ELEMENT REMOVAL METHOD FOR SINGULAR MINIMIZERS IN PROBLEMS OF HYPERELASTICITY

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ABSTRACT. A numerical method called element removal method is applied to calculate singular minimizers in problems of hyperelasticity. The method overcomes the difficulty in finite element approximations caused by restrictions, such as $det(I + \nabla u) > 0$, on admissible functions and can avoid Lavrentiev phenomenon if it does occur in the problem. The convergence of the method is proved.

1. Introduction

In this paper, I apply a numerical method called element removal method, which was designed to tackle singular minimizers in variational problems involving Lavrentiev phenomenon,¹ to solve the boundary value problems of hyperelasticity which can be given by the problem of minimizing

$$I(u) = \int_{\Omega} W(x, I + \nabla u) \, dx - \int_{\Omega} f \cdot u \, dx - \int_{\partial \Omega_1} g \cdot u dx \tag{1.1}$$

in the set of admissible functions

$$\mathbf{A} = \{ u \in \mathbf{W}^{1,p}(\Omega) : adj(I + \nabla u) \in \mathbf{L}^{q}(\Omega), det(I + \nabla u) \in L^{r}(\Omega), \\ det(I + \nabla u) > 0, \quad a.e. \text{ in } \Omega, u = u_{0} \text{ on } \partial\Omega_{0} \}$$
(1.2)

where $\Omega \subset R^3$ is a bounded open set with Lipschitz continuous boundary $\partial \Omega = \partial \Omega_0 \cup \partial \Omega_1$ with $area(\partial \Omega_0) > 0$; $\mathbf{W}^{1,p}(\Omega) = (W^{1,p}(\Omega))^3, \mathbf{L}^q(\Omega) = (L^q(\Omega))^{3\times 3}$; and where $W : \overline{\Omega} \times R^{3\times 3} \to R$, the stored energy function of the hyperelastic material in question, satisfies the following hypotheses

(H1) Polyconvexity: There exists a continuous function $G : \overline{\Omega} \times R^9 \times R^9 \times R_+ \to R$ such that $G(x, \cdot, \cdot, \cdot)$ is convex and

$$W(x,F) = G(x,F,adj\ F,det\ F), \ \forall F \in M^3_+$$

Typeset by $\mathcal{A}_{\mathcal{M}}\!\mathcal{S}\text{-}T_{\!E}\!X$

where *adj* F is the cofactor matrix of F, *det* F is the determinant of F, and $M_{+}^{3} = \{3 \times 3 \text{ matrices with positive determinant }\};$

(H2) Coerciveness: There exist constants $p \ge 2, q \ge \frac{p}{p-1}, r > 1$ and $c_1 \in R, c_2 > 0$ such that

$$G(x, F, H, \delta) \ge c_1 + c_2(||F||^p + ||H||^q + \delta^r), \text{ for } F, H \in M^3_+, \delta > 0;$$

(H3) Behavior as $\det F \to 0^+$: For any sequence $F_n, H_n \in M^3_+, \delta_n > 0$,

$$G(x, F_n, H_n, \delta_n) \to +\infty$$
, if $||F_n|| + ||H_n|| + \delta_n \to +\infty$,
or $\delta_n \to 0^+$ as $n \to \infty$;

and where f, g are in such function spaces that the functionals

$$(f,u) = \int_{\Omega} f \cdot u \, dx,$$

and

$$(g,u) = \int_{\partial\Omega_1} g \cdot u \, dx$$

are continuous in $\mathbf{W}^{1,p}(\Omega)$.

The existence of an absolute minimizer of $I(\cdot)$ in **A** was established by Ball.^{2,3} The finite element method for the problem was investigated by Li.⁴ But in,⁴ the convergence result was proved under certain additional hypothesis on the stored energy function W, the growth condition, and certain regularity conditions on minimizers. These additional conditions may not be removed when a standard finite element method is applied to solve the problem, because the restrictions such as $det(I + \nabla u) > 0$ can not be guaranteed to be satisfied by finite element approximations to a function in **A**, even if the minimizers does not have Lavrentiev phenomenon. However, with the element removal method, which is described in §2, the convergence result can be obtained without any additional hypothesis other than (H1)-(H3) (see §3).

2. Description of the Method

To avoid the difficulty in finite element approximations caused by the restriction $det(I + \nabla u) > 0$ and singularities of a function in **A**, a modified restriction on the admissible set of functions in finite element function spaces is to be introduced.

Let

$$\hat{\phi}(\delta) = \min_{(x,F,H)\in\bar{\Omega}\times R^9\times R^9} G(x,F,H,\delta).$$
(2.1)

By (H1)–(H3), $\hat{\phi}$ is a well defined continuous function on R^1_+ satisfying

$$\lim_{\delta \to 0^+} \hat{\phi}(\delta) = +\infty,$$
$$\lim_{\delta \to +\infty} \hat{\phi}(\delta) = +\infty.$$

For fixed M > 1, let $\bar{\delta}_M \in (0, \frac{1}{M}], \, \tilde{\delta}_M \in [M, +\infty)$ be such that

$$\hat{\phi}(\bar{\delta}_M) = \min_{\substack{0 < \delta \le \frac{1}{M}}} \hat{\phi}(\delta)$$
$$\hat{\phi}(\tilde{\delta}_M) = \min_{\substack{\delta > M}} \hat{\phi}(\delta).$$

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Let

$$\bar{\phi}_M(\delta) = \begin{cases} \min_{\delta' \ge \delta} \hat{\phi}(\delta'), & \text{if } \delta > \tilde{\delta}_M, \\ \hat{\phi}(\delta), & \text{if } \bar{\delta}_M \le \delta \le \tilde{\delta}_M, \\ \min_{0 < \delta' \le \delta} \hat{\phi}(\delta'), & \text{if } 0 < \delta \le \bar{\delta}_M. \end{cases}$$

It is easy to varify that $\bar{\phi}_M$ is a continuous function on R^1_+ , and it is nonincreasing on $(0, \bar{\delta}_M)$ and nondecreasing on $(\tilde{\delta}_M, +\infty)$, and satisfies

$$\bar{\phi}_M(\delta) \le \hat{\phi}(\delta), \quad \forall \delta > 0,$$

$$\bar{\phi}_M(\delta) \to +\infty, \quad \text{as } \delta \to 0^+,$$

$$\bar{\phi}_M(\delta) \to +\infty, \quad \text{as } \delta \to +\infty.$$
(2.2)

Let

$$\phi_M(\delta) = \begin{cases} \bar{\phi}_M(\tilde{\delta}_M), & \text{if } \delta \ge \tilde{\delta}_M, \\ \bar{\phi}_M(\delta), & \text{if } \bar{\delta}_M \le \delta \le \tilde{\delta}_M, \\ \bar{\phi}_M(\frac{\bar{\delta}_M}{k}(1+(k-1)\exp(\delta\bar{\delta}_M^{-1}-1))), & \text{if } \delta \le \bar{\delta}_M. \end{cases}$$

where k > 1 is a constant. It is easy to show that $\phi_M : \mathbb{R}^1 \to \mathbb{R}^1$ is a bounded continuous function, and it is nonincreasing on $(-\infty, \overline{\delta}_M)$ and satisfies

$$\phi_M(\delta) \le \bar{\phi}_M(\delta), \quad \forall \delta > 0, \tag{2.3}$$

$$\phi_M(\bar{\delta}_M) = \bar{\phi}_M(\bar{\delta}_M),\tag{2.4}$$

$$\phi_M(0) = \bar{\phi}_M(\frac{1}{k}(1 + (k-1)e^{-1})\bar{\delta}_M), \qquad (2.5)$$

$$\phi_M(\delta) \to \bar{\phi}_M(\frac{1}{k}\bar{\delta}_M), \text{ as } \delta \to -\infty.$$
 (2.6)

For each fixed M > 0 define $\Phi_M : W^{1,p}(\Omega) \to R^1$ by

$$\Phi_M(u) = \int_{\Omega} (|\nabla u|^p + \phi_M(\det(I + \nabla u))) \, dx.$$
(2.7)

Then, for $u \in \mathbf{A}$, by (2.1)–(2.3), (H1), (H2) and (H3)

$$\Phi_M(u) \le |u|_{1,p}^p + \int_{\Omega} W(x, I + \nabla u) \, dx$$

$$\le C_1 + C_2 |u|_{1,p}^p + I(u), \qquad (2.8)$$

where $C_1 > 0, C_2 > 0$ are constants independent of u and M.

Lemma 2.1. $\Phi_M: W^{1,p}(\Omega) \to R^1$ is continuous.

Proof. Suppose otherwise, i.e. there is a function $u \in W^{1,p}(\Omega)$ and a sequence of functions $u_n \in W^{1,p}(\Omega)$ such that

$$u_n \longrightarrow u$$
, in $W^{1,p}(\Omega)$,

but

$$|\Phi_M(u_n) - \Phi_M(u)| > \epsilon_0$$

for a constant $\epsilon_0 > 0$ and all $n \ge 1$. Without loss of generality, it may be assumed that

$$\nabla u_n \longrightarrow \nabla u, \quad a.e. \text{ in } \Omega.$$

Hence

$$\det(I + \nabla u_n) \longrightarrow \det(I + \nabla u), \quad a.e. \text{ in } \Omega.$$
(2.9)

By the boundedness of ϕ_M , there is a $C_M > 0$ such that

$$|\phi_M(u_n(x))| \le C_M, \quad \forall x \in \Omega, \forall n \ge 1.$$
(2.10)

It follows from (2.9), (2.10) and Lebesgue's dominated convergence theorem that

$$\Phi_M(u_n) \longrightarrow \Phi_M(u)$$

This contradicts the assumption.

Throughout the rest of this paper, for simplicity, it is assumed that Ω is a polyhedron, $\partial \Omega_0$ consists of the faces of the polyhedron and $u_0 = 0$ on $\partial \Omega_0$. Let \mathbf{T}_h be regular triangulations of Ω with h being the mesh size.⁵

To introduce the method, define

$$\mathbf{A}_h = \{ u \in C(\Omega) : u|_K \text{ is affine, } \forall K \in \mathbf{T}_h; u|_{\partial\Omega_0} = 0 \}$$
(2.11)

$$E_M^1(u) = \{ x \in \Omega : |\nabla u| \ge M \}$$

$$(2.12)$$

$$E_M^2(u) = \{ x \in \Omega : det(I + \nabla u) \le \frac{1}{M} \}$$
(2.13)

$$E_M(u) = E_M^1(u) \cup E_M^2(u)$$
(2.14)

$$\mathbf{A}_{M,h}(C) = \{ u \in \mathbf{A}_h : \Phi_M(u) \le C \}$$
(2.15)

where in (2.5) C is a constant to be decided later.

The element removal method to solve the problem of minimizing I in \mathbf{A} consists in finding an approximate solution by solving a finite problem of minimizing the functional

$$I_{M,h}(u) = \int_{\Omega \setminus E_M(u)} W(x, I + \nabla u(x)) \, dx$$
$$- \int_{\Omega} f \cdot u \, dx - \int_{\partial \Omega_1} g \cdot u \, dx \tag{2.16}$$

in the set of admissible functions $\mathbf{A}_{M,h}(C)$. The idea is to remove from the integral the contributions of those elements on which the difference between the value of the integral at a function in $\mathbf{W}^{1,p}(\Omega)$ and that at its interpolation in \mathbf{A}_h can be out of control and to restrict the admissible functions so that the total volume of the removed elements is sufficiently small.

Theorem 2.1. If $\mathbf{A}(C) = \{u \in \mathbf{A} : C_1 + C_2 |u|_{1,p}^p + I(u) < C\}$ is not empty, then for any fixed M > 0, there exists h(M) > 0 such that for $h \in (0, h(M))$

$$\mathbf{A}_{M,h}(C) \neq \emptyset.$$

Proof. Suppose otherwise. Then, for certain M, say M_0 , there would be a sequence of $h_j, j = 1, 2, \cdots$, with $\lim_{j\to\infty} h_j = 0$ and $\mathbf{A}_{M,h_j}(C) = \emptyset$.

Let $u \in \mathbf{A}(C)$. By the approximation properties of the finite element spaces \mathbf{A}_{h_j} , there exists a sequence of functions $u_{h_j} \in \mathbf{A}_{h_j}$ such that (see ⁵)

$$u_{h_j} \longrightarrow u$$
, in $W^{1,p}(\Omega)$, as $j \to \infty$.

By lemma 2.1 and (2.8),

$$\Phi_{M_0}(u_{h_j}) \longrightarrow \Phi_{M_0}(u) < C, \text{ as } j \to \infty.$$

This contradicts to $\mathbf{A}_{M_0,h_j}(C) = \emptyset$ for all j. \Box

Let

$$C_0 = 1 + C_1 + \int_{\Omega} W(x, I) \, dx,$$

where C_1 is the constant defined in (2.8). Then we have

Corollary 2.1. For any M > 0, there exists h(M) > 0 such that

$$\mathbf{A}_{M,h}(C_0) \neq \emptyset, \quad \forall h \in (0, h(M)).$$

Proof. Since $0 \in \mathbf{A}(C_0)$, the conclusion follows from theorem 2.1.

Since meas $(E_M(v) \setminus E_M(v_n))$ does not necessarily converge to zero for $v_n \to v$ in $W^{1,\infty}(\Omega)$, we see that $I_{M,h}(\cdot)$ is not continuous, in fact not even lower semi-continuous in general, in $\mathbf{A}_{M,h}(C_0)$. Thus it is not clear whether the finite dimensional problem of minimizing $I_{M,h}(\cdot)$ in $\mathbf{A}_{M,h}(C_0)$ always has a solution. However, we have the following result.

Theorem 2.2. There exists a constant $M_0 > 1$ such that for $M > M_0$ and $h \in (0, h(M))$, where h(M) is defined by corollary 2.1, there exists a solution to the problem of minimizing $I_{M,h}$ on $\mathbf{A}_{M,h}(C_0)$.

Proof. Let $u_j \in \mathbf{A}_{M,h}(C_0)$ be a minimizing sequence of $I_{M,h}$ in $\mathbf{A}_{M,h}(C_0)$. Since $|u_j|_{1,p}$ is uniformly bounded, $u_j = 0$ on $\partial\Omega_0$, there exists a subsequence of u_j , again denoted by u_j , and a function $u \in \mathbf{A}_h$ such that

$$u_j \rightharpoonup u$$
, in $W^{1,p}(\Omega)$.

As \mathbf{A}_h is of finite dimension, this implies

$$u_i \to u, \quad \text{in } W^{1,\infty}(\Omega).$$
 (2.17)

By the continuity of Φ_M , u is in $\mathbf{A}_{M,h}(C_0)$.

$$\begin{split} I_{M,h}(u) &= I_{M,h}(u_n) + I_{M,h}(u) - I_{M,h}(u_n) \\ &= I_{M,h}(u_n) \\ &+ \int_{\Omega \setminus (E_M(u) \cup E_M(u_n))} (W(x, I + \nabla u) - W(x, I + \nabla u_n)) \, dx \\ &+ \int_{E_M(u_n) \setminus E_M(u)} W(x, I + \nabla u) \, dx \\ &- \int_{E_M(u) \setminus E_M(u_n)} W(x, I + \nabla u_n) \, dx \\ &- \int_{\Omega} f \cdot (u - u_n) \, dx - \int_{\partial \Omega_1} g \cdot (u - u_n) \, dx \\ &= I_{M,h}(u_n) + I_1 + I_2 + I_3 + I_4 + I_5. \end{split}$$
(2.18)

By (H1), there is a constant C(M) > 0 such that

$$W(x, I + \nabla u(x)) < C(M), \forall x \in \Omega \setminus E_M(u),$$
(2.19)

$$W(x, I + \nabla u_n(x)) < C(M), \forall x \in \Omega \setminus E_M(u_n).$$
(2.20)

It follows from (H1), (2.17), (2.19), (2.20) and Lebesgue's dominated convergence theorem that

$$I_1 \longrightarrow 0, \quad \text{as } n \to \infty.$$
 (2.21)

By (2.17), we also have

$$I_4 \longrightarrow 0, \quad \text{as } n \to \infty.$$
 (2.22)

$$I_5 \longrightarrow 0, \quad \text{as } n \to \infty.$$
 (2.23)

Noticing that

$$\Omega \setminus E_M(u) = (\Omega \setminus E_{M-1}(u)) \cup_{i=1}^{\infty} (E_{M-\frac{1}{i}}(u) \setminus E_{M-\frac{1}{i+1}}(u)),$$

we have

$$meas(\bigcup_{i=k}^{\infty} (E_{M-\frac{1}{i}}(u) \setminus E_{M-\frac{1}{i+1}}(u)))$$

= $meas(E_{M-\frac{1}{k}}(u) \setminus E_M(u)) \to 0$, as $k \to \infty$, (2.24)

and

$$I_{2} = \int_{E_{M}(u_{n})\setminus E_{M-\frac{1}{k}}(u)} W(x, I + \nabla u) dx$$
$$+ \int_{E_{M}(u_{n})\cap (E_{M-\frac{1}{k}}(u)\setminus E_{M}(u))} W(x, I + \nabla u) dx$$
$$= I_{21} + I_{22}$$

By (2.19) and (2.24), for any $\epsilon > 0$, there is a $k(\epsilon) > 0$ such that

$$|I_{22}| < \epsilon/2, \forall k > k(\epsilon). \tag{2.25}$$

For fixed $k > k(\epsilon)$, by (2.17)

$$meas(E_M(u_n) \setminus E_{M-\frac{1}{k}}(u)) \to 0, \quad \text{as } n \to \infty.$$
(2.26)

Hence, by (2.19), for any $\epsilon > 0$ there is a $n(\epsilon, k) > 0$ such that

$$|I_{21}| < \epsilon/2, \quad \forall n > n(\epsilon, k). \tag{2.27}$$

It follows from (2.25) and (2.27) that

$$I_2 \longrightarrow 0, \quad \text{as } n \to \infty.$$
 (2.28)

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Now, we estimate I_3 .

$$-I_{3} = \int_{E_{M}(u) \setminus E_{M}(u_{n})} W(x, I + \nabla u_{n}) dx$$

=
$$\int_{(E_{M}(u) \setminus E_{M+1}(u)) \setminus E_{M}(u_{n})} W(x, I + \nabla u_{n}) dx$$

+
$$\int_{E_{M+1}(u) \setminus E_{M}(u_{n})} W(x, I + \nabla u_{n}) dx$$

=
$$I_{31} + I_{32}.$$
 (2.29)

By (2.17),

$$meas(E_{M+1}(u) \setminus E_M(u_n)) \to 0, \text{ as } n \to \infty,$$

hence, it follows from (2.20) that

$$|I_{32}| \longrightarrow 0, \quad \text{as } n \to \infty.$$
 (2.30)

For $x \in (E_M(u) \setminus E_{M+1}(u)) \setminus E_M(u_n)$, by (2.17), there exists a constant N(M) > 0, which is independent of x, such that

$$x \in E_{M-1}(u_n) \setminus E_M(u_n), \quad \forall n > N(M).$$

$$(2.31)$$

On the other hand, by (H2) and (H3), there exists $M_0 > 1$ such that for $F \in M_+^3$

$$W(x,F) \ge 0$$
, if $|F-I| \ge M_0 - 1$, or det $F \le \frac{1}{M_0 - 1}$. (2.32)

Thus, for $M \ge M_0$, we have

$$W(x, I + \nabla u_n) \ge 0, \quad \forall x \in E_{M-1}(u_n) \setminus E_M(u_n).$$

This implies

$$I_{31} \ge 0$$
, for $M > M_0$ and $n > N(M)$. (2.33)

It follows from (2.30) and (2.33) that for $M > M_0$

$$I_3 \le -I_{32} \to 0, \quad \text{as } n \to \infty.$$
 (2.34)

By (2.18) and (2.34), for $M > M_0$ and n > N(M)

$$I_{M,h}(u) \le I_{M,h}(u_n) + I_1 + I_2 - I_{32} + I_4 + I_5.$$
(2.35)

By (2.21), (2.22), (2.23), (2.28) and (2.34), and by passing to the limit in (2.35), we have

$$I_{M,h}(u) \leq \lim_{n \to \infty} I_{M,h}(u_n) = \inf_{v \in \mathbf{A}_{M,h}(C_0)} I_{M,h}(v).$$

This completes the proof.

The following lemma is essential to the method.

Lemma 2.2. For any $\gamma > 0$, there exists $M(\gamma) > 0$ such that

$$meas(E_M(u)) < \gamma, \quad \forall u \in \mathbf{A}_{M,h}(C_0), \tag{2.36}$$

for all $M > M(\gamma)$.

Proof. Let $u \in \mathbf{A}_{M,h}(C_0)$, then

$$\begin{split} C_0 &\geq \Phi_M(u) \\ &\geq \int_{E_M^1(u)} |\nabla u|^p \, dx + \int_{E_M^2(u)} \phi_M(\det(I + \nabla u)) \, dx \\ &\geq M^p \, meas(E_M^1(u)) + \min_{\delta \leq \frac{1}{M}} \phi_M(\delta) \, meas(E_M^2(u)) \\ &\geq \min\{M^p, \hat{\phi}(\bar{\delta}_M)\} \, meas(E_M(u)) \end{split}$$

Since $\bar{\delta}_M \in (0, \frac{1}{M})$,

$$\lim_{M \to \infty} \hat{\phi}(\bar{\delta}_M) = +\infty$$

The conclusion of the lemma now follows.

3. Convergence Theorem

For the element removal method described in §2, we have the following result.

Theorem 3.1. Let W satisfy (H1)-(H3). Then, for any $\epsilon > 0$ there exist $M(\epsilon) > 1$ and $h(\epsilon, M) > 0$ such that for $M > M(\epsilon)$ and $0 < h < h(\epsilon, M)$ there exists $u_h \in \mathbf{A}_{M,h}(C_0)$ satisfying

$$I_{M,h}(u_h) \le \inf_{v \in \mathbf{A}} I(v) + \epsilon.$$
(3.1)

Moreover, we can find sequences $M_j > 0, h_j > 0, u_j \in \mathbf{A}_{M_j,h_j}(C_0), j = 1, 2, \cdots$, with u_j being the minimizers of I_{M_j,h_j} in $\mathbf{A}_{M_j,h_j}(C_0)$, a nonincreasing sequence of measurable subsets $E_k \subset \Omega$ with

$$\lim_{k \to \infty} meas \ (E_k) = 0, \tag{3.2}$$

and $\bar{u} \in \mathbf{A}$ such that

$$I_{M_j,h_j}(u_j) \to \inf_{v \in \mathbf{A}} I(v) = I(\bar{u}); \tag{3.3}$$

$$u_j \rightharpoonup \bar{u} \quad in \ \mathbf{W}^{1,p}(\Omega);$$
 (3.4)

$$adj(I + \nabla u_j) \rightharpoonup adj(I + \nabla \bar{u}), \quad in \mathbf{L}^q(\Omega \setminus E_k), \text{ for each fixed } k;$$

$$(3.5)$$

$$det(I + \nabla u_j) \rightharpoonup det(I + \nabla \bar{u}), \quad in \ L^r(\Omega \setminus E_k), \text{ for each fixed } k, \tag{3.6}$$

where \rightarrow denotes weak convergence.

To prove the first part of the theorem, we need the following lemmas.

Let u be a minimizer of I in **A**. Let $\bar{u}_j \in \mathbf{A}_{h_j}$ be a sequence, which is not necessary a minimizing sequence, satisfying $h_j > 0$, $\lim_{j\to\infty} h_j = 0$ and

$$\|\bar{u}_j - u\|_{1,p} \to 0.$$
 (3,7)

Then, we have

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Lemma 3.1. For any $\epsilon > 0$, we can find $\gamma(\epsilon) > 0$ such that

$$\int_{\Omega'} |W(x, I + \nabla u)| \, dx < \epsilon$$

$$\forall \Omega' \subset \Omega, \ meas(\Omega') < \gamma(\epsilon). \tag{3.8}$$

Lemma 3.2. For any $\gamma > 0$ there exist $\overline{M}_1(\gamma) > 1$ and $\overline{N}_1(\gamma, M) > 1$ such that

$$\bar{u}_j \in \mathbf{A}_{M,h_j}(C_0), \tag{3.9}$$

$$meas \ E_M(\bar{u}_i) < \gamma, \tag{3.10}$$

for all $M \geq \overline{M}_1(\gamma)$ and $j \geq \overline{N}_1(\gamma, M)$.

Proof. The lemma follows directly from lemma 2.2 and theorem 2.1. $\hfill \Box$

Corollary 3.1. For any $\epsilon > 0$ there exist $M_1(\epsilon) > 1$ and $N_1(\epsilon, M) > 1$ such that

$$\int_{E_M(\bar{u}_j)} |W(x, I + \nabla u)| \, dx < \epsilon$$

$$\forall M \ge M_1(\epsilon), \text{ and } j \ge N_1(\epsilon, M). \tag{3.11}$$

Proof. (3.11) follows from lemma 3.1 and lemma 3.2 by taking $M_1(\epsilon) = \overline{M}_1(\gamma(\epsilon))$ and $N_1(\epsilon, M) = \overline{N}_1(\gamma(\epsilon), M)$.

Lemma 3.3. For given $\epsilon > 0$ and M > 1 there exists $\gamma(\epsilon) > \eta(\epsilon, M) > 0$ such that

$$\int_{\Omega'} |W(x, I + \nabla \bar{u}_j)| \, dx < \epsilon$$

$$\forall \Omega' \subset \Omega \setminus E_M(\bar{u}_j), \ meas(\Omega') < \eta(\epsilon, M). \tag{3.12}$$

Proof. By (H1) and (2.12) the integrands in (3.12) are bounded by a function of M. Hence the lemma follows.

Lemma 3.4. For any $\epsilon > 0$ let $M \ge M_1(\epsilon)$, then there exists $N(\epsilon, M) > 1$ such that

$$\left|\int_{\Omega \setminus E_{M}(\bar{u}_{j})} W(x, I + \nabla \bar{u}_{j}) \, dx - \int_{\Omega} f \cdot \bar{u}_{j} \, dx - \int_{\partial \Omega_{1}} g \cdot \bar{u}_{j} \, dx - \inf_{v \in \mathbf{A}} I(v)\right| < 5 \epsilon$$
$$\forall j \ge N(\epsilon, M). \tag{3.13}$$

Proof.

$$\begin{split} \int_{\Omega \setminus E_M(\bar{u}_j)} W(x, I + \nabla \bar{u}_j) \, dx &- \int_{\Omega} f \cdot \bar{u}_j \, dx - \int_{\partial \Omega_1} g \cdot \bar{u}_j \, dx \\ = &I(u) + \int_{\Omega \setminus (E_{M+1}(u) \cup E_M(\bar{u}_j))} [W(x, I + \nabla \bar{u}_j) \\ &- W(x, I + \nabla u)] \, dx \\ &+ \int_{E_{M+1}(u) \setminus E_M(\bar{u}_j)} [W(x, I + \nabla \bar{u}_j) \\ &- W(x, I + \nabla u)] \, dx \\ &- \int_{E_M(\bar{u}_j)} W(x, I + \nabla u) \, dx \\ &- \int_{\Omega} f \cdot (\bar{u}_j - u) \, dx - \int_{\partial \Omega_1} g \cdot (\bar{u}_j - u) \, dx \\ &= \inf_{v \in \mathbf{A}} I(v) + I_1 + I_2 + I_3 + I_4 + I_5. \end{split}$$
(3.14)

It follows from (3.7) and (H1) that there exists $N_2(\epsilon, M) > 0$ such that

$$|I_1| < \epsilon$$
, if $j \ge N_2(\epsilon, M)$.

Let $\eta(\epsilon, M)$ be as in lemme 3.3. It follows from (3.7) that there exists $N_3(\epsilon, M) > 1$ such that

meas $(E_{M+1}(u) \setminus E_M(\bar{u}_j)) < \eta(\epsilon, M)$, if $j \ge N_3(\epsilon, M)$.

Thus, by lemma 3.1 and lemma 3.3, we have

$$|I_2| < 2 \epsilon$$
, if $j \ge N_3(\epsilon, M)$.

By corollary 3.1, we have

$$|I_3| < \epsilon$$
, if $M \ge M_1(\epsilon)$, and $j \ge N_1(\epsilon, M)$

It follows from (3.7) that there exists $N_4(\epsilon) > 0$ such that

$$|I_4 + I_5| < \epsilon$$
, if $j \ge N_4(\epsilon)$.

Now, by taking

$$N(\epsilon, M) = \max\{N_1(\epsilon, M), N_2(\epsilon, M), N_3(\epsilon, M), N_4(\epsilon)\},\$$

we have (3.13).

Proof of Theorem 3.1. Since (3.1) i.e. the first part of the theorem is a direct conclusion of Lemma 3.1 – Lemma 3.4, we only need to establish (3.2) - (3.6).

Take a sequence $\gamma_j > 0$ such that

$$\sum_{j=1}^{\infty} \gamma_j < meas(\Omega). \tag{3.15}$$

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By lemma 2.2 and theorem 2.1, there exist $\overline{M}(\gamma) > 1$ and h(M) > 0 such that $\mathbf{A}_{M,h}(C_0) \neq \emptyset$ and

$$meas(E_M(v)) < \gamma_j \tag{3.16}$$

for all $v \in \mathbf{A}_{M,h}(C_0)$ provided that $M \ge \overline{M}(\gamma_j), h \in (0, h(M)).$

Take a sequence $\epsilon_j > 0$ satisfying $\lim_{j \to \infty} \epsilon_j = 0$. Take $M_j > \max\{\overline{M}(\gamma_j),$

 $M(\epsilon_j)$ and $0 < h_j < \min\{h(M_j), h(\epsilon_j, M_j)\}$. Let $u_j \in \mathbf{A}_{M_j, h_j}(C_0), j = 1, 2, \cdots$, be the minimizers of I_{M_j, h_j} in $\mathbf{A}_{M_j, h_j}(C_0)$. By (3.1), we have

$$I_{M_j,h_j}(u_j) \le \inf_{v \in \mathbf{A}} I(v) + \epsilon_j, \quad \forall j.$$
(3.17)

Let

$$E_k = \bigcup_{j=k}^{\infty} E_{M_j}(u_j), \quad k = 1, 2, \cdots.$$
 (3.18)

By (3.15) and (3.16)

$$\lim_{k \to \infty} meas(E_k) = 0. \tag{3.19}$$

By (H2), (H3) and (2.15), we have

$$\|u_j\|_{1,p} \le C, \quad \forall j \tag{3.20}$$

$$\|adj (I + \nabla u_j)\|_q \le C, \text{ on } \Omega \setminus E_k, \text{ for each } k, \text{ and } \forall j \ge k$$
(3.21)

$$\|\det (I + \nabla u_j)\|_q \le C$$
, on $\Omega \setminus E_k$, for each k , and $\forall j \ge k$ (3.22)

 $det (I + \nabla u_j) > 0, \text{ a.e. in } \Omega \setminus E_k, \text{ for each } k, \text{ and } \forall j \ge k$ (3.23)

for some constant C > 0. Thus, by the results in,^{2,3} there is a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, and a function $\bar{u} \in \mathbf{W}^{1,p}(\Omega)$ such that (3.2) - (3.6) hold.

By (H1), (H2) and (3.1) - (3.6), we have

$$\int_{\Omega \setminus E_k} W(x, I + \nabla \bar{u}) \, dx - \int_{\Omega} f \cdot \bar{u} \, dx - \int_{\partial \Omega_1} g \cdot \bar{u} \, dx$$

$$\leq \underline{\lim}_{j \to \infty} I_{M_j, h_j}(u_j)$$

$$\leq \inf_{v \in \mathbf{A}} I(v), \qquad \text{for each } k. \qquad (3.24)$$

This and (H3) imply that

 $det(I + \nabla \bar{u}) > 0$, a.e. in $\Omega \setminus E_k$, for each k,

and hence

$$det(I + \nabla \bar{u}) > 0, \text{ a.e. in } \Omega. \tag{3.25}$$

It follows from (3.5), (3.6), (3.21) and (3.22) that

$$adj (I + \nabla \bar{u}) \in \mathbf{L}^{q}(\Omega), \quad det (I + \nabla \bar{u}) \in L^{r}(\Omega).$$
(3.26)

This and (3.25) imply that $\bar{u} \in \mathbf{A}$. Let $k \to \infty$ in (3.24), by (3.19), (H2) and passing to the limit, we conclude that

$$I(\bar{u}) = \inf_{v \in \mathbf{A}} I(v). \tag{3.27}$$

This completes the proof.

Remark. If a strict polyconvexity assumption ⁶ is made on the stored energy function W(x, F), the weak convergence in (3.4) can be converted into a strong one.

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