# ERROR ANALYSIS OF A DUAL-PARAMETRIC BI-QUADRATIC FEM IN CAVITATION COMPUTATION IN ELASTICITY* 

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#### Abstract

The orientation-preservation conditions and approximation errors of a dual-parametric bi-quadratic finite element method for the computation of both radially-symmetric and general nonsymmetric cavity solutions in nonlinear elasticity are analyzed. The analytical results allow us to establish, based on an error equi-distribution principle, an optimal meshing strategy for the method in cavitation computation. Numerical results are in good agreement with the analytical results.


Key words. error analysis, dual-parametric bi-quadratic finite element, cavitation, optimal meshing strategy

AMS subject classifications. 65N15, 65N30, 65D05, 74B20, 74G15, 74 M 99

1. Introduction. Void formation on nonlinear elastic bodies under hydrostatic tension was observed and analyzed through a defect model by Gent and Lindley [8]. Ball [2] established a perfect model and studied a class of bifurcation problems in nonlinear elasticity, in which voids form in an intact body so that the total stored energy of the material is minimized in a class of radially-symmetric deformations. The work stimulated an intensive study on various aspects of radially-symmetric cavitations (see e.g., Sivaloganathan [19], Stuart [25], and a review paper by Horgan and Polignone [10] among many others).

Müller and Spector [16] later developed a general existence theory in nonlinear elasticity that allows for cavitation, which is not necessarily radially symmetric, by adding a surface energy term. Sivaloganathan and Spector [21] deduced the existence of hole creating deformations without the need for the surface energy term under the assumption that the points (a finite number) at which the cavities can form are prescribed. Optimal locations where cavities can arise are also studied analytically [22, 23] and numerically [14].

Numerically computing cavities based on the perfect model of Ball is very challenging, due to the so-called Lavrentiev phenomenon [11]. Though there are numerical methods developed to overcome the Lavrentiev phenomenon in some nonlinear elasticity problems $[1,3,12,17,18]$, they do not seem to be powerful enough for the cavitation problem. On the other hand, some numerical methods (see e.g., $[13,14,26]$ ) have been successfully developed for cavitation computation on general domains with single or multiple prescribed defects, based on the defect model or the regularized perfect model [9, 19, 20]. In these models, one considers to minimize the total elastic energy of the form

$$
\begin{equation*}
E(u)=\int_{\Omega_{\rho}} W(\nabla u(x)) d x \tag{1.1}
\end{equation*}
$$

in a set of admissible functions

$$
\begin{equation*}
U=\left\{u \in W^{1, p}\left(\Omega_{\rho} ; \mathbb{R}^{n}\right) \text { is one-to-one a.e. : }\left.u\right|_{\Gamma_{0}}=u_{0}, \operatorname{det} \nabla u>0 \text { a.e. }\right\} \tag{1.2}
\end{equation*}
$$

[^0]where $\Omega_{\rho}=\Omega \backslash \bigcup_{i=1}^{K} B_{\rho_{i}}\left(a_{i}\right) \subset \mathbb{R}^{n}(n=2,3)$ denotes the region occupied by an elastic body in its reference configuration, $B_{\rho_{i}}\left(a_{i}\right)=\left\{x \in \mathbb{R}^{n}:\left|x-a_{i}\right|<\rho_{i}\right\}$ are the pre-existing defects of radii $\rho_{i}$ centered at $a_{i}$, and where in (1.1) $W: M_{+}^{n \times n} \rightarrow \mathbb{R}^{+}$ is the stored energy density function of the material, $M_{+}^{n \times n}$ being the set of $n \times n$ matrices with positive determinant. In (1.2) $\Gamma_{0}$ denotes the boundary of $\Omega$.

The Euler-Lagrange equation of the minimization problem (1.1) is (see [13]):

$$
\begin{align*}
& \operatorname{div}\left(D_{F} W(\nabla u)\right)=0, \text { in } \Omega_{\rho}  \tag{1.3}\\
& D_{F} W(x, \nabla u) \nu=0, \text { on } \cup_{i=1}^{K} \partial B_{\rho_{i}}\left(a_{i}\right),  \tag{1.4}\\
& \quad u(x)=u_{0}(x), \text { on } \Gamma_{0} . \tag{1.5}
\end{align*}
$$

A typical class of the stored energy densities considered in the cavitation problem is the polyconvex isotropic functions of the form

$$
\begin{align*}
W(F) & =\omega|F|^{p}+g(\operatorname{det} F) \\
& =\omega\left(v_{1}^{2}+\cdots+v_{n}^{2}\right)^{p / 2}+g\left(v_{1} \cdots v_{n}\right), \quad \forall F \in M_{+}^{n \times n} \tag{1.6}
\end{align*}
$$

where $\omega>0$ is a material constant, $p \in(n-1, n), v_{1}, \ldots, v_{n}$ are the singular values of the deformation gradient $F$, and $g:(0, \infty) \rightarrow(0, \infty)$ is a $C^{2}$, strictly convex function satisfying

$$
\begin{equation*}
g(\delta) \rightarrow+\infty \text { as } \delta \rightarrow 0, \text { and } \frac{g(\delta)}{\delta} \rightarrow+\infty \text { as } \delta \rightarrow+\infty \tag{1.7}
\end{equation*}
$$

For the energy density of the form (1.6), one has:

$$
\begin{equation*}
D_{F} W(\nabla u)=p \omega|\nabla u|^{p-2} \nabla u+g^{\prime}(\operatorname{det} \nabla u) \operatorname{adj} \nabla u^{\mathrm{T}} \tag{1.8}
\end{equation*}
$$

One of the main difficulties in the computation of the growth of voids is the constraint of orientation preservation, i.e., $\operatorname{det} \nabla u>0$, for extremely large anisotropic finite element deformations. For the conforming piecewise affine finite element, this requirement demands an unbearably large amount of degrees of freedom ([26]). In $[13,14,26]$, other finite element methods are proposed to overcome this difficulty, and these methods have shown significant numerical success in the cavitation computation. In particular, Su and $\mathrm{Li}[24]$ analyzed the iso-parametric quadratic finite element method applied in [14]. Even though limited to the radially-symmetric cavitation solutions, the result, the first of its kind to our knowledge, nevertheless quantifies the theoretical as well as practical advantages of the method.

In this paper, we will introduce and analyze a dual-parametric bi-quadratic rectangular finite element method for the cavitation computation, including both radiallysymmetric and general nonsymmetric voids' growth, and establish a meshing strategy, which is optimal in the sense that the total number of degrees of freedom is minimized under certain mild constraints. It turns out that, for the cavitation computation, the dual-parametric bi-quadratic rectangular finite element method is definitely much more efficient than the iso-parametric quadratic triangular finite element method, especially when the prescribed defects are very small. In fact, in the radially-symmetric case, the optimal mesh of the new method is essentially solely determined by the approximation accuracy, while the orientation-preservation condition plays a leading role in the iso-parametric finite element method in the vicinity of the void.

We would like to point out that, even though our analysis in the present paper is focused on the problem defined on the unit ball with only one-prescribed-defect in the
center, the result should work also for general multiple-prescribed-defects problems, since in general the cavitation solution is highly stressed only locally in a neighborhood of each defect, where our method applies, while elsewhere the stress field is finite and the standard method of finite element analysis applies. The important issue of the error estimate for the finite element cavitation solutions is not touched here in this work and it remains, to the knowledge of the authors, an open problem, even for the radially-symmetric case where the cavitation solution can be characterized by a boundary value problem of an ordinary differential equation. The main difficulties are: (1) the elastic energy $W(F)$ is non-convex, and lacks certain kind of coerciveness; (2) there is no monotonicity or other useful structures in the stress $D_{F} W(F) ;(3)$ the material subjects to very large anisotropic deformation. For more discussions in this aspect, we refer to $[4,5,6,7]$. However, based on our numerical results, it seems reasonable to conjecture that, with some more delicate manipulations and calculations than we have done here, such a result could be established.

The structure of the paper is as follows. In § 2, we introduce the dual-parametric bi-quadratic rectangular finite element. § 3 is devoted to deriving the orientationpreservation conditions on the mesh distributions. In $\S 4$, we present some results on the interpolation errors of the cavitation solutions. An optimal meshing strategy is established in $\S 5$. Numerical results are presented in $\S 6$. The paper is ended with some concluding remarks in § 7.
2. Dual-parametric bi-quadratic finite element. For simplicity, we restrict ourselves to a simplified problem with $\Omega_{\epsilon_{0}}=B_{1}(0) \backslash B_{\epsilon_{0}}(0)$ in $\mathbb{R}^{2}$. Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be a standard bi-quadratic rectangular element as shown in Figure 1(a) (here $n=2$ ), where $\hat{P}=Q_{2}(\hat{T}):=\left\{u\left(\hat{x}_{1}, \hat{x}_{2}\right)=\sum_{0 \leq i, j \leq 2} c_{i j} \hat{x}_{1}^{i} \hat{x}_{2}^{j}\right\}, \hat{\Sigma}=\left\{\hat{p}\left(\hat{a}_{i}\right), 0 \leq i \leq 8\right\}$. For a given set of four points $a_{i}=\left(R_{i} \cos \theta_{i}, R_{i} \sin \theta_{i}\right), 0 \leq i \leq 3$ satisfying $R_{0}=R_{3}<R_{1}=R_{2}$, $\theta_{0}=\theta_{1}<\theta_{2}=\theta_{3}$, define $F_{T}: \hat{T} \rightarrow \mathbb{R}^{2}$ as

$$
\left\{\begin{array}{l}
R=R_{0}+\frac{\hat{x}_{1}+1}{2}\left(R_{1}-R_{0}\right)  \tag{2.1}\\
\theta=\theta_{0}+\frac{\hat{x}_{2}+1}{2}\left(\theta_{3}-\theta_{0}\right) \\
x_{1}=R \cos \theta, x_{2}=R \sin \theta
\end{array}\right.
$$

It is easily seen that $F_{T}$ is an injection, thus $T=F_{T}(\hat{T})$ defines an element. Now define the dual-parametric bi-quadratic finite element $\left(T, P_{T}, \Sigma_{T}\right)$ as follows

$$
\left\{\begin{array}{l}
T=F_{T}(\hat{T})  \tag{2.2}\\
P_{T}=\left\{p: T \rightarrow \mathbb{R}^{2} \mid p=\hat{p} \circ F_{T}^{-1}, \hat{p} \in \hat{P}\right\} \\
\Sigma_{T}=\left\{p\left(a_{i}\right), a_{i}=F_{T}\left(\hat{a}_{i}\right), 0 \leq i \leq 8\right\}
\end{array}\right.
$$

3. On the orientation-preservation conditions. Let $\mathcal{J}$ be a subdivision on $\Omega_{\epsilon_{0}}$ as Figure 2(a). A typical curved element in a prescribed circular ring with inner radius $\epsilon$ and thickness $\tau$ is shown in Figure 2(b). Let $N$ be the number of the evenly spaced elements on each layer. Then, the dual-parametric bi-quadratic finite element function space is given by

$$
\left.X_{h}:=\left\{u_{h} \in C\left(\bar{\Omega}_{\epsilon_{0}}\right):\left.u_{h}\right|_{T} \in P_{T}, u_{h}(x)=u_{0}(x), \forall x \in \Gamma_{0} \bigcap\left(\bigcup_{T \in \mathcal{J}} \Sigma_{T}\right)\right\}\right\}
$$



Fig. 1. The dual-parametric element.
where $\Gamma_{0}=\partial B_{1}(0)$.


Fig. 2. The subdivision of the mesh.
We are concerned with the orientation preservation of large expansion dominant finite element deformations around a small prescribed void. Without loss of generality, we restrict ourselves to the curved rectangular element as shown in Figure 2(b), for which $F_{T}$ can be simplified as

$$
\left\{\begin{array}{l}
R=\epsilon+\frac{\hat{x}_{1}+1}{2} \tau,  \tag{3.1}\\
\theta=\frac{\pi}{N} \hat{x}_{2}, \\
x_{1}=R \cos \theta, x_{2}=R \sin \theta .
\end{array}\right.
$$

It is easily verified that $\operatorname{det} \nabla x=\operatorname{det} \frac{\partial F_{T}}{\partial \hat{x}}=\frac{R \pi \tau}{2 N}>0$.

Let $u$ be the cavitation solution, and let $\mathcal{J}$ be a given mesh (see Figure 2(a)) consisting of well defined curved rectangular elements. To have $u$ well resolved by functions in the finite element function space defined on $\mathcal{J}$, a necessary condition is that the finite element interpolation function $\Pi u(x)$ is an admissible function, i.e., $\operatorname{det} \nabla \Pi u(x)>0$ on each of the curved rectangular elements. We will investigate in this section the conditions that ensure $\operatorname{det} \nabla \Pi u(x)>0$ for smooth deformations $u(x)=\left(u_{1}(x), u_{2}(x)\right)$ defined on $\Omega_{\epsilon_{0}}$. Since $\operatorname{det} \nabla \Pi u(x) \cdot \operatorname{det} \nabla x=\operatorname{det} \frac{\partial \Pi u}{\partial \hat{x}}$, it suffices to ensure det $\frac{\partial \Pi u}{\partial \hat{x}}>0$. For simplicity, we denote $\Omega_{(\epsilon, \tau)}=\{x: \epsilon \leq|x| \leq \epsilon+\tau\}$.

Theorem 3.1. Suppose $x=(R \cos \theta, R \sin \theta), u(x)=\left(u_{1}, u_{2}\right)=(r \cos \phi, r \sin \phi)$, where $r=r(R, \theta), \phi=\phi(R, \theta)$ are smooth functions in the domain $B_{1}(0) \backslash\{0\}$ satisfying det $\frac{\partial u}{\partial x} \geq c>0$ and that the derivatives $\frac{\partial^{i+j} r}{\partial R^{i} \partial \theta^{j}}, \frac{\partial^{i+j} \phi}{\partial R^{i} \theta^{j}}, i+j \leq 4$, are bounded. Then, there exists a positive constant $\tau_{0}=C_{1} \epsilon^{1 / 2}$ and an integer $N_{0}=C_{2} \epsilon^{-1 / 2}$, such that $\operatorname{det} \frac{\partial \Pi u}{\partial x}>0$ on the circular annulus $\Omega_{(\epsilon, \tau)}$ if $\tau \leq \tau_{0}$ and $N \geq N_{0}$, where $C_{1}, C_{2}$ depend on $c$ and $\left\|\frac{\partial^{3} u_{i}}{\partial R^{3}}\right\|_{\infty},\left\|\frac{\partial^{3} u_{i}}{\partial \theta^{3}}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R^{3} \partial \theta}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R^{2} \partial \theta^{2}}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R \partial \theta^{3}}\right\|_{\infty}$. Moreover, the error between the Jacobian determinants of $\nabla u(x)$ and $\nabla \Pi u(x)$ is given by

$$
\begin{equation*}
\operatorname{det} \frac{\partial \Pi u}{\partial x}(x)=\operatorname{det} \frac{\partial u}{\partial x}(x)+\frac{1}{|x|} O\left(\tau^{2}+N^{-2}\right) \tag{3.2}
\end{equation*}
$$

Proof. Let $\hat{x}=F_{T}^{-1}(x)$, with $x=\left(x_{1}, x_{2}\right)=(R \cos \theta, R \sin \theta)$. Then the dualparametric bi-quadratic finite element interpolation function can be written as

$$
\begin{equation*}
\Pi u(x)=\sum_{i=0}^{8} u\left(a_{i}\right) \hat{p}_{i}(\hat{x})=\left(X_{1}, X_{2}\right) \tag{3.3}
\end{equation*}
$$

with $X_{k}$ being bi-quadratic functions of the form
$X_{k}=u_{k}\left(a_{8}\right)+f_{k} \hat{x}_{1}+h_{k} \hat{x}_{2}+\frac{d_{k}}{2} \hat{x}_{1}^{2}+e_{k} \hat{x}_{1} \hat{x}_{2}+\frac{g_{k}}{2} \hat{x}_{2}^{2}+\frac{b_{k}}{2} \hat{x}_{1} \hat{x}_{2}^{2}+\frac{c_{k}}{2} \hat{x}_{1}^{2} \hat{x}_{2}+\frac{j_{k}}{2} \hat{x}_{1}^{2} \hat{x}_{2}^{2}, k=1,2$.
We estimate $j_{k}, b_{k}, c_{k}, d_{k}, e_{k}, f_{k}, g_{k}, h_{k}$ below in (3.5) - (3.15), where some of them are given in two ways for the convenience of later manipulations in (3.18)- (3.21). On the representative element given by (3.1), regarding $u_{k}(x)$ as a function of $R$ and $\theta$, and making Taylor expansions at appropriate points, we obtain

$$
\begin{align*}
j_{k}= & \frac{u_{k}\left(a_{1}\right)+u_{k}\left(a_{2}\right)-2 u_{k}\left(a_{5}\right)}{2}+\frac{u_{k}\left(a_{0}\right)+u_{k}\left(a_{3}\right)-2 u_{k}\left(a_{7}\right)}{2}+2 u_{k}\left(a_{8}\right)  \tag{3.4}\\
& -u_{k}\left(a_{4}\right)-u_{k}\left(a_{6}\right)=\frac{\pi^{2}}{2 N^{2}}\left(\frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{5}\right)+\frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{7}\right)-2 \frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{8}\right)\right)+O\left(N^{-3}\right) \\
= & O\left(N^{-3}+\tau^{2} N^{-2}\right),
\end{align*}
$$

or alternatively, by regrouping $u_{k}\left(a_{i}\right)$ in $j_{k}$,

$$
\begin{align*}
j_{k}= & \frac{u_{k}\left(a_{0}\right)+u_{k}\left(a_{1}\right)-2 u_{k}\left(a_{4}\right)}{2}+\frac{u_{k}\left(a_{2}\right)+u_{k}\left(a_{3}\right)-2 u_{k}\left(a_{6}\right)}{2}+2 u_{k}\left(a_{8}\right)  \tag{3.5}\\
& -u_{k}\left(a_{5}\right)-u_{k}\left(a_{7}\right)=\frac{1}{2} \frac{\tau^{2}}{4}\left(\frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{4}\right)+\frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{6}\right)-2 \frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{8}\right)\right)+O\left(\tau^{3}\right) \\
= & O\left(\tau^{3}+\tau^{2} N^{-2}\right)
\end{align*}
$$

similarly,

$$
\begin{align*}
b_{k} & =\frac{u_{k}\left(a_{1}\right)+u_{k}\left(a_{2}\right)-2 u_{k}\left(a_{5}\right)}{2}-\frac{u_{k}\left(a_{0}\right)+u_{k}\left(a_{3}\right)-2 u_{k}\left(a_{7}\right)}{2}  \tag{3.6}\\
& =\frac{\pi^{2}}{2 N^{2}}\left(\frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{5}\right)-\frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{7}\right)\right)+O\left(N^{-3}\right) \\
& =\frac{\pi^{2}}{N^{2}} \frac{\partial^{3} u_{k}}{\partial R \partial \theta^{2}}\left(a_{8}\right) \frac{\tau}{2}+O\left(N^{-3}+\tau^{2} N^{-2}\right)
\end{align*}
$$

or alternatively, by regrouping $u_{k}\left(a_{i}\right)$ in $b_{k}$,

$$
\begin{align*}
b_{k} & =\frac{u_{k}\left(a_{1}\right)-u_{k}\left(a_{0}\right)}{2}+\frac{u_{k}\left(a_{2}\right)-u_{k}\left(a_{3}\right)}{2}+u_{k}\left(a_{7}\right)-u_{k}\left(a_{5}\right)  \tag{3.7}\\
& =\frac{\tau}{2}\left(\frac{\partial u_{k}}{\partial R}\left(a_{4}\right)+\frac{\partial u_{k}}{\partial R}\left(a_{6}\right)-2 \frac{\partial u_{k}}{\partial R}\left(a_{8}\right)\right)+O\left(\tau^{3}\right) \\
& =\frac{\pi^{2}}{N^{2}} \frac{\partial^{3} u_{k}}{\partial R \partial \theta^{2}}\left(a_{8}\right) \frac{\tau}{2}+O\left(\tau^{3}+\tau N^{-3}\right) \\
c_{k} & =\frac{u_{k}\left(a_{2}\right)-u_{k}\left(a_{1}\right)}{2}+\frac{u_{k}\left(a_{3}\right)-u_{k}\left(a_{0}\right)}{2}+u_{k}\left(a_{4}\right)-u_{k}\left(a_{6}\right)  \tag{3.8}\\
& =\frac{\pi}{N}\left(\frac{\partial u_{k}}{\partial \theta}\left(a_{5}\right)+\frac{\partial u_{k}}{\partial \theta}\left(a_{7}\right)-2 \frac{\partial u_{k}}{\partial \theta}\left(a_{8}\right)\right)+O\left(N^{-3}\right) \\
& =\frac{\pi}{N} \frac{\partial^{3} u_{k}}{\partial \theta \partial R^{2}}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2}+O\left(N^{-3}+\tau^{3} N^{-1}\right),
\end{align*}
$$

or alternatively, by regrouping $u_{k}\left(a_{i}\right)$ in $c_{k}$,

$$
\begin{align*}
c_{k} & =\frac{u_{k}\left(a_{2}\right)+u_{k}\left(a_{3}\right)-2 u_{k}\left(a_{6}\right)}{2}-\frac{u_{k}\left(a_{0}\right)+u_{k}\left(a_{1}\right)-2 u_{k}\left(a_{4}\right)}{2}  \tag{3.9}\\
& =\frac{1}{2} \frac{\tau^{2}}{4}\left(\frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{6}\right)-\frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{4}\right)\right)+O\left(\tau^{3}\right) \\
& =\frac{\pi}{N} \frac{\partial^{3} u_{k}}{\partial \theta \partial R^{2}}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2}+O\left(\tau^{3}+\tau^{2} N^{-2}\right) \\
d_{k} & =u_{k}\left(a_{5}\right)+u_{k}\left(a_{7}\right)-2 u_{k}\left(a_{8}\right)=\frac{\partial^{2} u_{k}}{\partial R^{2}}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2}+O\left(\tau^{3}\right)  \tag{3.10}\\
e_{k} & =\frac{u_{k}\left(a_{2}\right)-u_{k}\left(a_{1}\right)}{4}-\frac{u_{k}\left(a_{3}\right)-u_{k}\left(a_{0}\right)}{4}  \tag{3.11}\\
& =\frac{\pi}{2 N}\left(\frac{\partial u_{k}}{\partial \theta}\left(a_{5}\right)-\frac{\partial u_{k}}{\partial \theta}\left(a_{7}\right)\right)+O\left(N^{-3}\right) \\
& =\frac{\pi}{N} \frac{\partial^{2} u_{k}}{\partial R \partial \theta}\left(a_{8}\right) \frac{\tau}{2}+O\left(N^{-3}+\tau^{3} N^{-1}\right),
\end{align*}
$$

or alternatively, by regrouping $u_{k}\left(a_{i}\right)$ in $e_{k}$,

$$
\begin{align*}
e_{k} & =\frac{u_{k}\left(a_{2}\right)-u_{k}\left(a_{3}\right)}{4}-\frac{u_{k}\left(a_{1}\right)-u_{k}\left(a_{0}\right)}{4}  \tag{3.12}\\
& =\frac{\tau}{4}\left(\frac{\partial u_{k}}{\partial R}\left(a_{6}\right)-\frac{\partial u_{k}}{\partial R}\left(a_{4}\right)\right)+O\left(\tau^{3}\right) \\
& =\frac{\pi}{N} \frac{\partial^{2} u_{k}}{\partial R \partial \theta}\left(a_{8}\right) \frac{\tau}{2}+O\left(\tau^{3}+\tau N^{-3}\right)
\end{align*}
$$

$$
\begin{align*}
f_{k} & =\frac{u_{k}\left(a_{5}\right)-u_{k}\left(a_{7}\right)}{2}=\frac{\partial u_{k}}{\partial R}\left(a_{8}\right) \frac{\tau}{2}+O\left(\tau^{3}\right)  \tag{3.13}\\
g_{k} & =u_{k}\left(a_{4}\right)+u_{k}\left(a_{6}\right)-2 u_{k}\left(a_{8}\right)=\frac{\partial^{2} u_{k}}{\partial \theta^{2}}\left(a_{8}\right)\left(\frac{\pi}{N}\right)^{2}+O\left(N^{-3}\right)  \tag{3.14}\\
h_{k} & =\frac{u_{k}\left(a_{6}\right)-u_{k}\left(a_{4}\right)}{2}=\frac{\partial u_{k}}{\partial \theta}\left(a_{8}\right) \frac{\pi}{N}+O\left(N^{-3}\right) \tag{3.15}
\end{align*}
$$

Hence, by definition

$$
\frac{\partial \Pi u}{\partial \hat{x}}=\left(\begin{array}{ll}
A_{11} & A_{12}  \tag{3.16}\\
A_{21} & A_{22}
\end{array}\right)
$$

where, by (3.6), (3.8), (3.10), (3.10), (3.13), (3.13), we have

$$
\begin{align*}
& A_{11}=j_{1} \hat{x}_{1} \hat{x}_{2}^{2}+\frac{1}{2} b_{1} \hat{x}_{2}^{2}+c_{1} \hat{x}_{1} \hat{x}_{2}+d_{1} \hat{x}_{1}+e_{1} \hat{x}_{2}+f_{1}  \tag{3.17}\\
& =\frac{\partial u_{1}}{\partial R}\left(a_{8}\right) \frac{\tau}{2}+\frac{\partial^{2} u_{1}}{\partial R^{2}}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2} \hat{x}_{1}+\frac{\partial^{2} u_{1}}{\partial R \partial \theta}\left(a_{8}\right) \frac{\tau}{2} \frac{\pi}{N} \hat{x}_{2}+\frac{\partial^{3} u_{1}}{\partial R^{2} \partial \theta}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2} \frac{\pi}{N} \hat{x}_{1} \hat{x}_{2} \\
& +\frac{1}{2} \frac{\partial^{3} u_{1}}{\partial R \partial \theta^{2}}\left(a_{8}\right) \frac{\tau}{2}\left(\frac{\pi}{N}\right)^{2} \hat{x}_{2}^{2}+O\left(\tau^{3}+\tau^{2} N^{-2}+\tau N^{-3}\right) \\
& =\frac{\tau}{2} \frac{\partial u_{1}}{\partial R}(x)+O\left(\tau^{3}+\tau^{2} N^{-2}+\tau N^{-3}\right),
\end{align*}
$$

and by (3.5), (3.7), (3.9), (3.12), (3.14), (3.15), we have

$$
\begin{equation*}
A_{12}=j_{1} \hat{x}_{1}^{2} \hat{x}_{2}+\frac{1}{2} c_{1} \hat{x}_{1}^{2}+b_{1} \hat{x}_{1} \hat{x}_{2}+e_{1} \hat{x}_{1}+g_{1} \hat{x}_{2}+h_{1} \tag{3.18}
\end{equation*}
$$

$$
\begin{aligned}
= & \frac{\partial u_{1}}{\partial \theta}\left(a_{8}\right) \frac{\pi}{N}+\frac{\partial^{2} u_{1}}{\partial R \partial \theta}\left(a_{8}\right) \frac{\tau}{2} \frac{\pi}{N} \hat{x}_{1}+\frac{\partial^{2} u_{1}}{\partial \theta^{2}}\left(a_{8}\right)\left(\frac{\pi}{N}\right)^{2} \hat{x}_{2}+\frac{\partial^{3} u_{1}}{\partial R \partial \theta^{2}} \frac{\tau}{2}\left(\frac{\pi}{N}\right)^{2}\left(a_{8}\right) \hat{x}_{1} \hat{x}_{2} \\
& +\frac{1}{2} \frac{\partial^{3} u_{1}}{\partial R^{2} \partial \theta}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2} \frac{\pi}{N} \hat{x}_{1}^{2}+O\left(N^{-3}+\tau^{2} N^{-2}+\tau^{3} N^{-1}\right) \\
= & \frac{\pi}{N} \frac{\partial u_{1}}{\partial \theta}(x)+O\left(N^{-3}+\tau^{2} N^{-2}+\tau^{3} N^{-1}\right),
\end{aligned}
$$

and similarly, we have

$$
\begin{align*}
A_{21} & =j_{2} \hat{x}_{1} \hat{x}_{2}^{2}+\frac{1}{2} b_{2} \hat{x}_{2}^{2}+c_{2} \hat{x}_{1} \hat{x}_{2}+d_{2} \hat{x}_{1}+e_{2} \hat{x}_{2}+f_{2}  \tag{3.19}\\
& =\frac{\tau}{2} \frac{\partial u_{2}}{\partial R}(x)+O\left(\tau^{3}+\tau^{2} N^{-2}+\tau N^{-3}\right) \\
A_{22} & =j_{2} \hat{x}_{1}^{2} \hat{x}_{2}+\frac{1}{2} c_{2} \hat{x}_{1}^{2}+b_{2} \hat{x}_{1} \hat{x}_{2}+e_{2} \hat{x}_{1}+g_{2} \hat{x}_{2}+h_{2}  \tag{3.20}\\
& =\frac{\pi}{N} \frac{\partial u_{2}}{\partial \theta}(x)+O\left(N^{-3}+\tau^{2} N^{-2}+\tau^{3} N^{-1}\right) .
\end{align*}
$$

Thus

$$
\begin{equation*}
\operatorname{det} \frac{\partial \Pi u}{\partial \hat{x}}\left(\hat{x}_{1}, \hat{x}_{2}\right)=\left.\frac{\tau}{2} \frac{\pi}{N}\left(\frac{\partial u_{1}}{\partial R} \frac{\partial u_{2}}{\partial \theta}-\frac{\partial u_{2}}{\partial R} \frac{\partial u_{1}}{\partial \theta}\right)\right|_{x}+O\left(\tau^{3} N^{-1}+\tau N^{-3}\right) \tag{3.21}
\end{equation*}
$$

It is easily verified that, in (3.5)-(3.15) as well as (3.18)-(3.21), the constants in $O(\cdot)$ depend on $\left\|\frac{\partial^{3} u_{i}}{\partial R^{3}}\right\|_{\infty},\left\|\frac{\partial^{3} u_{i}}{\partial \theta^{3}}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R^{3} \partial \theta}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R^{2} \partial \theta^{2}}\right\|_{\infty},\left\|\frac{\partial^{4} u_{i}}{\partial R \partial \theta^{3}}\right\|_{\infty}$. Note that $R \geq \epsilon$ and

$$
\frac{\partial u_{1}}{\partial R} \frac{\partial u_{2}}{\partial \theta}-\frac{\partial u_{1}}{\partial \theta} \frac{\partial u_{2}}{\partial R}=R\left(\frac{\partial u_{1}}{\partial x_{1}} \frac{\partial u_{2}}{\partial x_{2}}-\frac{\partial u_{1}}{\partial x_{2}} \frac{\partial u_{2}}{\partial x_{1}}\right)=R \operatorname{det} \frac{\partial u}{\partial x}
$$

which yields the first part of the conclusion. Estimate (3.2) is a direct consequence of (3.21) and the fact that $\operatorname{det} \frac{\partial x}{\partial \hat{x}}=\frac{R \tau}{2} \frac{\pi}{N}$.

REMARK 3.2. As a consequence of Theorem 3.1, we see that, for a general cavity deformation, the interpolation function is orientation-preserving on a mesh $\Omega_{\epsilon_{0}}=$ $\bigcup_{i=0}^{m} \Omega_{\left(\epsilon_{i}, \tau_{i}\right)}$, where $\epsilon_{i+1}=\epsilon_{i}+\tau_{i}$, which satisfies $\tau_{i} \leq C_{1} \sqrt{\epsilon_{i}}$ and $N_{i} \geq C_{2} \epsilon_{i}^{-1 / 2}$ for some constants $C_{1}$ and $C_{2}$. If restricted to a radially-symmetric cavity deformation, we may expect to have a more relaxed orientation-preservation condition.

THEOREM 3.3. For a radially-symmetric deformation $u(x)=\frac{r(R)}{R} x$, where $r>0$ is an increasing convex function satisfying

$$
\begin{equation*}
4 r(\epsilon+\tau / 2)>3 r(\epsilon)+r(\epsilon+\tau) \tag{3.22}
\end{equation*}
$$

then the interpolation function $\Pi u(x)$ preserves orientation on the circular ring $\Omega_{(\epsilon, \tau)}$. Moreover, if

$$
\begin{equation*}
\operatorname{det} \frac{\partial u}{\partial x}(x)=\frac{r(R) r^{\prime}(R)}{R} \leq M \tag{3.23}
\end{equation*}
$$

where $R=|x|$, then

$$
\begin{equation*}
\operatorname{det} \frac{\partial \Pi u}{\partial x}(x)=\operatorname{det} \frac{\partial u}{\partial x}(x)+O\left(N^{-2}\right)+\frac{1}{|x|} O\left(\tau^{2}\right) . \tag{3.24}
\end{equation*}
$$

Proof. For $u(x)=\frac{r(R)}{R} x$, a direct but tedious calculation yields

$$
\begin{equation*}
\Pi u(x)=\left(X_{1}, X_{2}\right)=C\left(\hat{x}_{1}\right)\left(1-2 \hat{x}_{2}^{2} \sin ^{2} \frac{\pi}{2 N}, \hat{x}_{2} \sin \frac{\pi}{N}\right) \tag{3.25}
\end{equation*}
$$

where

$$
\begin{equation*}
C\left(\hat{x}_{1}\right)=\frac{1}{2} \hat{x}_{1}\left(\hat{x}_{1}-1\right) r(\epsilon)+\frac{1}{2} \hat{x}_{1}\left(\hat{x}_{1}+1\right) r(\epsilon+\tau)+\left(1-\hat{x}_{1}^{2}\right) r(\epsilon+\tau / 2) . \tag{3.26}
\end{equation*}
$$

Hence

$$
\operatorname{det} \frac{\partial \Pi u}{\partial \hat{x}}=C\left(\hat{x}_{1}\right) C^{\prime}\left(\hat{x}_{1}\right) \sin \frac{\pi}{N}\left(1+2 \hat{x}_{2}^{2} \sin ^{2} \frac{\pi}{2 N}\right) .
$$

Since $r(R)$ is increasing and convex, it is easily seen that $C\left(\hat{x}_{1}\right)>0$ on $[-1,1]$. On the other hand, $C^{\prime}\left(\hat{x}_{1}\right)=\left(\hat{x}_{1}-\frac{1}{2}\right) r(\epsilon)+\left(\hat{x}_{1}+\frac{1}{2}\right) r(\epsilon+\tau)-2 \hat{x}_{1} r(\epsilon+\tau / 2)$ is a linear function of $\hat{x}_{1}$ with $C^{\prime}(1)=\frac{1}{2}(r(\epsilon)+3 r(\epsilon+\tau)-4 r(\epsilon+\tau / 2))>0$, and $C^{\prime}(-1)=\frac{1}{2}(4 r(\epsilon+\tau / 2)-3 r(\epsilon)-r(\epsilon+\tau))>0$, thus $C^{\prime}\left(\hat{x}_{1}\right)>0$ on $[-1,1]$. Hence, the first part of the theorem follows.

Making a Taylor expansion of $r(\epsilon), r(\epsilon+\tau / 2), r(\epsilon+\tau)$ at $R=\epsilon+\frac{\hat{x}_{1}+1}{2} \tau$, one gets

$$
C\left(\hat{x}_{1}\right)=r(R)+O\left(\tau^{3}\right), \quad C^{\prime}\left(\hat{x}_{1}\right)=\frac{r^{\prime}(R) \tau}{2}+O\left(\tau^{3}\right)
$$

Thus, it follows from $\operatorname{det}\left(\frac{\partial \Pi u}{\partial x} \frac{\partial x}{\partial \tilde{x}}\right)=\operatorname{det} \frac{\partial \Pi u}{\partial \hat{x}}$ and $\operatorname{det} \frac{\partial x}{\partial \tilde{x}}=\frac{R \tau}{2} \frac{\pi}{N}$ that

$$
\begin{aligned}
\operatorname{det} \frac{\partial \Pi u}{\partial x} & =\frac{2 C\left(\hat{x}_{1}\right) C^{\prime}\left(\hat{x}_{1}\right)}{R \tau}\left(1+O\left(N^{-2}\right)\right) \\
& =\frac{\left(r(R)+O\left(\tau^{3}\right)\right)\left(r^{\prime}(R)+O\left(\tau^{2}\right)\right)}{R}\left(1+O\left(N^{-2}\right)\right) \\
& =\frac{r(R) r^{\prime}(R)}{R}+\frac{O\left(\tau^{2}\right)}{R}+O\left(N^{-2}\right),
\end{aligned}
$$

which gives (3.24) and completes the proof of the theorem.
REMARK 3.4. For the energy minimizers among radially-symmetric cavity deformations, an extended version of condition (3.23) $0<m \leq \frac{r(R) r^{\prime}(R)}{R} \leq M$ is generally satisfied, and there exists a positive constant $C$ such that (3.22) holds whenever $\epsilon \geq C \tau^{2}$ (see [24]). As a consequence, we see that, for the dual-parametric bi-quadratic finite element method, the orientation preservation adds no further restrictions on the number of elements $N$ on an annular ring, which means that a much smaller number of total degrees of freedom is required as compared with the methods in [24, 26].
4. Interpolation errors of cavity deformations. In this section, the interpolation errors are estimated, including those on the interpolation function and the elastic energy in the dual-parametric bi-quadratic finite element function spaces defined on the meshes described in § 3. Throughout this section, $u(x)$ is supposed to be a smooth cavitation deformation in the regularized domain $\Omega_{\epsilon_{0}}$, and $p$ is a real number satisfying $1<p<2$. We also assume that the meshes are so given that Theorem 3.1 holds.
4.1. The error of the interpolation function. Let $\hat{x}=F_{T}^{-1}(x) \in \hat{T}$, where $x \in T$ is a point on the mesh. We will estimate, in this subsection, the errors between $u(x)$ and its interpolation function $\Pi u(x)$.

ThEOREM 4.1. Under the assumptions of Theorem 3.1, the error between a cavity deformation $u(x)$ and its interpolation function $\Pi u(x)$ satisfies

$$
\begin{equation*}
\|u(x)-\Pi u(x)\|_{\infty}=O\left(\tau^{3}+N^{-3}\right) \tag{4.1}
\end{equation*}
$$

Proof. For a typical element as used in § 3, denote $X=\Pi u(x)=\left(X_{1}, X_{2}\right)$, where $x=F_{T}(\hat{x})$. With the same notation as used in Theorem 3.1, and making Taylor expansions at appropriate points, one gets

$$
\begin{aligned}
& X_{1}=\sum_{i=0}^{8} u_{1}\left(a_{i}\right) \hat{p}_{i}(\hat{x}) \\
= & u_{1}\left(a_{8}\right)+f_{1} \hat{x}_{1}+h_{1} \hat{x}_{2}+\frac{d_{1}}{2} \hat{x}_{1}^{2}+e_{1} \hat{x}_{1} \hat{x}_{2}+\frac{g_{1}}{2} \hat{x}_{2}^{2}+\frac{b_{1}}{2} \hat{x}_{1} \hat{x}_{2}^{2}+\frac{c_{1}}{2} \hat{x}_{1}^{2} \hat{x}_{2}+\frac{j_{1}}{2} \hat{x}_{1}^{2} \hat{x}_{2}^{2} \\
= & u_{1}\left(a_{8}\right)+\frac{\partial u_{1}}{\partial R}\left(a_{8}\right) \frac{\tau}{2} \hat{x}_{1}+\frac{\partial u_{1}}{\partial \theta}\left(a_{8}\right) \frac{\pi}{N} \hat{x}_{2}+\frac{1}{2} \frac{\partial^{2} u_{1}}{\partial R^{2}}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2} \hat{x}_{1}^{2} \\
& +\frac{\partial^{2} u_{1}}{\partial R \partial \theta}\left(a_{8}\right) \frac{\tau}{2} \frac{\pi}{N} \hat{x}_{1} \hat{x}_{2}+\frac{1}{2} \frac{\partial^{2} u_{1}}{\partial \theta^{2}}\left(a_{8}\right)\left(\frac{\pi}{N}\right)^{2} \hat{x}_{2}^{2}+\frac{1}{2} \frac{\partial^{3} u_{1}}{\partial R \partial \theta^{2}}\left(a_{8}\right) \frac{\tau}{2}\left(\frac{\pi}{N}\right)^{2} \hat{x}_{1} \hat{x}_{2}^{2} \\
& +\frac{1}{2} \frac{\partial^{3} u_{1}}{\partial R^{2} \partial \theta}\left(a_{8}\right)\left(\frac{\tau}{2}\right)^{2} \frac{\pi}{N} \hat{x}_{1}^{2} \hat{x}_{2}+O\left(\tau^{3}+N^{-3}+\tau^{2} N^{-2}\right) \\
= & u_{1}(x)+O\left(\tau^{3}+N^{-3}\right) .
\end{aligned}
$$

Similarly, $X_{2}=u_{2}(x)+O\left(\tau^{3}+N^{-3}\right)$. Hence, the conclusion follows.
THEOREM 4.2. Denote $\Omega_{\epsilon_{0}}=\bigcup_{i=0}^{m} \Omega_{\left(\epsilon_{i}, \tau_{i}\right)}$, where $\Omega_{(\epsilon, \tau)}=\{x: \epsilon \leq|x| \leq \epsilon+\tau\}$, $\epsilon_{i+1}=\epsilon_{i}+\tau_{i}, \epsilon_{m+1}=1.0$. Let $N_{i}$ be the number of elements in the layer $\Omega_{\left(\epsilon_{i}, \tau_{i}\right)}$. If $\epsilon_{i}, \tau_{i}, N_{i}$ satisfy the assumptions of Theorem 3.1, and $\tau_{i}=O(h), N_{i}^{-1}=O(h)$, as $h \rightarrow 0$, where $h$ is a global reference mesh size, then the error between a cavity deformation $u(x)$ and its interpolation function $\Pi u$ satisfies

$$
\begin{equation*}
\|u(x)-\Pi u(x)\|_{1, p}=O\left(h^{2}\right) \tag{4.2}
\end{equation*}
$$

Proof. On a representative element as shown in Figure 2(b), by (3.1), (3.16) and $\frac{\partial \Pi u}{\partial x} \frac{\partial x}{\partial \hat{x}}=\frac{\partial \Pi u}{\partial \hat{x}}$, we have

$$
\frac{\partial \Pi u(x)}{\partial x}=\frac{2 N}{\pi R \tau}\left(\begin{array}{ll}
A_{11} R \frac{\pi}{N} \cos \theta-A_{12} \frac{\tau}{2} \sin \theta & A_{11} R \frac{\pi}{N} \sin \theta+A_{12} \frac{\tau}{2} \cos \theta  \tag{4.3}\\
A_{21} R \frac{\pi}{N} \cos \theta-A_{22} \frac{\tau}{2} \sin \theta & A_{21} R \frac{\pi}{N} \sin \theta+A_{22} \frac{\tau}{2} \cos \theta
\end{array}\right)
$$

where $A_{i j}$ are given by (3.18)-(3.21). Denote

$$
\frac{\partial \Pi u(x)}{\partial x}-\frac{\partial u}{\partial x}=\left(\begin{array}{ll}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}\right)
$$

Then, it follows from (4.3), (3.18) and (3.19) that

$$
\begin{aligned}
B_{11} & =\frac{2 N}{\pi R \tau}\left(A_{11} R \frac{\pi}{N} \cos \theta-A_{12} \frac{\tau}{2} \sin \theta\right)-\frac{\partial u_{1}}{\partial R} \cos \theta+\frac{\partial u_{1}}{\partial \theta} \frac{\sin \theta}{R} \\
& =O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)+\frac{1}{R} O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right)
\end{aligned}
$$

Since $\tau=O(h), N^{-1}=O(h)$, this yields $B_{11}=\frac{1}{R} O\left(h^{2}\right)$. Similarly, $B_{i j}=\frac{1}{R} O\left(h^{2}\right)$. As a consequence $\left|\frac{\partial \Pi u}{\partial x}-\frac{\partial u}{\partial x}\right|^{p}=\frac{1}{R^{p}} O\left(h^{2 p}\right)$. Thus $\left\|\frac{\partial \Pi u}{\partial x}-\frac{\partial u}{\partial x}\right\|_{p}=\left(\int_{\epsilon_{0}}^{1} R^{1-p} d R\right)^{\frac{1}{p}} O\left(h^{2}\right)$, which completes the proof, since $1<p<2$.
4.2. The error on the elastic energy. Let $\mathcal{J}\left(\Omega_{(\epsilon, \tau)}\right)$ be a dual-parametric bi-quadratic finite element division of $\Omega_{(\epsilon, \tau)}$ consisting of only one layer of evenly distributed elements, denoted by $T_{j}, j=1, \cdots, N$. For the energy density function of the form (1.6), denote $E\left(u ; \Omega_{(\varepsilon, \tau)}\right)=E_{1}\left(u ; \Omega_{(\varepsilon, \tau)}\right)+E_{2}\left(u ; \Omega_{(\varepsilon, \tau)}\right)$ with

$$
\begin{align*}
E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right) & =\int_{\Omega_{(\epsilon, \tau)}} \omega\left|\frac{\partial u}{\partial x}\right|^{p} d x  \tag{4.4}\\
E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right) & =\int_{\Omega_{(\epsilon, \tau)}} g\left(\operatorname{det} \frac{\partial u}{\partial x}\right) d x  \tag{4.5}\\
A(\epsilon, \tau) & =(2-p) \int_{\epsilon}^{\epsilon+\tau} R^{1-p} d R=(\epsilon+\tau)^{2-p}-\epsilon^{2-p} \tag{4.6}
\end{align*}
$$

and let $\operatorname{err}\left(E_{i}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=\left|E_{i}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)-E_{i}\left(u ; \Omega_{(\epsilon, \tau)}\right)\right|$ be the absolute interpolation error of $\left.E_{i}\left(u ; \Omega_{(\epsilon, \tau)}\right)\right), i=1,2$, respectively. We have the following result.

THEOREM 4.3. Under the assumptions of Theorem 3.1, the elastic energy of a cavity deformation $u(x)$ and its interpolation function $\Pi u$ satisfy

$$
\begin{align*}
& E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)=O(A(\epsilon, \tau)), \quad E\left(u ; \Omega_{(\epsilon, \tau)}\right)=O(A(\epsilon, \tau)),  \tag{4.7}\\
& \operatorname{err}\left(E_{1}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=A(\epsilon, \tau) O\left(\tau^{2}+N^{-2}\right)  \tag{4.8}\\
& \operatorname{err}\left(E_{2}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=O\left(\tau^{3}+\tau N^{-2}\right)  \tag{4.9}\\
& \operatorname{err}\left(E\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=A(\epsilon, \tau) O\left(\tau^{2}+N^{-2}\right) \tag{4.10}
\end{align*}
$$

where $A(\epsilon, \tau)$ is defined as (4.6). Moreover, if there exist positive constants $0<l_{1} \leq l_{2}$ such that $l_{1} \leq\left|\frac{\partial u}{\partial \theta}\right| \leq l_{2}$, then

$$
\begin{align*}
& E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right) \sim A(\epsilon, \tau), \quad E\left(u ; \Omega_{(\epsilon, \tau)}\right) \sim A(\epsilon, \tau)  \tag{4.11}\\
& \operatorname{err}\left(E_{1}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right) O\left(\tau^{2}+N^{-2}\right)  \tag{4.12}\\
& \operatorname{err}\left(E\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right)=E\left(u ; \Omega_{(\epsilon, \tau)}\right) O\left(\tau^{2}+N^{-2}\right) \tag{4.13}
\end{align*}
$$

Proof. Since $\frac{\partial u_{i}}{\partial R}, \frac{\partial u_{i}}{\partial \theta}$ are bounded, it follows that $\left|\frac{\partial u}{\partial x}(x)\right|^{p}=O\left(R^{-p}\right)$. Thus $E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)=O\left(\int_{\epsilon}^{\epsilon+\tau} R^{1-p} d R\right)=O(A(\epsilon, \tau))$. Note that $g$ is bounded and $A(\epsilon, \tau)>$ $(2-p) \tau$, then $E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right)=O(\tau)=O(A(\epsilon, \tau))$, so that we deduce (4.7). In view of (4.3), we find that

$$
\begin{aligned}
& \left|\frac{\partial \Pi u}{\partial x}\right|^{2}=\frac{4}{\tau^{2}}\left(A_{11}^{2}+A_{21}^{2}\right)+\frac{N^{2}}{\pi^{2} R^{2}}\left(A_{12}^{2}+A_{22}^{2}\right) \\
= & \left(\frac{\partial u_{1}}{\partial R}(x)\right)^{2}+\left(\frac{\partial u_{2}}{\partial R}(x)\right)^{2}+\frac{1}{R^{2}}\left(\left(\frac{\partial u_{1}}{\partial \theta}(x)\right)^{2}+\left(\frac{\partial u_{2}}{\partial \theta}(x)\right)^{2}\right) \\
& +\sum_{i=1}^{2}\left|\frac{\partial u_{i}}{\partial R}\right| O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)+\frac{1}{R^{2}} \sum_{i=1}^{2}\left|\frac{\partial u_{i}}{\partial \theta}\right| O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right) \\
= & \left|\frac{\partial u}{\partial x}\right|^{2}+\sum_{i=1}^{2}\left|\frac{\partial u_{i}}{\partial R}\right| O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)+\frac{1}{R^{2}} \sum_{i=1}^{2}\left|\frac{\partial u_{i}}{\partial \theta}\right| O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right)
\end{aligned}
$$

Since $\frac{\partial u_{i}}{\partial R}$ is bounded and $\left|\frac{\partial u}{\partial x}\right| \geq \frac{1}{R}\left|\frac{\partial u_{i}}{\partial \theta}\right|$, this implies

$$
\begin{aligned}
\left|\frac{\partial \Pi u}{\partial x}\right|^{p} & =\left|\frac{\partial u}{\partial x}\right|^{p}\left(1+O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)\right)+ \\
& \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{1}{R^{2}} \sum_{i=1}^{2}\left|\frac{\partial u_{i}}{\partial \theta}\right| O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right) \\
& =\left|\frac{\partial u}{\partial x}\right|^{p}\left(1+O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)\right)+\left|\frac{\partial u}{\partial x}\right|^{p-1} \frac{1}{R} O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right)
\end{aligned}
$$

Obviously, the first term will lead to a relative error of the order $O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)$ to the first part of the energy $E_{1}$. What remains to consider is the second term. Applying the Hölder inequality, we deduce that

$$
\int_{T}\left|\frac{\partial u}{\partial x}\right|^{p-1} \frac{1}{R} d x \leq\left(\int_{T}\left|\frac{\partial u}{\partial x}\right|^{p} d x\right)^{1-\frac{1}{p}}\left(\int_{T} R^{-p} d x\right)^{\frac{1}{p}}=\left(\frac{2 \pi A(\epsilon, \tau)}{N(2-p)}\right)^{\frac{1}{p}}\left(\int_{T}\left|\frac{\partial u}{\partial x}\right|^{p} d x\right)^{1-\frac{1}{p}}
$$

Applying the Hölder inequality again yields

$$
\sum_{j=1}^{N} \int_{T_{j}}\left|\frac{\partial u}{\partial x}\right|^{p-1} \frac{1}{R} d x \leq\left(\frac{2 \pi}{2-p} A(\epsilon, \tau)\right)^{1 / p} E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)^{1-1 / p} \omega^{1 / p-1}
$$

Hence, we obtain

$$
\begin{aligned}
& E_{1}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)=E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)\left(1+O\left(\tau^{2}+\tau N^{-2}+N^{-3}\right)\right)+ \\
& \quad E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)^{1-\frac{1}{p}} A(\epsilon, \tau)^{\frac{1}{p}} O\left(\tau^{3}+\tau^{2} N^{-1}+N^{-2}\right)
\end{aligned}
$$

which together with (4.7) yields (4.8).

On the other hand, by (3.2) and the fact that $g$ is strictly convex,

$$
\begin{align*}
E_{2}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right) & =\sum_{j=1}^{N} \int_{T_{j}} g(\operatorname{det} \nabla u) d x+\sum_{j=1}^{N} \int_{T_{j}} g^{\prime}\left(\eta_{x}\right)|x|^{-1} d x O\left(\tau^{2}+N^{-2}\right)  \tag{4.14}\\
& =E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right)+\sum_{j=1}^{N} \int_{T_{j}}|x|^{-1} d x O\left(\tau^{2}+N^{-2}\right) \\
& =E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right)+O\left(\tau^{3}+\tau N^{-2}\right) .
\end{align*}
$$

(4.10) is a direct consequence of (4.8), (4.9) and $A(\epsilon, \tau)>(2-p) \tau$, for $\epsilon \in[0,1-\tau]$.

Finally, if there exists a positive constant $l_{1}>0$ such that $\left|\frac{\partial u}{\partial \theta}\right| \geq l_{1}$ and thus $\left|\frac{\partial u}{\partial x}(x)\right|^{p} \sim R^{-p}$, meaning that there exist positive constants $L_{1}$ and $L_{2}$ such that $L_{1} R^{-p} \leq\left|\frac{\partial u}{\partial x}(x)\right|^{p} \leq L_{2} R^{-p}$, this gives (4.11). This together with (4.8) and (4.10) yield (4.12) and (4.13).

THEOREM 4.4. For the radially-symmetric cavity solution, the error of the energy satisfies $E\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)=E\left(u ; \Omega_{(\epsilon, \tau)}\right)\left(1+O\left(\max \{\epsilon, \tau\}^{p-1} \tau^{2}+N^{-2}\right)\right)$.

Proof. For the radially-symmetric solution, it follows from (3.25) that

$$
\frac{\partial \Pi u}{\partial \hat{x}}=\left(\begin{array}{cc}
C^{\prime}\left(\hat{x}_{1}\right)\left(1-2 \hat{x}_{2}^{2} \sin ^{2} \frac{\pi}{2 N}\right) & -4 C\left(\hat{x}_{1}\right) \hat{x}_{2} \sin ^{2} \frac{\pi}{2 N} \\
C^{\prime}\left(\hat{x}_{1}\right) \hat{x}_{2} \sin \frac{\pi}{N} & C\left(\hat{x}_{1}\right) \sin \frac{\pi}{N} .
\end{array}\right)
$$

Thus, by (3.26) and the facts that $r(R) \geq r(0)>0$ and $r^{\prime}(R) \leq M R$, one gets

$$
\begin{aligned}
\left|\frac{\partial \Pi u}{\partial x}\right|^{2} & =\frac{4}{\tau^{2}} C^{\prime}\left(\hat{x}_{1}\right)^{2}\left(1+O\left(N^{-4}\right)\right)+\frac{N^{2}}{\pi^{2} R^{2}} C\left(\hat{x}_{1}\right)^{2} \sin ^{2} \frac{\pi}{N}\left(1+O\left(N^{-2}\right)\right) \\
& =\left(r^{\prime}(R)+O\left(\tau^{2}\right)\right)^{2}\left(1+O\left(N^{-4}\right)\right)+\frac{\left(r(R)+O\left(\tau^{3}\right)\right)^{2}}{R^{2}}\left(1+O\left(N^{-2}\right)\right) \\
& =\frac{r(R)^{2}}{R^{2}}\left(1+O\left(\tau^{3}+N^{-2}\right)\right)+r^{\prime}(R)^{2}+r^{\prime}(R) O\left(\tau^{2}\right) \\
& =\left(r^{\prime}(R)^{2}+\frac{r(R)^{2}}{R^{2}}\right)\left(1+O\left(\tau^{3}+N^{-2}+r^{\prime}(R) R^{2} \tau^{2}\right)\right) \\
& =\left(r^{\prime}(R)^{2}+\frac{r(R)^{2}}{R^{2}}\right)\left(1+O\left(\tau^{3}+N^{-2}+(\epsilon+\tau)^{3} \tau^{2}\right)\right)
\end{aligned}
$$

It follows that

$$
E_{1}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)=E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)\left(1+O\left(\epsilon^{3} \tau^{2}+\tau^{3}+N^{-2}\right)\right)
$$

On the other hand, by (3.24) and with similar arguments as in the proof of Theorem 4.3 (see (4.15)), one has that

$$
E_{2}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)=E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right)+O\left(\tau^{3}+(\epsilon+\tau) \tau N^{-2}\right)
$$

Recalling that

$$
\begin{aligned}
& E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right)=2 \pi \int_{\epsilon}^{\epsilon+\tau}\left(r^{\prime}(R)^{2}+\frac{r(R)^{2}}{R^{2}}\right)^{p / 2} R d R \sim A(\epsilon, \tau)>(2-p) \tau \\
& E_{2}\left(u ; \Omega_{(\epsilon, \tau)}\right)=O(\tau)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\operatorname{err}\left(E_{2}\left(\Pi u ; \Omega_{(\epsilon, \tau)}\right)\right. & =E_{1}\left(u ; \Omega_{(\epsilon, \tau)}\right) \frac{O\left(\tau^{3}+(\epsilon+\tau) \tau N^{-2}\right)}{A(\epsilon, \tau)} \\
& =E\left(u ; \Omega_{(\epsilon, \tau)}\right) O\left(\max \{\epsilon, \tau\}^{p-1} \tau^{2}+(\epsilon+\tau) N^{-2}\right)
\end{aligned}
$$

which completes the proof.
5. A meshing strategy. The aim of this section is to establish, for a given reference mesh size $h>0$, a meshing strategy on the domain $\Omega_{\epsilon_{0}}=B_{1}(0) \backslash B_{\epsilon_{0}}(0)$, i.e. to design a method of calculating $\epsilon_{i}, \tau_{i}, N_{i}$, where $\epsilon_{i+1}=\epsilon_{i}+\tau_{i}$, and $N_{i}$ is the number of the elements in the layer $\Omega_{\left(\epsilon_{i} \tau_{i}\right)}$, so that, on $\Omega_{\epsilon_{0}}=\bigcup_{i=0}^{m} \Omega_{\left(\epsilon_{i}, \tau_{i}\right)}$, a cavity solution $u$ and its finite element interpolation function $\Pi u$ satisfy
(C1) the orientation-preservation condition: $\operatorname{det} \nabla \Pi u>0$;
(C2) the approximation condition: $\|u-\Pi u\|_{1, p}=O\left(h^{2}\right)$;
(C3) error sub-equi-distribution condition (see (4.11)-(4.13)): $A\left(\epsilon_{i}, \tau_{i}\right)=O(h)$;
(C4) least total degrees of freedom condition: $N_{d}=\sum_{i=0}^{m} N_{i}$ is minimized under the restriction that $N_{i+1}=N_{i}$ or $N_{i+1}=2 N_{i}$.
By Theorems 3.1 and 3.3 , for a radially-symmetric cavity solution, condition C1 can be ensured by setting $\tau_{i} \leq C_{1} \epsilon_{i}^{1 / 2}$, while in the nonsymmetric case, setting in addition $N_{i} \geq C_{2} \epsilon_{i}^{-1 / 2}$ meets the requirement. By Theorem 4.2, for condition C2 to hold it suffices that $N_{i}^{-1}=O(h), \tau_{i}=O(h)$.

The idea of error equi-distribution is often used in mesh adaptivity and mesh redistribution. By Theorem 4.3, $A\left(\epsilon_{i}, \tau_{i}\right)$ can serve as a monitor for the energy error equi-distribution, especially in the neighborhood of the void. Without loss of generality, assume $\epsilon_{m}>\frac{1}{2}$. Since $\epsilon_{m}+\tau_{m}=1$ for the layer $m$, this implies $A\left(\epsilon_{m}, \tau_{m}\right)=1-\left(1-\tau_{m}\right)^{2-p} \leq(2-p) 2^{p-1} \tau_{m}$. Thus, it is easily verified that, for a given constant $C \geq(2-p) 2^{p-1}$, a reference mesh size $0<h \leq h_{0} \leq \frac{2-p}{2^{2-p} C}$, $A\left(\epsilon_{m}, \tau_{m}\right) \leq C h$, provided that $\tau_{m} \leq \frac{C}{(2-p) 2^{p-1}} h$. Hence, it is natural to require C3: $A\left(\epsilon_{i}, \tau_{i}\right) \leq C h$, for all $0 \leq i \leq m$, which imposes an implicit condition on the layer's thickness $\tau_{i}$. In fact, given $C \geq(2-p) 2^{p-1}$ and $h_{0} \geq h>0$, denoting $d(x, h)=\left(x^{2-p}+C h\right)^{\frac{1}{2-p}}-x$, we have $A(x, d(x, h))=C h$. On the other hand, since $p \in(1,2)$, we have $C h=A\left(\epsilon_{i}, d\left(\epsilon_{i}, h\right)\right) \triangleq(2-p) \int_{\epsilon_{i}}^{\epsilon_{i}+d\left(\epsilon_{i}, h\right)} R^{1-p} d R \geq \frac{(2-p) d\left(\epsilon_{i}, h\right)}{\left(1+d\left(\epsilon_{i}, h\right)\right)^{p-1}}$, which implies that $d\left(\epsilon_{i}, h\right) \leq 1$, as long as $0<h \leq h_{1} \leq \frac{2-p}{2^{p-1} C}$, and consequently $\frac{(2-p) d\left(\epsilon_{i}, h\right)}{\left(1+d\left(\epsilon_{i}, h\right)\right)^{p-1}} \leq C h$ yields $d\left(\epsilon_{i}, h\right) \leq \frac{2^{p-1} C}{(2-p)} h$. Thus, for condition C3 and $\tau_{i}=O(h)$ to hold, it suffices to require $\tau_{i} \leq d\left(\epsilon_{i}, h\right)$.

Finally, assuming an optimized distribution of layers is given, then, condition C4 can be achieved easily by taking the least admissible $N_{i}, 0 \leq i \leq m$. It is in this sense that the total degrees of freedom are minimized.

For given positive constants $C_{1}, C_{2}, C \geq(2-p) 2^{p-1}, h \leq \min \left\{h_{0}, h_{1}\right\}, A_{1}<A_{2}$ satisfying $\left[\left(A_{2} h\right)^{-1},\left(A_{1} h\right)^{-1}\right] \cap \mathbb{Z}_{+} \neq \emptyset$, the analysis above leads to the following meshing strategy satisfying C1-C4 for non-radially-symmetric cavitation solutions.

A meshing strategy of $\left\{\Omega_{\left(\epsilon_{i}, \tau_{i}\right)}\right\}_{i=0}^{m}$ :
(1) Take $\tilde{N}_{m} \in\left[\left(A_{2} h\right)^{-1},\left(A_{1} h\right)^{-1}\right] \cap \mathbb{Z}_{+}$. Let $\bar{N}_{0}=\min \left\{N \in \mathbb{Z}_{+}: N \geq C_{2} \epsilon_{0}^{-1 / 2}\right\}$. Set $k=\min \left\{j: 2^{j} \tilde{N}_{m} \geq \bar{N}_{0}\right\}$, and $N_{0}=2^{k} \tilde{N}_{m}$. Set $\tau_{0}=\min \left\{C_{1} \epsilon_{0}^{1 / 2}, d\left(\epsilon_{0}, h\right)\right\}$.
(2) Set $k_{0}=0$. For $i \geq 1$, set $\epsilon_{i}=\epsilon_{i-1}+\tau_{i-1}$, and

$$
\begin{equation*}
\tau_{i}=\min \left\{1-\epsilon_{i}, C_{1} \epsilon_{i}^{1 / 2}, d\left(\epsilon_{i}, h\right)\right\} \tag{5.1}
\end{equation*}
$$

If $\tau_{i}=1-\epsilon_{i}$, set $m=i$. The least admissible $N_{i} \geq C_{2} \epsilon_{i}^{-1 / 2}$ is determined as follows:

1. If $k_{i-1}<k$, set $\bar{N}_{i}=\frac{N_{i-1}}{2}$. If $\bar{N}_{i} \geq C_{2} \epsilon_{i}^{-1 / 2}$, then set $k_{i}=k_{i-1}+1$, $N_{i}=\bar{N}_{i}$; otherwise, set $k_{i}=k_{i-1}, N_{i}=N_{i-1}$.
2. If $k_{i-1}=k$, set $k_{i}=k_{i-1}, N_{i} \underset{\sim}{=} N_{i-1}$.

REMARK 5.1. By setting $k=0, N_{i}=\tilde{N}_{m}, 0 \leq i \leq m$, the meshing strategy above can be adapted to create a mesh for the radially-symmetric solutions, for which the preservation of orientation adds no restrictions on $N_{i}$ (see Theorem 3.3 and Remark 3.4). As a consequence, the total degrees of freedom of a dual-parametric bi-quadratic FE approximation, in which a triangulation is introduced on the domain by local polar coordinates maps and the shape functions are bi-quadratic with respect to $(r, \theta)$, are significantly less than that of an iso-parametric quadratic FE approximation, where both the elements in the triangulation and the shape functions are given by quadratic functions defined on a reference triangle and where the orientationpreservation condition plays a leading role in determining $N_{i}$, especially when $\epsilon_{i} \ll h$ [24].

THEOREM 5.2. Let $u$ be a cavity solution satisfying the assumptions of Theorem 3.1. Then, for a given constant $C \geq(2-p) 2^{p-1}$, there exists $0<\hat{h} \leq$ $\min \left\{\frac{2-p}{2^{2-p} C}, \frac{2-p}{2^{p-1} C}\right\}$ such that, for a reference mesh size $0<h \leq \hat{h}$, on a mesh $\left\{\Omega_{\left(\epsilon_{i}, \tau_{i}\right)}\right\}_{i=0}^{m}$ with $\epsilon_{i}, \tau_{i}$, $N_{i}$ produced by the above meshing strategy, we have $\operatorname{det} \nabla \Pi u(x)$ $>0$, and

$$
\begin{align*}
\|u-\Pi u\|_{\infty} & =O\left(h^{3}\right)  \tag{5.2}\\
\|u-\Pi u\|_{1, p} & =O\left(h^{2}\right)  \tag{5.3}\\
\operatorname{err}\left(E\left(u ; \Omega_{\epsilon_{0}}\right)\right) & =O\left(h^{2}\right) \tag{5.4}
\end{align*}
$$

Proof. The claims $\operatorname{det} \nabla \Pi u(x)>0$, (5.2) and (5.3) follow from Theorem 3.1, Theorem 4.1 and Theorem 4.2 respectively; (5.4) is a direct consequence of (4.10) and $\sum_{i=0}^{m} A\left(\epsilon_{i}, \tau_{i}\right)=(2-p) \int_{\epsilon}^{1} R^{1-p} d R \leq 1$. What remains to show is $N_{i}^{-1}=O(h)$, which is a consequence of $\tilde{N}_{m} \sim 1 / h$ and $N_{i} \geq \tilde{N}_{m}$.

To estimate the total degrees of freedom, we need the following lemma.
Lemma 5.3. Let $f(x)=C_{1} x^{1 / 2}+x-\left(x^{2-p}+C h\right)^{\frac{1}{2-p}}, x \in[0,1]$, where $C_{1}>0$, $C>0$ and $1<p<2$ are given constants. Then, there exist positive constants $a_{1}<a_{2}$ independent of $h<\bar{h}_{0}=\min \left\{\frac{\left(1+C_{1}\right)^{2-p}-1}{C}, \frac{(2-p) C_{1}}{C}\left(1+\frac{C}{s}\right)^{\frac{1-p}{2-p}}\right\}$, where $s$ is the bigger root of the equation $(2-p) C_{1} x^{2}-C x-C^{2}=0$, such that $f(x)<0$ if $x \in\left[0, a_{1} h^{\frac{2}{2-p}}\right]$, and $f(x) \geq 0$ if $x \in\left[a_{2} h^{\frac{2}{2-p}}, 1\right]$.

Proof. Since $x \geq 0$ and $1<p<2$, it follows that

$$
f(x) \leq \bar{f}(x)=C_{1} x^{1 / 2}+x-C^{\frac{1}{2-p}} h^{\frac{1}{2-p}}
$$

thus, $f(x)<0$, if $x<x_{1} \triangleq 4\left(\sqrt{C_{1}^{2}+4(C h)^{\frac{1}{2-p}}}+C_{1}\right)^{-2} C^{\frac{2}{2-p}} h^{\frac{2}{2-p}}$, which is the bigger root of $\bar{f}(x)=0$.

For $x^{2-p}>s h$, it is easily verified that

$$
\begin{aligned}
f(x) & =C_{1} x^{1 / 2}+x-x\left(1+x^{p-2} C h\right)^{\frac{1}{2-p}} \\
& =C_{1} x^{1 / 2}+x-x\left(1+\xi x^{p-2} C h\right),
\end{aligned}
$$

where $\xi=\frac{1}{2-p}(1+\eta)^{\frac{p-1}{2-p}}$ and $0<\eta<x^{p-2} C h<C s^{-1}$, thus, we have

$$
\begin{equation*}
f(x)>C_{1} x^{1 / 2}-\frac{1}{2-p}\left(1+C s^{-1}\right)^{\frac{p-1}{2-p}} C h x^{p-1} . \tag{5.5}
\end{equation*}
$$

If $p \geq 3 / 2$, then $x^{p-1} \leq x^{1 / 2}$, hence $f(x)>\left(C_{1}-\frac{1}{2-p}\left(1+C s^{-1}\right)^{\frac{p-1}{2-p}} C h\right) x^{1 / 2}>0$, since $h<\bar{h}_{0}$. If $p<3 / 2$, then $\frac{p-1}{2-p}<1$ and $x^{p-1} h<x^{2-p} x^{p-1} / s=x / s<x^{1 / 2} / s$, thus, it follows from (5.5) that

$$
f(x)>\left(C_{1}-\frac{C s^{-1}}{2-p}\left(1+C s^{-1}\right)^{\frac{p-1}{2-p}}\right) x^{1 / 2}>\left(C_{1}-\frac{C s^{-1}}{2-p}\left(1+C s^{-1}\right)\right) x^{1 / 2}
$$

By the definition of $s$, this leads to $f(x)>0$.
On the other hand, for $x^{2-p} \leq s h$, denoting $M=(C+s)^{\frac{1}{2-p}}$, we have

$$
\begin{aligned}
f(x) & =C_{1} x^{1 / 2}+x-(C h)^{\frac{1}{2-p}}\left(1+x^{2-p} C^{-1} h^{-1}\right)^{\frac{1}{2-p}} \\
& \geq C_{1} x^{1 / 2}+x-(C h)^{\frac{1}{2-p}}\left(1+s C^{-1}\right)^{\frac{1}{2-p}} \\
& =\tilde{f}(x)=C_{1} x^{1 / 2}+x-M h^{\frac{1}{2-p}}
\end{aligned}
$$

Hence $f(x) \geq 0$, if $x \geq x_{2} \triangleq 4 M^{2}\left(C_{1}+\sqrt{C_{1}^{2}+4 M h^{\frac{1}{2-p}}}\right)^{-2} h^{\frac{2}{2-p}}$, which is the bigger root of the equation $\tilde{f}(x)=0$. The proof is completed by taking $a_{1}=$ $4\left(\sqrt{C_{1}^{2}+4 C^{\frac{1}{2-p}}}+C_{1}\right)^{-2} C^{\frac{2}{2-p}}$ and $a_{2}=M^{2} C_{1}^{-2}=(C+s)^{\frac{2}{2-p}} C_{1}^{-2}$.

THEOREM 5.4. Let $C_{1}>0, C_{2}>0,1<p<2$ and $C \geq(2-p) 2^{p-1}$ be given. Let $a_{1}<a_{2}, \bar{h}_{0}$ and $\hat{h}$ be the constants given in Lemma 5.3 and Theorem 5.2 respectively. For given $\epsilon_{0}<1$, let $\left\{\epsilon_{i}, \tau_{i}, N_{i}\right\}_{i=0}^{m}$ be defined by the meshing strategy with $h<\bar{h}_{1}=\min \left\{\bar{h}_{0}, \hat{h}, C_{1}^{2-p} a_{1}^{1-p / 2}\left(a_{2}-a_{1}\right)^{p-2}, C a_{2}^{p-2}\right\}$. Then, we have

$$
\begin{align*}
& m \leq M_{2}= \begin{cases}{\left[\log _{2} \log _{b(h)} \epsilon_{0} C_{1}^{-2}\right]+3+\left[(C h)^{-1}\right],} & \text { if } a_{2} h^{\frac{2}{2-p}} \geq \epsilon_{0} \\
{\left[(C h)^{-1}\right],} & \text { otherwise }\end{cases}  \tag{5.6}\\
& m \geq M_{1}=\frac{1-\max \left\{\epsilon_{0}^{2-p}, a_{2}^{2-p} h\right\}}{C h}-1, \tag{5.7}
\end{align*}
$$

where $b(h)=a_{1} C_{1}^{-2} h^{\frac{2}{2-p}}$. Consequently, the total degrees of freedom $N_{d}$ satisfies

$$
\begin{equation*}
\tilde{N}_{m} M_{1} \leq N_{d} \leq N_{0} M_{2} \tag{5.8}
\end{equation*}
$$

where $N_{0}$ and $\tilde{N}_{m}$ are given as in the meshing strategy (1).
Proof. By Lemma 5.3, $C_{1} x^{1 / 2}<d(x, h)$ if $x \leq a_{1} h^{\frac{2}{2-p}}$, and $C_{1} x^{1 / 2} \geq d(x, h)$ if $x \geq a_{2} h^{\frac{2}{2-p}}$. Hence, by (5.1), $\epsilon_{i+1}=\epsilon_{i}+C_{1} \epsilon_{i}^{1 / 2}$ if $\epsilon_{i} \leq a_{1} h^{\frac{2}{2-p}}$, and $\epsilon_{i+1}=$ $\epsilon_{i}+d\left(\epsilon_{i}, h\right)$ if $\epsilon_{i} \geq a_{2} h^{\frac{2}{2-p}}$. Let $\epsilon_{m_{1}}$ be the biggest $\epsilon_{i}$ such that $\epsilon_{i} \leq a_{1} h^{\frac{2}{2-p}}$, then for all $i \leq m_{1}, \epsilon_{i+1}=\epsilon_{i}+C_{1} \epsilon_{i}^{1 / 2}$. We have that $\epsilon_{m_{1}}=\epsilon_{m_{1}-1}+C_{1} \epsilon_{m_{1}-1}^{1 / 2}>C_{1} \epsilon_{m_{1}-1}^{1 / 2}>$ $C_{1}^{1+1 / 2} \epsilon_{m_{1}-2}^{1 / 2^{2}}>\cdots>C_{1}^{1+1 / 2+\cdots+1 / 2^{m_{1}-1}} \epsilon_{0}^{1 / 2^{m_{1}}}=C_{1}^{2}\left(\frac{\epsilon_{0}}{C_{1}^{2}}\right)^{1 / 2^{m_{1}}}$. Let $j$ be the smallest integer $i$ such that $C_{1}^{2}\left(\frac{\epsilon_{0}}{C_{1}^{2}}\right)^{1 / 2^{i}} \geq a_{1} h^{\frac{2}{2-p}}$, then $m_{1} \leq j$. By the definition of $j$, one has

$$
m_{1} \leq j= \begin{cases}\log _{2} \log _{b(h)} \frac{\epsilon_{0}}{C_{1}^{2}}, & \text { if } \log _{2} \log _{b(h)} \frac{\epsilon_{0}}{C_{1}^{2}} \text { is an integer }  \tag{5.9}\\ {\left[\log _{2} \log _{b(h)} \frac{\epsilon_{0}}{C_{1}^{2}}\right]+1,} & \text { otherwise }\end{cases}
$$

Let $\epsilon_{m_{2}}$ be the smallest $\epsilon_{i}$ such that $\epsilon_{i} \geq a_{2} h^{\frac{2}{2-p}}$, then, for all $m_{2} \leq i<m$, $\epsilon_{i+1}^{2-p}=\epsilon_{i}^{2-p}+C h=\epsilon_{m_{2}}^{2-p}+C\left(i+1-m_{2}\right) h$. It follows from the facts that $\epsilon_{m+1}=1.0$
and $\tau_{m}=\min \left\{1-\epsilon_{m}, d\left(\epsilon_{m}, h\right), C_{1} \epsilon_{m}^{1 / 2}\right\}$ that $m$ is the smallest integer $j$ such that $\epsilon_{j}^{2-p}+C h=\epsilon_{m_{2}}^{2-p}+C\left(j+1-m_{2}\right) h \geq 1$. Hence

$$
m= \begin{cases}m_{2}-1+\frac{1-\epsilon_{m_{2}}^{2-p}}{C h}, & \text { if } \frac{1-\epsilon_{m_{2}}^{2-p}}{C h} \text { is an integer }  \tag{5.10}\\ m_{2}+\left[\frac{1-\epsilon_{m_{2}}^{2-p}}{C h}\right], & \text { otherwise }\end{cases}
$$

Next, we estimate $m_{2}-m_{1}$. By the definition of $m_{1}, \epsilon_{m_{1}+1}=\epsilon_{m_{1}}+C_{1} \epsilon_{m_{1}}^{1 / 2}>a_{1} h^{\frac{2}{2-p}}$. Thus, $\tau_{m_{1}+1} \geq \min \left\{C_{1} a_{1}^{\frac{1}{2}} h^{\frac{1}{2-p}}, d\left(\epsilon_{m_{1}+1}, h\right)\right\}$. This implies $\epsilon_{m_{1}+2}>\min \left\{C_{1} a_{1}^{1 / 2} h^{\frac{1}{2-p}}\right.$ $\left.+a_{1} h^{\frac{2}{2-p}},(C h)^{\frac{1}{2-p}}\right\}$. Hence, by the definition of $\bar{h}_{1}, \epsilon_{m_{1}+2}>a_{2} h^{\frac{2}{2-p}}$. Consequently, by the definition of $\epsilon_{m_{2}}$, we conclude $m_{2} \leq m_{1}+2$, which together with (5.9) and (5.10) yields (5.6).

If $\epsilon_{0} \geq a_{2} h^{\frac{2}{2-p}}$, then (5.7) follows directly from (5.10), since in this case $m_{2}=0$. If $\epsilon_{0}<a_{2} h^{\frac{2}{2-p}}$, then $m_{2} \geq 1$ and $\epsilon_{m_{2}-1}<a_{2} h^{\frac{2}{2-p}}$ by the definition of $m_{2}$. Thus, for all $m_{2} \leq i \leq m, \epsilon_{i}^{2-p}=\epsilon_{m_{2}}^{2-p}+C\left(i-m_{2}\right) h \leq \epsilon_{m_{2}-1}^{2-p}+C\left(i+1-m_{2}\right) h$. Set

$$
j_{0}= \begin{cases}m_{2}-2+\frac{1-\epsilon_{m_{2}-1}^{2-p}}{C h}, & \text { if } \frac{1-\epsilon_{m_{2}}^{2-p}}{C h} \text { is an integer }  \tag{5.11}\\ m_{2}-1+\left[\frac{1-\epsilon_{m_{2}-1}^{2-1}}{C h}\right], & \text { otherwise }\end{cases}
$$

Then, it is easily verified that $\epsilon_{i}<1$, for all $i \leq j_{0}$. Hence, by the definition of $m$, we conclude $m \geq j_{0}$, which implies (5.7), since $m_{2} \geq 1$ and $\epsilon_{m_{2}-1}<a_{2} h^{\frac{2}{2-p}}$.

It is worth noticing that there are two solution-dependent constants $C_{1}$ and $C_{2}$, which are not known a priori, used in the meshing strategy. In applications, we can always start with $C_{1}:=d\left(\epsilon_{0}, h\right) \epsilon_{0}^{-1 / 2}$ and $C_{2}:=\tilde{N}_{m} \epsilon_{0}^{1 / 2}$, which are the least $C_{1}$ and greatest $C_{2}$ such that the orientation-preservation conditions will practically not affect the mesh produced. It is of vital importance to know what would happen if the constants are not properly given, and how to adjust the mesh in an a posteriori fashion so that the conditions C1-C4 are satisfied in the end. To specify this, we present below two examples in both the radially-symmetric and the nonsymmetric cases, where the energy density is given by (1.6) with $p=3 / 2, \omega=2 / 3$, and $g(x)=$ $2^{-1 / 4}\left(\frac{1}{2}(x-1)^{2}+\frac{1}{x}\right)$, and we take $A=A_{1}=A_{2}=0.8, h=0.05$.

EXAMPLE 5.5. In the radially-symmetric case, let $\epsilon_{0}=0.0001$, $u_{0}(x)=2 x$, and $N_{i}=N_{h}=A / h$. For $C=2, C_{1}=1.0$ and 0.9 , the mesh strategy produces two meshes shown in Table 5.5. While the numerical solutions obtained on both meshes successfully capture the cavitation, the solution with a $C_{1}$ marginally too big fails to be orientation preserving. In fact, $\operatorname{det} \nabla u_{h}(x)<0$ is detected on the element vertices on the inner boundary $\left\{x:|x|=\epsilon_{0}\right\}$, where the preservation of orientation is found most easily broken. Our numerical experiments show that, whenever this happens, instead of reducing $C_{1}$, a proper mesh can be obtained simply by dividing the innermost layer into two, repeating if necessary, according to condition C3.

TABLE 1
Radially-symmetric case: $\epsilon_{0}=0.0001, N_{i}=16,0 \leq i \leq 9, \epsilon_{10}=1.0$.

| $C_{1}$ | $\epsilon_{1}$ | $\epsilon_{2}$ | $\epsilon_{3}$ | $\epsilon_{4}$ | $\epsilon_{5}$ | $\epsilon_{6}$ | $\epsilon_{7}$ | $\epsilon_{8}$ | $\epsilon_{9}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 0.0101 | 0.0402 | 0.0903 | 0.1604 | 0.2505 | 0.3606 | 0.4907 | 0.6408 | 0.8109 |
| 0.9 | 0.0091 | 0.0382 | 0.0873 | 0.1563 | 0.2454 | 0.3545 | 0.4836 | 0.6327 | 0.8017 |

EXAMPLE 5.6. Let $\epsilon_{0}=0.0005$ and $u_{0}(x)=\left(2.5 x_{1}, 2 x_{2}\right)^{T}$, then, the corresponding cavity solution is non-radially-symmetric. Now, we are facing the problem of choosing $C_{1}$ and $C_{2}$ appearing in the conditions $\tau \leq C_{1} \epsilon^{1 / 2}$ and $N \geq C_{2} \epsilon^{-1 / 2}$. For $C=3, C_{1}=1.0$ and, $C_{2}=1.1$, the meshing strategy produces a mesh shown in "Test 1" in Table 2, where $N_{i}=N_{0} / 2, \forall i \geq 1$, which holds also for other three tests. It turns out that the numerical solution obtained on this mesh is indeed orientation preserving, while for $C_{1}=1.25$ (see Test 2) or $C_{2}=0.9$ (see Test 3), the numerical solutions obtained on the corresponding meshes will fail to be orientation preserving. Again, it is found that the failure is most likely to happen on the element vertices on the inner boundary of the domain $\Omega_{\epsilon_{0}}$. And again, instead of reducing $C_{1}$ or increasing $C_{2}$, a proper mesh can usually be obtained simply by dividing the inner most layer into two (see Test 4 where $\epsilon_{8}=1.0$ ), according to condition C3, or doubling $N_{0}$, or both, and repeating the process if necessary.

TABLE 2
Non-radially-symmetric case: $\epsilon_{0}=0.0005, u_{0}(x)=\left(2.5 x_{1}, 2 x_{2}\right)^{T}$.

| Test | $\epsilon_{1}$ | $\epsilon_{2}$ | $\epsilon_{3}$ | $\epsilon_{4}$ | $\epsilon_{5}$ | $\epsilon_{6}$ | $\epsilon_{7}$ | $N_{0}$ | Result |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 0.0229 | 0.0907 | 0.2036 | 0.3614 | 0.5643 | 0.7946 | 1.0 | 50 | Succeed |
| 2 | 0.0285 | 0.1016 | 0.2198 | 0.3829 | 0.5911 | 0.8442 | 1.0 | 50 | Fail |
| 3 | 0.0229 | 0.0907 | 0.2036 | 0.3614 | 0.5643 | 0.7946 | 1.0 | 40 | Fail |
| 4 | 0.0091 | 0.0285 | 0.1016 | 0.2198 | 0.3829 | 0.5911 | 0.8442 | 50 | Succeed |

REMARK 5.7. In our code, the condition $\operatorname{det} \nabla u_{h}>0$ is firstly only checked on quadrature nodes in a gradient flow iteration; after the iteration converges, the condition $\operatorname{det} \nabla u_{h}>0$ is checked on elements vertices, particulary those on the inner boundary of the domain $\Omega_{\epsilon_{0}}$, where the condition is most easily broken; the mesh layer, where $\operatorname{det} \nabla u_{h}<0$ is detected, is then refined accordingly. Such a modified meshing strategy practically makes the whole process more efficient, though a mesh so obtained is not necessarily "optimal".
6. Numerical experiments and results. In our numerical experiments, the energy density is given by (1.6) with $p=3 / 2, \omega=2 / 3$, and $g(x)=2^{-1 / 4}\left(\frac{1}{2}(x-1)^{2}+\right.$ $\frac{1}{x}$ ), the domain $\Omega_{\epsilon_{0}}=B_{1}(0) \backslash B_{\epsilon_{0}}(0)$ with a displacement boundary $\Gamma_{0}=\partial B_{1}(0)$ and a traction free boundary $\Gamma_{1}=\partial B_{\epsilon_{0}}(0)$, and the meshes used are shown in Table 3 and Table 4, which are produced by the meshing strategy with $C=2, C_{1}=0.9$, $A=0.8$ for $\epsilon_{0}=0.01, \epsilon_{0}=0.0001$ and various $h$. It happens that, in all these meshes, $N_{i}=N_{h}=A / h$ on each of the $m+1$ mesh layers. Figure 3 shows that the total degrees of freedom $N_{d}$ is basically a quadratic function of $h^{-1}$.

Table 3
$\epsilon_{0}=0.01$.

| $h$ | $\min \tau_{i}$ | $\max \tau_{i}$ | $m$ | $N_{h}$ |
| :--- | :--- | :--- | :--- | :--- |
| 0.04 | 0.0224 | 0.1504 | 11 | 20 |
| 0.03 | 0.0156 | 0.1164 | 14 | 30 |
| 0.02 | 0.0096 | 0.0768 | 22 | 40 |
| 0.01 | 0.0044 | 0.0396 | 44 | 80 |
| 0.005 | 0.0021 | 0.0199 | 89 | 160 |

Table 4
$\epsilon_{0}=0.0001$.

| $h$ | $\min \tau_{i}$ | $\max \tau_{i}$ | $m$ | $N_{h}$ |
| :--- | :--- | :--- | :--- | :--- |
| 0.04 | 0.008 | 0.1488 | 12 | 20 |
| 0.03 | 0.0048 | 0.1128 | 16 | 30 |
| 0.02 | 0.0024 | 0.076 | 24 | 40 |
| 0.01 | 0.0008 | 0.0392 | 49 | 80 |
| 0.005 | 0.0003 | 0.0197 | 98 | 160 |

6.1. Radially-symmetric case with $\left.u(x)\right|_{\Gamma_{0}}=\lambda x$. The convergence behavior of the radially-symmetric numerical cavity solutions corresponding to $\lambda=2.0$ obtained by the dual-parametric bi-quadratic FEM are shown in Figure 4-Figure 6, where the
high precision numerical solutions to the equivalent 1-D ODE problem are taken as the exact solutions [2, 14].


Fig. 3. $N_{d} \sim K h^{-2}$.


Fig. 4. The energy error.

Figure 4 shows that the energy error $\left|E(u)-E\left(u_{h}\right)\right|=O\left(N_{d}^{-2}\right)=O\left(h^{4}\right)$, which is even better than our energy error estimate on the interpolation function (see (5.4)). In Figure 5 and Figure 6, it is shown that $\left\|u-u_{h}\right\|_{0,2}=O\left(N_{d}^{-3 / 2}\right)=O\left(h^{3}\right)$ and $\left\|u-u_{h}\right\|_{1, p}=O\left(N_{d}^{-1}\right)=O\left(h^{2}\right)$ respectively, which are in good agreement with our interpolation error estimates (see (5.2) and (5.3)).

A comparison between $W^{1, p}$ errors of the iso-parametric triangular FEM ([24]) and the dual-parametric bi-quadratic FEM is also shown in Figure 6, which demonstrates that the latter should be a more efficient method in cavitation computations.


Fig. 5. The $L^{2}$ errors of $u_{h}$.


Fig. 6. The $W^{1, p}$ errors of $u_{h}$.
6.2. Non-radially-symmetric case with $\left.u(x)\right|_{\Gamma_{0}}=\left(\lambda_{1} x_{1}, \lambda_{2} x_{2}\right)^{T}$. The numerical results for $\lambda_{1}=2.5, \lambda_{2}=2.0, \epsilon_{0}=0.01$ obtained on the mesh given in Table 3 are shown in Figure 7-Figure 10.

Figure 7 shows the numerical solution with $h=0.02$, where the cavity is seen to be approximately an ellipse. The convergence behaviors of the energy, semi-major axis, and semi-minor axis of the cavity, with respect to the mesh size $h$, are displayed respectively in Figure 8, Figure 9(a) and Figure 9(b). The convergence behavior of $\left\|u_{h}-u_{h / 2}\right\|_{0,2}$ and $\left|u_{h}-u_{h / 2}\right|_{1, p}$, in terms of $N_{d} \sim h^{-2}$, are demonstrated respectively


Fig. 7. The numerical solution.

(a) Convergence of semi-major axis.


Fig. 8. Convergence of the energy.

(b) Convergence of semi-minor axis.

Fig. 9. The convergence behavior of the cavity of $u_{h}$.


Fig. 10. The convergence behavior of $u_{h}-u_{h / 2}$.
in Figure 10(a) and Figure 10(b). The numerical results are clearly in good agreement with our analytical results (see (5.2)-(5.4)).
7. Conclusion. We derived the orientation-preservation conditions and interpolation errors of the dual-parametric bi-quadratic rectangular FEM for both radiallysymmetric and general non-symmetric cavity solutions, which is the first theoretical result of its kind in this field, and established an optimal meshing strategy for the method in computing void's growth based on an error equi-distribution principle. We would like to emphasize here that, even though there are two a priori unknown solution dependent constants $C_{1}$ and $C_{2}$ involved in the meshing strategy, we can always make a posteriori adjustments so that the strategy works well (see $\S 5$ for details). Numerical results obtained on the meshes produced by our meshing strategy verified the efficiency of the method. In fact, our numerical experiments showed that the convergence rates of the finite element cavitation solutions are in good agreement with our interpolation error estimates, and the total degrees of freedom needed for the method to achieve a given level of approximation accuracy is of an optimal order, and is much less than that of the iso-parametric quadratic triangular FEM in practical cavitation computations.

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