AN INTEGRAL REPRESENTATION THEOREM FOR LOWER SEMICONTINUOUS ENVELOPES OF INTEGRAL FUNCTIONALS

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Abstract. In this paper, an integral representation theorem on sequentially weakly lower semicontinuous envelope of multiple integral functionals is proved for integrands which satisfy growth conditions of order p and have lower compactness property. The result generalizes the standard results in the area.

1. Introduction

In calculus of variations, the problem of minimizing an integral functional

$$F(u;\ \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx, \tag{1.1}$$

on a set of admissible functions

$$\mathbb{A} = \{u \in W^{1,p}(\Omega;\ R^m) : u = u_0 \text{ on } \partial\Omega\}, \tag{1.2}$$

where $\Omega \subset R^n$ is a bounded open set with Lipschitz continuous boundary $\partial\Omega$ and $1 \le p < \infty$, can often be replaced by the relaxed problem of minimizing the relaxed functional

$$\hat{F}(u;\ \Omega) = \int_\Omega \hat{f}(x, u(x), Du(x))\, dx, \tag{1.3}$$

on $\mathbb{A}$, where $\hat{f}(x,\ s,\ \cdot)$ is the quasiconvex envelope of $f(x,\ s,\ \cdot)$, *i.e.* the greatest quasiconvex function less than or equal to $f(x,\ s,\ \cdot)$ (see [1, 2, 3]). In fact, the solutions to the two problems coincide whenever $\hat{F}(\cdot,\ \Omega)$ happens to be the sequentially weakly lower semicontinuous envelope of $F(\cdot,\ \Omega)$, *i.e.* the

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greatest sequentially weakly lower semicontinuous functional defined on $\mathbb{A}$ less than or equal to $F(\cdot,\ \Omega)$, which is identified by (see [3, 4, 5])

$$\Gamma\text{-}\lim F(u;\ \Omega) = \min\{\liminf_{\alpha\to\infty} F(u_\alpha;\ \Omega) : u_\alpha \in \mathbb{A},\ \ u_\alpha \rightharpoonup u \text{ in } W^{1,p}(\Omega;\ R^m)\},$$

where "$\rightharpoonup$" means "converges weakly to". This is useful in numerical analysis and computations (see for example [6, 7, 8, 9]) as well as theoretically important. Thus, it is naturally desirable to know under what conditions this is true. The following well known result is given by Acerbi and Fusco [3].

Theorem 1.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies the hypotheses*

- **(H1):** $0 \le f(x,s,\xi) \le a(x) + C(|s|^p + |\xi|^p)$ *for every* $x \in \Omega$, $s \in R^m$, $\xi \in R^{n\times m}$, *where* C *is a nonnegative constant and* $a(x) \in L^1(\Omega)$ *is a nonnegative function;*
- **(H2):** $|f(x,s_1,\xi) - f(x,s_2,\xi)| \le \omega(x,\ |s_1 - s_2|)\beta(|\xi|)$, *where* $\omega : \Omega \times R \to R^+$ *is a Carathéodory function,* $\omega(x,0) = 0$, *and* $\beta(\cdot)$ *is increasing and nonnegative.*

Then, for every open subset $\Omega' \subset \Omega$, $\hat{F}(\cdot;\ \Omega')$ *is the sequentially weakly lower semicontinuous envelope of* $F(\cdot;\ \Omega')$ *in* $W^{1,p}(\Omega';\ R^m)$.

The main result of this paper (Theorem 3.1) generalizes theorem 1.1 by replacing the hypothesis (H1) by two weaker hypotheses :

- **(H1a):** $|f(x,s,\xi)| \le a(x) + C(|s|^p + |\xi|^p)$ for every $x \in \Omega$, $s \in R^m$, $\xi \in R^{n\times m}$, where C is a nonnegative constant and $a(x) \in L^1(\Omega)$ is a nonnegative function;
- **(H1b):** f has the lower compactness property, *i.e.* $f^-(x, u_\alpha(x), Du_\alpha(x))$ is precompact in $L^1(\Omega')$ whenever Ω' is an open subset of Ω, u, $u_\alpha \in W^{1,p}(\Omega';\ R^m)$ are such that $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega';\ R^m)$ and $F(u_\alpha;\ \Omega') \le \hat{C} < +\infty$, where $f^- = \min\{f,\ 0\}$.

The proof of the main result makes fully use of theorem 1.1 and the results developed in section 2, where it is shown under the hypothesis (H1a) that

$$\hat{f}(x,s,\xi) = \lim_{\beta\to\infty} \hat{f}_\beta(x,s,\xi),$$

for all $(x,s,\xi) \in \Omega \times R^m \times R^{n\times m}$ and

$$\hat{F}(u;\ \Omega') = \lim_{\beta\to\infty} \hat{F}_\beta(u;\ \Omega'),$$

for every open subset $\Omega' \subset \Omega$ and for all $u \in W^{1,p}(\Omega';\ R^m)$, where $\hat{f}_\beta$ is the quasiconvex envelope of

$$f_\beta(x, s, \xi) = \max\{f(x, s, \xi),\ -\beta\} \tag{1.4}$$

and

$$\hat{F}_\beta(u;\ \Omega') = \int_{\Omega'} \hat{f}_\beta(x, u(x), Du(x))\, dx. \tag{1.5}$$

As a by-product, we obtain a theorem on lower semicontinuity of integral functionals (Theorem 3.2) which is not covered by the more general results of the kind given recently by Li [10, 11].

Remark 1.1. Here and throughout this paper, assumptions and statements are referred to sets with measure-negligible projections on Ω, *i.e.* they hold on a subset $\Omega' \subset \Omega$ with $\text{meas}(\Omega') = \text{meas}(\Omega)$ where $\text{meas}(\cdot)$ denotes the Lebesgue measure in R^n.

2. Quasiconvex envelope of $f(x,\ s,\ \cdot)$

Let $\Omega \subset R^n$ be a bounded open set with Lipschitz continuous boundary $\partial\Omega$ and let $1 \le p < \infty$. Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function. Define

$$f_\beta = \max\{f,\ -\beta\}, \quad \forall \beta \in \mathbb{N} = \{1, 2, 3, \ldots\}. \tag{2.1}$$

Denote the quasiconvex envelopes of f_β and f by $\hat{f}_\beta$ and $\hat{f}$ respectively.

Lemma 2.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a). Then, $\{\hat{f}_\beta\}_{\beta=1}^\infty$ is a nonincreasing sequence of Carathéodory functions which satisfy*

$$-\beta \le \hat{f}_\beta(x, s, \xi) \le a(x) + C(|s|^p + |\xi|^p) \quad \forall (x, s, \xi) \in \Omega \times R^m \times R^{n\times m}, \tag{2.2}$$

where, as given in (H1a), C is a nonnegative constant and $a(x) \in L^1(\Omega)$ is a nonnegative function.

Proof. By the inequality,

$$\hat{f}_\beta(x, s, \xi) \le f_\beta(x, s, \xi) \le f_\alpha(x, s, \xi) \quad \forall (x, s, \xi) \in \Omega \times R^m \times R^{n\times m} \text{ and } \forall \beta \ge \alpha,$$

which follows directly form the definitions, we conclude that $\hat{f}_\beta \le \hat{f}_\alpha$ for all $\beta \ge \alpha$. That is $\{\hat{f}_\beta\}_{\beta=1}^\infty$ is a nonincreasing sequence of functions.

Next, let

$$g_\beta(x, s, \xi) = f_\beta(x, s, \xi) + \beta.$$

Then, $\{g_\beta\}_{\beta=1}^{\infty}$ are Carathéodory functions satisfying (H1). As a consequence (see [3, 4, 5]), for each $\beta \in \mathbb{N}$, $\hat{g}_\beta$, the quasiconvex envelope of g_β, is a Carathéodory function and satisfies

$$0 \le \hat{g}_\beta(x,s,\xi) \le a(x) + C(|s|^p + |\xi|^p) + \beta.$$

Since $\hat{f}_\beta = \hat{g}_\beta - \beta$, (2.2) follows. □

Lemma 2.2. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a). Then, for every $(x,s,\xi) \in \Omega \times R^m \times R^{n\times m}$,*

$$\bar{f}(x,s,\xi) > -\infty, \tag{2.3}$$

where

$$\bar{f}(x,s,\xi) = \inf\{\liminf_{\alpha\to\infty} \frac{1}{\text{meas}(\Omega)} \int_\Omega f(x,s,\xi + D\varphi_\alpha(x'))\,dx' :$$
$$\varphi_\alpha \rightharpoonup 0 \ \textit{in}\ W^{1,p}(\Omega;\ R^m)\}.$$

Proof. By the definition of $\bar{f}(x,s,\xi)$, there exists a sequence of functions $\{\varphi_\alpha\}_{\alpha=1}^{\infty} \subset W^{1,p}(\Omega;\ R^m)$ such that

$$\varphi_\alpha \rightharpoonup 0 \quad \text{in } W^{1,p}(\Omega;\ R^m), \tag{2.4}$$

$$\bar{f}(x,s,\xi) = \lim_{\alpha\to\infty} \frac{1}{\text{meas}(\Omega)} \int_\Omega f(x,s,\xi + D\varphi_\alpha(x'))\,dx'. \tag{2.5}$$

It follows from (H1a) that

$$f(x,s,\xi + D\varphi_\alpha(x')) \ge -[a(x) + C(|s|^p + |\xi + D\varphi_\alpha(x')|^p)].$$

This gives

$$\int_\Omega f(x,s,\xi + D\varphi_\alpha(x'))\,dx'$$
$$\ge -\,[a(x) + C(|s|^p + 2^{p-1}|\xi|^p)]\,\text{meas}(\Omega) - 2^{p-1}\int_\Omega |D\varphi_\alpha(x')|^p\,dx'. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we obtain (2.3). □

Lemma 2.3. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a). Then, for every $(x,s,\xi) \in \Omega \times R^m \times R^{n\times m}$ the function $\tilde{f}$ defined by $\tilde{f}(x,s,\xi) = \lim_{\beta\to\infty} \hat{f}_\beta(x,s,\xi)$ satisfies*

$$-\infty < \tilde{f}(x,s,\xi) \le a(x) + C(|s|^p + |\xi|^p). \tag{2.7}$$

Proof. Since, by lemma 2.2, $\{\hat{f}_\beta\}$ is a nonincreasing sequence satisfying (2.2), it only remains to show that, for every $(x, s, \xi) \in \Omega \times R^m \times R^{n\times m}$, $\{\hat{f}_\beta(x, s, \xi)\}$ is bounded from below.

Fix $(x, s, \xi) \in \Omega \times R^m \times R^{n\times m}$, denote $g_\beta(D\varphi) = f_\beta(x, s, \xi + D\varphi) + \beta$. Then g_β, $\beta \in \mathbb{N}$, satisfy (H1) and (H2) and $\hat{g}_\beta = \hat{f}_\beta + \beta$. Recall that the sequentially weakly lower semicontinuous envelope of $G_\beta(\varphi) = \int_\Omega g_\beta(x, s, \xi + D\varphi(x'))\,dx'$ in $W^{1,p}(\Omega;\ R^m)$ is (see [4, 5])

$$\begin{aligned}\hat{I}_\beta(\varphi) &= \Gamma^{-}\text{-}\lim G_\beta(\varphi)\\ &= \min\{\liminf_{\alpha\to\infty} G_\beta(\varphi_\alpha) : \varphi_\alpha \rightharpoonup \varphi,\ \text{in } W^{1,p}(\Omega;\ R^m)\}.\end{aligned} \tag{2.8}$$

By lemma 2.1, (2.8) and theorem 1.1, there exists a sequence $\varphi_\alpha \in W^{1,p}(\Omega; R^m)$ such that

$$\varphi_\alpha \rightharpoonup 0 \qquad \text{in } W^{1,p}(\Omega;\ R^m), \tag{2.9}$$

$$\hat{g}_\beta(0)\,\text{meas}(\Omega) = \hat{I}_\beta(0) = \lim_{\alpha\to\infty} \int_\Omega g_\beta(D\varphi_\alpha(x'))\,dx'. \tag{2.10}$$

Since

$$\begin{aligned}\int_\Omega g_\beta(D\varphi_\alpha(x'))\,dx' &= \int_\Omega (f_\beta(x, s, \xi + D\varphi_\alpha(x')) + \beta)\,dx'\\ &\geq \int_\Omega f(x, s, \xi + D\varphi_\alpha(x'))\,dx' + \beta\,\text{meas}(\Omega),\end{aligned}$$

by (2.9), (2.10) and lemma 2.2, we have

$$\hat{f}_\beta(x, s, \xi) = (\hat{g}_\beta(0) - \beta) \geq \bar{f}(x, s, \xi) > -\infty \quad \forall \beta \in \mathbb{N}, \tag{2.11}$$

where $\bar{f}(x, s, \xi)$ is defined by (2.3). This completes the proof. □

Theorem 2.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a). Then*

$$\hat{f}(x, s, \xi) = \lim_{\beta\to\infty} \hat{f}_\beta(x, s, \xi) \quad \forall (x, s, \xi) \in \Omega \times R^m \times R^{n\times m}, \tag{2.12}$$

and

$$\int_{\Omega'} \hat{f}(x, u(x), v(x))\,dx = \lim_{\beta\to\infty} \int_{\Omega'} \hat{f}_\beta(x, u(x), v(x))\,dx \tag{2.13}$$

for every measurable subset $\Omega' \subset \Omega$, $u \in L^p(\Omega';\ R^m)$ and $v \in L^p(\Omega';\ R^{n\times m})$.

Proof. By lemma 2.3, $\tilde{f}(x,s,\xi) = \lim_{\beta\to\infty} \hat{f}_\beta(x,s,\xi)$ satisfies (2.7). We are going to show that $\tilde{f}(x,s,\cdot)$ is the quasiconvex envelope of $f(x,s,\cdot)$.

For fixed $(x,s,\xi) \in \Omega \times R^m \times R^{n\times m}$, by lemma 2.1 and the quasiconvexity of $\hat{f}_\beta(x,s,\xi)$ respect to ξ (see [1, 2, 3]),

$$\begin{aligned} \tilde{f}(x,s,\xi) &\le \hat{f}_\beta(x,s,\xi) \\ &\le \frac{1}{\text{meas}(\Omega)} \int_\Omega \hat{f}_\beta(x,s,\xi + D\varphi(x'))\,dx' \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned} \tag{2.14}$$

Let

$$g_\beta(x') = a(x) + C(|s|^p + |\xi + D\varphi(x')|^p) - \hat{f}_\beta(x,s,\xi + D\varphi(x')).$$

By lemma 2.1, $\{g_\beta\}$ is a nondecreasing sequence of nonnegative functions. By lemma 2.3,

$$\lim_{\beta\to\infty} g_\beta(x') = g(x')$$

with

$$g(x') = a(x) + C(|s|^p + |\xi + D\varphi(x')|^p) - \tilde{f}(x,s,\xi + D\varphi(x')).$$

Hence, by Beppo Levi's theorem [12, 13], we have

$$\int_\Omega g(x')\,dx' = \lim_{\beta\to\infty} \int_\Omega g_\beta(x')\,dx'. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\tilde{f}(x,s,\xi) \le \frac{1}{\text{meas}(\Omega)} \int_\Omega \tilde{f}(x,s,\xi + D\varphi(x'))\,dx' \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.16}$$

This implies that $\tilde{f}(x,s,\xi)$ is quasiconvex respect to ξ (see [1, 2, 3]). Thus, it follows from

$$\tilde{f}(x,s,\xi) \le \hat{f}_\beta(x,s,\xi) \le f_\beta(x,s,\xi), \quad \forall \beta \in \mathbb{N}$$

and hence $\tilde{f}(x,s,\xi) \le f(x,s,\xi)$ that

$$\tilde{f}(x,s,\xi) \le \hat{f}(x,s,\xi), \tag{2.17}$$

since $\hat{f}(x,s,\cdot)$ is the greatest quasiconvex function less than or equal to $f(x,s,\cdot)$. On the other hand, since

$$\hat{f}(x,s,\xi) \le f(x,s,\xi) \le f_\beta(x,s,\xi), \quad \forall \beta \in \mathbb{N},$$

we have $\hat{f}(x,s,\xi) \le \hat{f}_\beta(x,s,\xi)$ for all β and hence, by (2.7)

$$\hat{f}(x,s,\xi) \le \tilde{f}(x,s,\xi). \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$\hat{f}(x,s,\xi) = \tilde{f}(x,s,\xi) \quad \forall (x,s,\xi) \in \Omega \times R^m \times R^{n\times m}. \tag{2.19}$$

This proves (2.12).

Applying Beppo Levi's theorem [12, 13] to the sequence

$$g_\beta(x) = a(x) + C(|u(x)|^p + |v(x)|^p) - \hat{f}_\beta(x,u(x),v(x)), \quad \beta \in \mathbb{N},$$

which, by lemma 2.1, is nonnegative and nondecreasing and which, by (2.12), converges to

$$g(x) = a(x) + C(|u(x)|^p + |v(x)|^p) - \hat{f}(x,u(x),v(x))$$

as $\beta \to \infty$ for all $x \in \Omega'$, we obtain (2.13). □

Corollary 2.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a). Then*

$$\hat{F}(u;\, \Omega') = \lim_{\beta\to\infty} \hat{F}_\beta(u;\, \Omega') \tag{2.20}$$

for every open subset $\Omega' \subset \Omega$ and for all $u \in W^{1,p}(\Omega';\ R^m)$.

Proof. The conclusion follows directly from (2.13). □

3. Integral representation of $\Gamma^{-}\text{-}\lim F(u;\,\Omega)$

In section 2, the relationship between the quasiconvex envelopes of f and f_β is discussed. In this section, we will see how the sequentially weakly lower semicontinuous envelope of $F(\cdot;\,\Omega')$ relates to those of $F_\beta(\cdot;\,\Omega')$. First, recall that they are defined by (see [4, 5])

$$\Gamma^{-}\text{-}\lim F(u;\,\Omega') = \min\{\liminf_{\alpha\to\infty} F(u_\alpha;\,\Omega') : u_\alpha \rightharpoonup u \text{ in } W^{1,p}(\Omega';\ R^m)\}, \tag{3.1}$$

$$\Gamma^{-}\text{-}\lim F_\beta(u;\,\Omega') = \min\{\liminf_{\alpha\to\infty} F_\beta(u_\alpha;\,\Omega') : u_\alpha \rightharpoonup u \text{ in } W^{1,p}(\Omega';\ R^m)\}. \tag{3.2}$$

Lemma 3.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a) and (H2). Then*

$$\Gamma^{-}\text{-}\lim F_\beta(u;\,\Omega') = \int_{\Omega'} \hat{f}_\beta(x,u(x),Du(x))\,dx \tag{3.3}$$

for each $\beta \in \mathbb{N}$ and every open subset $\Omega' \subset \Omega$.

Proof. For each $\beta \in \mathbb{N}$, let $g_\beta = f_\beta + \beta$. It is easily seen that g_β is a Carathéodory function satisfying (H1) and (H2). Thus, by theorem 1.1,

$$\Gamma^{\text{-}}\text{-}\lim G_\beta(u;\,\Omega') = \int_{\Omega'} \hat{g}_\beta(x, u(x), Du(x))\, dx \tag{3.4}$$

for every open subset $\Omega' \subset \Omega$, where

$$G_\beta(u;\,\Omega') = \int_{\Omega'} g_\beta(x, u(x), Du(x))\, dx$$

and $\hat{g}_\beta(x, s, \cdot)$ is the quasiconvex envelope of $g_\beta(x, s, \cdot)$. Since (see [4, 5])

$$\Gamma^{\text{-}}\text{-}\lim F_\beta(u;\,\Omega') = \Gamma^{\text{-}}\text{-}\lim G_\beta(u;\,\Omega') - \beta\,\text{meas}(\Omega'),$$

$$\hat{f}_\beta(x, s, \xi) = \hat{g}_\beta(x, s, \xi) - \beta,$$

(3.3) follows from (3.4). □

Lemma 3.2. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ be a Carathéodory function which satisfies (H1a), (H1b) and (H2). Then*

$$\Gamma^{\text{-}}\text{-}\lim F(u;\,\Omega') = \lim_{\beta\to\infty} \Gamma^{\text{-}}\text{-}\lim F_\beta(u;\,\Omega') \tag{3.5}$$

for every open subset $\Omega' \subset \Omega$ and $u \in W^{1,p}(\Omega';\ R^m)$.

Proof. Let an open subset $\Omega' \subset \Omega$ and a function $u \in W^{1,p}(\Omega';\ R^m)$ be given. It follows from lemma 2.1 and lemma 3.1 that $\Gamma^{\text{-}}\text{-}\lim F_\beta(u;\,\Omega')$, $\beta \in \mathbb{N}$, are nonincreasing. Since obviously $\Gamma^{\text{-}}\text{-}\lim F(u;\,\Omega') \le \Gamma^{\text{-}}\text{-}\lim F_\beta(u;\,\Omega')$ for every $\beta \in \mathbb{N}$, we have

$$\Gamma^{\text{-}}\text{-}\lim F(u;\,\Omega') \le \lim_{\beta\to\infty} \Gamma^{\text{-}}\text{-}\lim F_\beta(u;\,\Omega'). \tag{3.6}$$

To show the inverse inequality, let $\{u_\alpha\}_{\alpha=1}^{\infty} \subset W^{1,p}(\Omega';\ R^m)$ be such that (see (3.1)

$$u_\alpha \rightharpoonup u \quad \text{in } W^{1,p}(\Omega';\ R^m), \tag{3.7}$$

$$\Gamma^{\text{-}}\text{-}\lim F(u;\,\Omega') = \lim_{\alpha\to\infty} F(u_\alpha;\,\Omega'). \tag{3.8}$$

For any $\epsilon > 0$, by (H1b), (3.7) and (3.8), there exists $\delta(\epsilon) > 0$ such that

$$\left|\int_E f^-(x, u_\alpha(x), Du_\alpha(x))\, dx\right| < \epsilon \quad \forall \alpha \in \mathbb{N}, \tag{3.9}$$

whenever $E \subset \Omega'$ is measurable and $\text{meas}(E) < \delta(\epsilon)$. Denote

$$\Omega'_{\beta,\alpha} = \{x \in \Omega' : f(x, u_\alpha(x), Du_\alpha(x)) \leq -\beta\}.$$

Then, (3.9) implies that

$$\text{meas}(\Omega'_{\beta,\alpha}) \to 0 \quad \text{uniformly for all } \alpha \text{ as } \beta \to \infty.$$

This in turn implies that for any $\epsilon > 0$ there exists $\beta(\epsilon) > 0$ such that

$$|\int_{\Omega'_{\beta,\alpha}} f^-(x, u_\alpha(x), Du_\alpha(x))\, dx| < \epsilon \quad \forall \alpha \in \mathbb{N} \text{ and } \beta > \beta(\epsilon). \tag{3.10}$$

Since

$$\begin{aligned} F(u_\alpha;\, \Omega') &= F_\beta(u_\alpha;\, \Omega') + (F(u_\alpha;\, \Omega') - F_\beta(u_\alpha;\, \Omega')) \\ &\geq F_\beta(u_\alpha;\, \Omega') + \int_{\Omega'_{\beta,\alpha}} f^-(x, u_\alpha(x), Du_\alpha(x))\, dx \quad \forall \alpha,\ \beta \in \mathbb{N}, \end{aligned}$$

by (3.8) and (3.10),

$$\Gamma^{-}\text{-}\lim F(u;\, \Omega') \geq \Gamma^{-}\text{-}\lim F_\beta(u;\, \Omega') - \epsilon \quad \forall \beta > \beta(\epsilon),$$

and hence, by the arbitrariness of $\epsilon > 0$, we conclude

$$\Gamma^{-}\text{-}\lim F(u;\, \Omega') \geq \lim_{\beta \to \infty} \Gamma^{-}\text{-}\lim F_\beta(u;\, \Omega').$$

This completes the proof. □

Theorem 3.1. *Let $f : \Omega \times R^m \times R^{n \times m} \to R$ be a Carathéodory function which satisfies (H1a), (H1b) and (H2). Then, for every open subset $\Omega' \subset \Omega$,*

$$\hat{F}(u;\, \Omega') = \Gamma^{-}\text{-}\lim F(u;\, \Omega') \quad \forall u \in W^{1,p}(\Omega';\ R^m). \tag{3.11}$$

In other words, $\hat{F}(\cdot;\, \Omega')$ is the sequentially weakly lower semicontinuous envelope of $F(\cdot;\, \Omega')$ in $W^{1,p}(\Omega';\ R^m)$.

Proof. The theorem follows from theorem 2.1, lemma 3.1 and lemma 3.2 by combing (2.13), (3.3) and (3.5). □

As a corollary of theorem 3.1, we have

Theorem 3.2. *Let $f : \Omega \times R^m \times R^{n \times m} \to R$ be a Carathéodory function which satisfies (H1a), (H1b) and (H2). Let $u_\alpha,\ u \in W^{1,p}(\Omega';\ R^m)$ be such that*

$$u_\alpha \rightharpoonup u \quad \text{in } W^{1,p}(\Omega';\ R^m).$$

Then

$$\int_{\Omega} \hat{f}(x, u(x), Du(x))\, dx \leq \liminf_{\alpha\to\infty} \int_{\Omega} \hat{f}(x, u_\alpha(x), Du_\alpha(x))\, dx. \tag{3.12}$$

That is the functional $\hat{F}(\cdot;\, \Omega)$ *is sequentially weakly lower semicontinuous on* $W^{1,p}(\Omega;\ R^m)$.

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