))

$$E_h^{**}(\mathbf{m}_h) = \sum_{K\in\Gamma_h^1(\Omega)} \hat{\varphi}^{**}(\mathbf{m}_h|_K)|K| - \sum_{K\in\Gamma_h^1(\Omega)} \mathbf{H}_K\cdot\mathbf{m}_h|_K|K| + \frac{1}{2}\sum_{K\in\Gamma_h^{(i)}}\int_K \mathbf{m}_h\chi_\Omega\cdot\nabla u^h_{\mathbf{m}_h}\,dx \tag{4.9}$$

in the set $\mathcal{A}_h^{**} = \{\mathbf{m}_h : \mathbf{m}_h|_K \text{ is constant } \forall K\in\Gamma_h^1 \text{ and } |\mathbf{m}_h|\le 1\}$, where $\mathbf{H}_K = \frac{1}{|K|}\int_K H(x)dx$, and $u^h_{\mathbf{m}_h}$ is the finite element solution of problem (3.18).

Lemma 4.2. *For the uniaxial energy density $\varphi(\mathbf{m}) = c_1m_1^2 + c_2(1-m_2^2)$, we have that, for any $k\ge 2$,*

(a): *if $\nu^{h,k}$ is a minimizer of E_h^μ in $\mathcal{A}_{h,k}^\mu$, then $\mathbf{m}_h = \int_{S^{n-1}} A\nu^{h,k}(dA)$ is a minimizer of E_h^{**} in $\mathcal{A}_h^{**}$, and*

$$\varphi^{**}(\mathbf{m}_h) = \int_{S^{n-1}} \varphi(A)\nu^{h,k}(dA). \tag{4.10}$$

(b): *if $\mathbf{m}_h$ is a minimizer of E_h^{**} in $\mathcal{A}_h^{**}$, then there exists a minimizer $\nu^{h,k}$ of E_h^μ in $\mathcal{A}_{h,k}^\mu$ such that $\mathbf{m}_h = \int_{S^{n-1}} A\nu^{h,k}(dA)$ and (4.10) holds.*

As a consequence, we have

$$\inf_{\mathbf{m}_h\in\mathcal{A}_h^{**}} \mathrm{E}_h^{**}(\mathbf{m}_h) = \inf_{\nu^{h,k}\in\mathcal{A}_{h,k}^\mu} \mathrm{E}_h^\mu(\nu^{h,k}),\ \ \forall k\ge 2. \tag{4.11}$$

Proof. The lemma follows from similar arguments as we used in the proof of Lemma 2.1. □

Lemma 4.3. *The potential of the stray field energy $u^h_{\mathbf{m}_h}$ is uniquely determined by the minimizers of the full discrete problem* (**FDRP1**).

Proof. The lemma is a direct consequence of Lemma 4.1 and Lemma 4.2. □

Lemma 4.4. *For the uniaxial energy density* $\varphi(\mathbf{m}) = c_1 m_1^2 + (1 - m_2^2)^2$*, the discrete relaxed energy functional* $\mathrm{E}_h^{**}(\cdot)$ *has a unique minimizer in* $\mathcal{A}_h^{**}$*.*

Proof. Suppose $\mathbf{m}_h^{(1)}$ and $\mathbf{m}_h^{(2)}$ are two minimizers of $E_h^{**}(\cdot)$ in $\mathcal{A}_h^{**}$. By the equation (3.18), and the uniqueness of the potential of the stray field energy $u_{\mathbf{m}_h}^h$ for the full discrete problem (**FDRP1**), we have

$$\sum_{K\in\Gamma_h^{(i)}} \int_K (\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)})\chi_\Omega \cdot \nabla v_h dx = 0, \quad \forall v_h \in V_h. \tag{4.12}$$

Let $\boldsymbol{\delta}_h = (\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)})\chi_\Omega$, by the discrete Helmholtz decomposition theorem[1], there exists $\alpha_h \in V_{h,0}$ and $\beta_h \in \hat{V}_h/R$ such that

$$\boldsymbol{\delta}_h|_K = \nabla\alpha_h|_K + \mathbf{curl}\,\beta_h|_K \ a.e., \quad \forall K \in \Gamma_h^{(i)}, \tag{4.13}$$

where $V_{h,0} = \{v \in V_h \mid v(b) = 0, \ \text{ if } b \text{ is a midpoint of an edge on } \partial\Omega_i\}$, $\hat{V}_h = \{v \in C(\bar{\Omega}_i) \mid v|_K \in P_1(K), \ \forall K \in \Gamma_h^{(i)}\}$. Noticing that

$$\sum_{K\in\Gamma_h^{(i)}} \int_K \nabla v_h \cdot \mathbf{curl}\beta_h dx = 0, \quad \forall v_h \in V_{h,0}, \tag{4.14}$$

by (4.12), we have

$$\sum_{K\in\Gamma_h^{(i)}} \int_K \nabla v_h \cdot \nabla\alpha_h dx = 0, \quad \forall v_h \in V_{h,0}. \tag{4.15}$$

This implies that $\alpha_h = 0$ and

$$\boldsymbol{\delta}_h = \mathbf{curl}\,\beta_h = (\partial_y\beta_h, -\partial_x\beta_h)^T. \tag{4.16}$$

Since $\boldsymbol{\delta}_h = 0$ in $\Omega_i \setminus \bar{\Omega}$, (4.16) yields $\beta_h = C$ in $\Omega_i \setminus \bar{\Omega}$, where C is a constant, without loss of generality, we may assume $C = 0$. Next, we show that the first element of the vector $\boldsymbol{\delta}_h$ vanishes almost everywhere, *i.e.*

$$\partial_y\beta_h = 0, \ a.e. \ \text{ in } \Omega_i. \tag{4.17}$$

Suppose otherwise, then $m_{h,1}^{(1)} \neq m_{h,1}^{(2)}$ on some $K \subset \Omega$. Define $\mathbf{m}_h^{(\xi)} = \xi\mathbf{m}_h^{(1)} + (1-\xi)\mathbf{m}_h^{(2)}$. By the strict convexity of $\hat{\varphi}^{**}(\mathbf{m}) = c_1(m_1)^2 + c_2(m_1)^4$ with respect

to m_1 and the uniqueness of the stray field energy, this would lead to

$$E_h^{**}(\mathbf{m}_h^{(\xi)}) < \xi E_h^{**}(\mathbf{m}_h^{(1)}) + (1-\xi)E_h^{**}(\mathbf{m}_h^{(2)}), \quad \forall \xi \in (0,1),$$

which again contradicts the assumption that $\mathbf{m}_h^{(1)}$ and $\mathbf{m}_h^{(2)}$ are the minimizers of $E_h^{**}(\cdot)$ in $\mathcal{A}_h^{**}$. Since $\beta_h = 0$ in $\Omega_i \setminus \bar{\Omega}$, (4.17) implies $\beta_h \equiv 0$ in Ω_i, and by (4.16) this gives $\boldsymbol{\delta}_h \equiv 0$, i.e. $\mathbf{m}_h^{(1)} = \mathbf{m}_h^{(2)}$. □

As a consequence of Lemma 4.2, Lemma 4.4, we have the following uniqueness theorem for the macroscopic magnetization field $\mathbf{m}_h$.

Theorem 4.3. *In the uniaxial case, where $\varphi(\mathbf{m}) = c_1 m_1^2 + c_2(1-m_2^2)^2$, all minimizers of $\mathrm{E}_h^{\mu}(\cdot)$ in $\mathcal{A}_{h,k}^{\mu}$ have the same macroscopic magnetization field $\mathbf{m}_h$.*

5. The Algorithm and Numerical Examples

First, we rewrite the set $\mathcal{A}_{h,k}^{\mu}$ and deduce the discrete problem to an unconstrained nonconvex optimization problem.

Let $k = 2^j$ where $j \geq 1$ is an integer. For a $K \in \Gamma_h^1(\Omega)$ and $i = 1, 2, \cdots, 2^j$, let $\theta_{K,i} \in [-\pi, \pi]/\{-\pi, \pi\}$, i.e. $-\pi$ and π are considered to be the same point in the set, define $A(\theta_{K,i}) \in S^1$ by

$$A(\theta_{K,i}) = \begin{pmatrix} \cos(\theta_{K,i}) \\ \sin(\theta_{K,i}) \end{pmatrix}. \tag{5.1}$$

For a $K \in \Gamma_h^1(\Omega)$, let $\alpha_K = \{\alpha_{K,l}\}_{l=1}^j$ with $\alpha_{K,l} \in [-\pi/2, \pi/2]/\{-\pi/2, \pi/2\}$, and let $i = 1 + i_1 2^0 + i_2 2^2 + \cdots + i_j 2^j$ with $i_l \in \{0, 1\}$ for $l = 1, 2, \cdots, j$, define

$$\lambda(\alpha_K, i) = \prod_{l=1}^{j} cs(i_l, \alpha_{K,l}) \tag{5.2}$$

where

$$cs(\xi, \beta) = \begin{cases} \cos^2(\beta), & \text{if } \xi = 0\ ; \\ \sin^2(\beta), & \text{if } \xi = 1\ . \end{cases} \tag{5.3}$$

It is not difficult to see that $\lambda(\alpha_K, i)$ satisfy

$$0 \leq \lambda(\alpha_K, i) \leq 1, \quad \text{and} \quad \sum_{i=1}^{k} \lambda(\alpha_K, i) = 1.$$

Denote $\boldsymbol{\theta} = \{\theta_{K,i} \mid K \in \Gamma_h^1(\Omega)\ ,\ i = 1, \cdots, k\}$ and $\boldsymbol{\alpha} = \{\alpha_{K,l} \mid K \in \Gamma_h^1(\Omega), l = 1, \cdots, j\}$. It is easily verified that

$$\mathcal{A}_{h,k}^{\mu} = \mathcal{A}_{h,k}^{\mu}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \Big\{\nu^{h,k} = \big\{\nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})|_K\big\}_{K \in \Gamma_h^1} : \nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})|_K = \sum_{i=1}^{k} \lambda(\alpha_K, i)\delta_{A(\theta_{K,i})}\Big\}. \quad (5.4)$$

Now, the full discrete problem (**FDPR**) can be rewritten as

$$(\mathbf{FDRP}'): \text{ To minmize } E_h^{\nu}(\boldsymbol{\theta}, \boldsymbol{\alpha}) := E_h^{\nu}(\nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})) \text{ in } \mathcal{A}_{h,k}^{\mu}(\boldsymbol{\theta}, \boldsymbol{\alpha}). \quad (5.5)$$

The following algorithm can be applied to this unconstrained nonconvex optimization problem [20]:

(1) set $j = j_0 \geq 1$, set $k = 2^j$, give the initial mesh;
(2) set $(\boldsymbol{\theta}, \boldsymbol{\alpha}) = (\boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)$;
(3) compute $E_h^{\nu}(\boldsymbol{\theta}, \boldsymbol{\alpha})$ by (5.1)-(5.3) and by solving (3.18);
(4) compute $\mathbf{d}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \frac{\partial E_h^{\nu}(\boldsymbol{\theta}, \boldsymbol{\alpha})}{\partial(\boldsymbol{\theta}, \boldsymbol{\alpha})}$
(5) if $\|\mathbf{d}(\boldsymbol{\theta}, \boldsymbol{\alpha})\| < TOL$, go to step 7;
(6) search for a minimizer $(\boldsymbol{\theta}_1, \boldsymbol{\alpha}_1)$ of E_h^{ν} along the conjugate gradient direction. Let $(\boldsymbol{\theta}, \boldsymbol{\alpha}) = (\boldsymbol{\theta}_1, \boldsymbol{\alpha}_1)$, go to step 3;
(7) if j is not sufficiently large, then set $j = j + 1$ and $k = 2^j$, distribute the new atoms accordingly, then go to Step 3;
(8) if h is not sufficiently small, set $h = h/2$, $TOL = TOL/2$ and initiate the data on the refined mesh, then go to step 3.

Notice that the equation (3.18) is in fact a system of linear equations of the form

$$Tu_{\mathbf{m}} = G\mathbf{m}, \quad (5.6)$$

where T is symmetric and positive definite, and thus we have

$$\int_{\Omega} \mathbf{m} \cdot \nabla u_{\mathbf{m}} dx = u_{\mathbf{m}}^T T u_{\mathbf{m}} = \mathbf{m}^T G^T T^{-1} G\mathbf{m}, \quad (5.7)$$

and

$$\frac{\partial \int_{\Omega} \mathbf{m} \cdot \nabla u_{\mathbf{m}} dx}{\partial(\boldsymbol{\theta}, \boldsymbol{\alpha})} = 2G^T T^{-1} G\mathbf{m}. \quad (5.8)$$

In step 3 and 4, Equation (5.7) and (5.8) are used to compute the corresponding items.

The criteria for enlarging j in step 7 and reducing h in step 8 may depend on the problem we solve. In general, we may enlarge j on an element, if the number of the active atoms is greater than 2^{j-1}, and we may reduce h if $u^h - u^{h/2}$ is not sufficiently small. For the uniaxial case we can set $j = 1$ and omit step 7.

In the following, we present some numerical examples, which show that our new method is efficient and avoids the artificial oscillations.

Example 1. Let $\Omega = (-0.1, 0.1) \times (-0.5, 0.5)$ and $\Omega_i = \{x \in R^n : |x| < 1\}$, $\varphi(\mathbf{m}) = 10^{-2}(m_1^2 + (1 - m_2^2)^2)$ and $\mathbf{H} = (10^{-2}, 0)$, we set j=1 and TOL= 10^{-10}, In (3.18), we set $N = 9$.

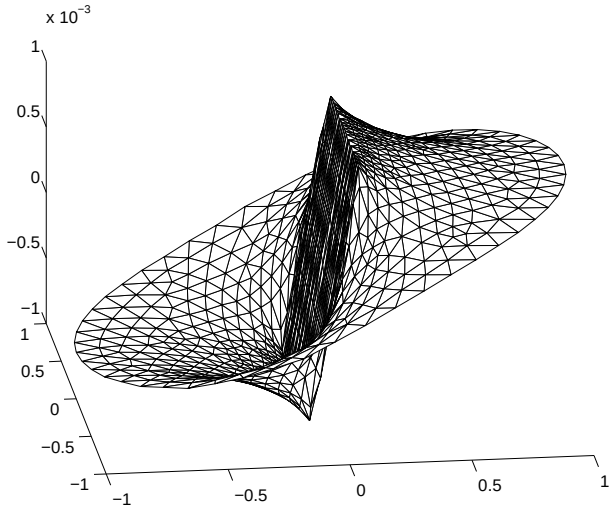

FIGURE 2. The potential $u^h_{\mathbf{m}_h}$ for Example 1.

Figure 1 shows the potential $u_{\mathbf{m}_h}$, which is plotted using area averages of the vertices' values of $u_{\mathbf{m}_h}$ on the adjacent elements. Figure 3 and Figure 5 in the following examples are drawn in the same way. Compare with the numerical results obtained in [20], the accuracy is no significant difference. Figure 2 shows the magnetization $\mathbf{m}_h$ together with the counter map for the potential of the stray field energy and some typical two atomic Young measures, where the arrows indicate the position of the atoms on S^1, and the ratios of the areas of

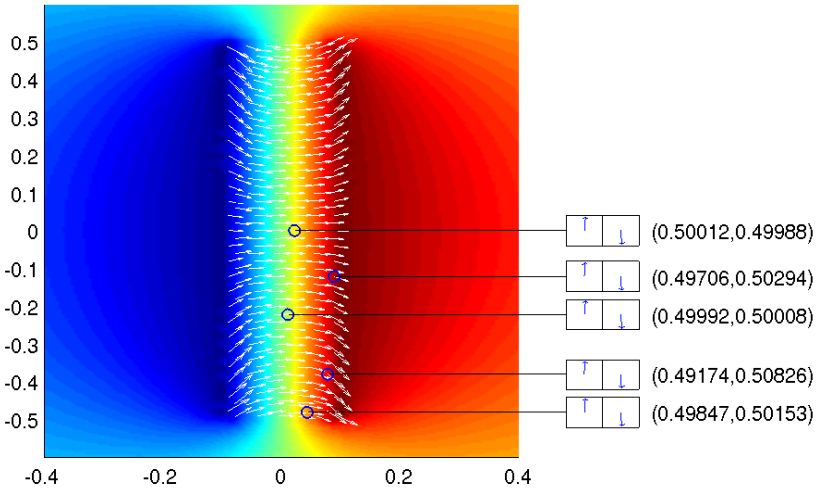

FIGURE 3. The magnetization distribution $\mathbf{m}_h$, the potential $u^h_{\mathbf{m}_h}$ and the two-atomic Young measures for Example 1.

the rectangles give their volume fractions and we marked respective values on the rightside, which are all very close to 1/2.

Example 2. Let $\Omega = (-0.1, 0.1)\times(-0.5, 0.5)$, $\Omega_i = \{x \in R^n : |x| < 1\}$. Let $\varphi(\mathbf{m}) = 10^{-2}(m_1^2+(1-m_2^2)^2)$ and $\mathbf{H} = 10^{-2}(\cos(y\pi)\sin(2.5x\pi), \sin(y\pi)\cos(2.5x\pi))$. We set j=1 and TOL$= 10^{-10}$, and again set $N = 9$ in (3.18).

Figure 3 shows the potential $u_{\mathbf{m}_h}$ and Figure 4 shows the magnetization $\mathbf{m}_h$ together with the counter map for the potential of the stray field energy $u_{\mathbf{m}_h}$ and some typical two atomic Young measures, where it is clearly seen that the volume fractions are well apart in this case .

The numerical experiments on the above examples clearly show that, in sharp contrast to the numerical results obtained by the conforming finite element approach combined with a smoothing process taking local averages to eliminate the numerical oscillations [20], the microscopic magnetization field $\mathbf{m}_h$ obtained by the nonconforming finite element approximation, as shown in Figure 2 and Figure 4, is much smoother and shows no trace of numerical oscillations, which verifies our theoretical result on the stability of the method.

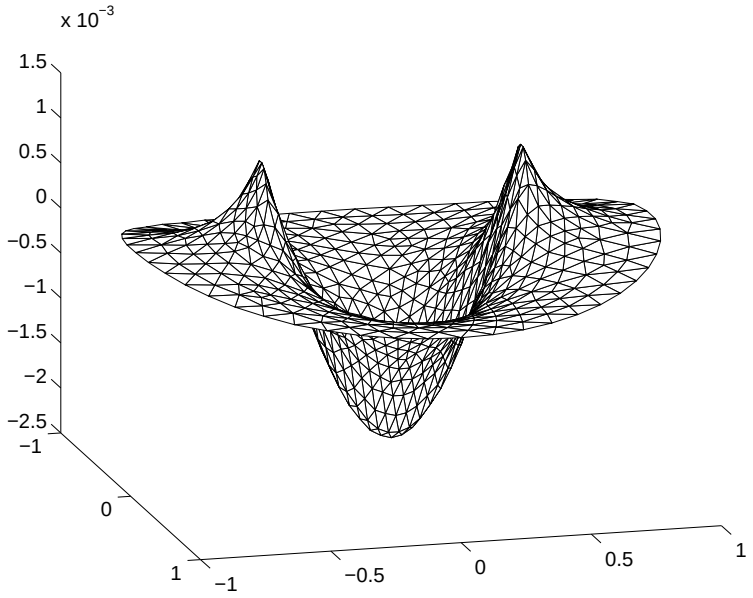

FIGURE 4. The potential $u^h_{\mathbf{m}_h}$ for Example 2.

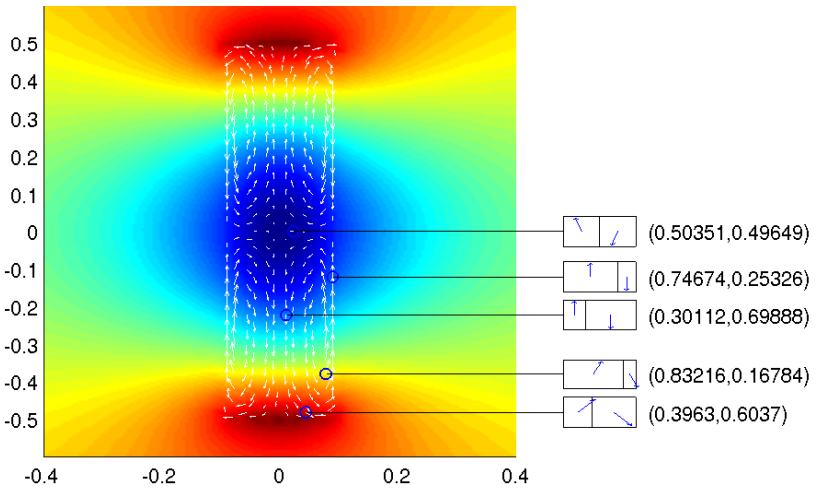

FIGURE 5. The magnetization distribution $\mathbf{m}_h$, the potential $u^h_{\mathbf{m}_h}$ and the two-atomic Young measures for Example 2.

Example 3. Let $\Omega = \{(x, y) : 4x^2 + 25y^2 \leq 1\}$ and everything else be the same as in Example 1.

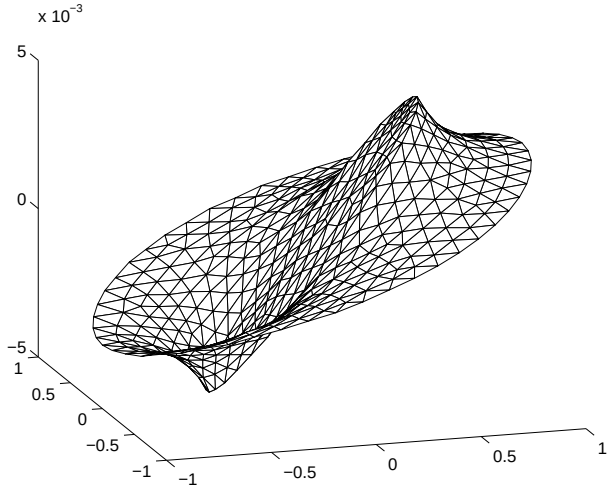

FIGURE 6. The potential $u^h_{\mathbf{m}_h}$ for Example 3.

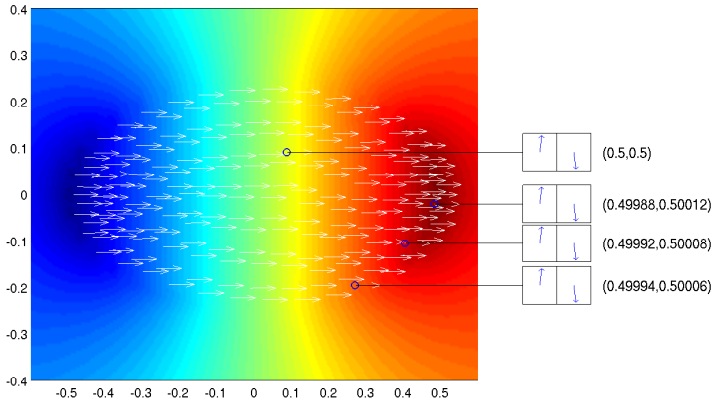

FIGURE 7. The magnetization distribution $\mathbf{m}_h$, the potential $u^h_{\mathbf{m}_h}$ and the two-atomic Young measures for Example 3.

Figure 5 shows the potential $u_{\mathbf{m}_h}$. Figure 6 shows the magnetization $\mathbf{m}_h$, the potential's counter map and some typical two-atomic Young measures, where it is clearly seen that the magnetization is almost uniform, which agrees with the observation in physics.

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