Numerical Solutions to
Partial Differential Equations

Zhiping Li

LMAM and School of Mathematical Sciences
Peking University
Céa Lemma — an Abstract Error Estimate Theorem

1. Consider the variational problem of the form
   \[
   \begin{cases}
   \text{Find } u \in \mathbb{V} \text{ such that } \\
   a(u, v) = f(v), \quad \forall v \in \mathbb{V}.
   \end{cases}
   \]

2. Consider the conforming finite element method of the form
   \[
   \begin{cases}
   \text{Find } u_h \in \mathbb{V}_h \subset \mathbb{V} \text{ such that } \\
   a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h.
   \end{cases}
   \]

3. The problem: how to estimate the error \( \| u - u_h \| \)?

4. The method used for FDM is not an ideal framework for FEM.

5. The standard approach for the error estimations of a finite element solution is to use an abstract error estimate to reduce the problem to a function approximation problem.
Céa Lemma — an Abstract Error Estimate Theorem

**Theorem**

Let $\mathcal{V}$ be a Hilbert space, $\mathcal{V}_h$ be a linear subspace of $\mathcal{V}$. Let the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ satisfy the conditions of the Lax-Milgram lemma (see Theorem 5.1). Let $u \in \mathcal{V}$ be the solution to the variational problem, and $u_h \in \mathcal{V}_h$ satisfy the equation

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathcal{V}_h.$$

Then, there exist a constant $C$ independent of $\mathcal{V}_h$, such that

$$\|u - u_h\| \leq C \inf_{v_h \in \mathcal{V}_h} \|u - v_h\|,$$

where $\|\cdot\|$ is the norm of $\mathcal{V}$. 
Proof of the Céa Lemma

1. Since \( u \) and \( u_h \) satisfy the equations, and \( \nabla_h \subset \nabla \), we have
\[
a(u-u_h, w_h) = a(u, w_h) - a(u_h, w_h) = f(w_h) - f(w_h) = 0, \quad \forall w_h \in \nabla_h.
\]

2. In particular, taking \( w_h = u_h - v_h \) leads to
\[
a(u-u_h, u_h - v_h) = 0.
\]

3. The \( \nabla \)-ellipticity \( \Rightarrow \) \( \alpha \| u - u_h \|^2 \leq a(u - u_h, u - u_h) \).

4. The boundedness \( \Rightarrow \) \( a(u - u_h, u - v_h) \leq M \| u - u_h \| \| u - v_h \| \).

5. Hence, \( \alpha \| u - u_h \|^2 \leq a(u - u_h, u - v_h) \leq M \| u - u_h \| \| u - v_h \| \).

6. Take \( C = M/\alpha \), we have
\[
\| u - u_h \| \leq C \| u - v_h \|, \quad \forall v_h \in \nabla_h.
\]

7. The conclusion of the theorem follows. \( \square \)
Remarks on the Céa Lemma

1. The Céa lemma reduces the error estimation problem of $\|u - u_h\|$ to the optimal approximation problem of $\inf_{v_h \in V_h} \|u - v_h\|$.  

2. Error of the finite element solution $\|u - u_h\|$ is of the same order as the optimal approximation error $\inf_{v_h \in V_h} \|u - v_h\|$.  

3. Suppose the $V_h$-interpolation function $\Pi_h u$ of $u$ is well defined in the finite element function space $V_h$, then,  
$$\|u - u_h\| \leq C \inf_{v_h \in V_h} \|u - v_h\| \leq C \|u - \Pi_h u\|.$$  

4. Therefore, the error estimation problem of $\|u - u_h\|$ can be further reduced to the error estimation problem for the $V_h$-interpolation error $\|u - \Pi_h u\|$. 
For Symmetric $a(\cdot, \cdot)$, $u_h$ Is a Orthogonal Projection of $u$ on $\nabla_h$

1. If the $\nabla$-elliptic bounded bilinear form $a(\cdot, \cdot)$ is symmetric, then, $a(\cdot, \cdot)$ defines an inner product on $\nabla$, with the induced norm equivalent to the $\nabla$-norm.

2. Denote $P_h : \nabla \to \nabla_h$ as the orthogonal projection operator induced by the inner product $a(\cdot, \cdot)$. Then,
$$a(u - P_h u, v_h) = 0, \quad \forall v_h \in \nabla_h.$$

3. Therefore, the finite element solution $u_h = P_h u$, i.e. it is the orthogonal projection of $u$ on $\nabla_h$ with respect to the inner product $a(\cdot, \cdot)$. 
Céa Lemma for Symmetric $a(\cdot, \cdot)$

**Corollary**

Under the conditions of the Céa Lemma, if the bilinear form $a(\cdot, \cdot)$ is in addition symmetric, then, the solution $u_h$ is the orthogonal projection, which is induced by the inner product $a(\cdot, \cdot)$, of the solution $u$ on the subspace $V_h$, meaning $u_h = P_h u$.

Furthermore, we have

$$a(u - u_h, u - u_h) = \inf_{v_h \in V_h} a(u - v_h, u - v_h).$$

The proof follows the same lines as the proof of the Céa lemma. The only difference here is that $\alpha = M = 1$. 

Céa Lemma in the Form of Orthogonal Projection Error Estimate

Denote $\tilde{P}_h : \mathbb{V} \to \mathbb{V}_h$ as the orthogonal projection operator induced by the inner product $(\cdot, \cdot)_\mathbb{V}$ of $\mathbb{V}$, then,

$$
\| u - \tilde{P}_h u \| = \| (I - \tilde{P}_h) u \| = \inf_{v_h \in \mathbb{V}_h} \| u - v_h \|.
$$

Therefore, as a corollary of the Céa lemma, we have

**Corollary**

Let $\mathbb{V}$ be a Hilbert space, and $\mathbb{V}_h$ be a linear subspace of $\mathbb{V}$. Let $a(\cdot, \cdot)$ be a symmetric bilinear form on $\mathbb{V}$ satisfying the conditions of the Lax-Milgram lemma. Let $P_h$ and $\tilde{P}_h$ be the orthogonal projection operators from $\mathbb{V}$ to $\mathbb{V}_h$ induced by the inner products $a(\cdot, \cdot)$ and $(\cdot, \cdot)_\mathbb{V}$ respectively. Then, we have

$$
\| I - \tilde{P}_h \| \leq \| I - P_h \| \leq \frac{M}{\alpha} \| I - \tilde{P}_h \|.
$$
1-D Example on Linear Interpolation Error Estimation for $H^2$ Functions

1. $\hat{\Omega} = (0, 1), \Omega = (b, b + h), h > 0$.

2. $F : \hat{x} \in [0, 1] \rightarrow [b, b + h], F(\hat{x}) = h\hat{x} + b$: an invertible affine mapping from $\hat{\Omega}$ to $\Omega$.

3. $\hat{\Pi} : C([0, 1]) \rightarrow P_1([0, 1])$: the interpolation operator with $\hat{\Pi}\hat{v}(0) = \hat{v}(0), \hat{\Pi}\hat{v}(1) = \hat{v}(1)$.

4. $\Pi : C([b, b + h]) \rightarrow P_1([b, b + h])$: the interpolation operator with $\Pi v(b) = v(b), \Pi v(b + h) = v(b + h)$. 
1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

Let $u \in \mathbb{H}^2(\Omega)$, denote $\hat{u}(\hat{x}) = u \circ F(\hat{x}) = u(h\hat{x} + b)$, then, it can be shown $\hat{u} \in \mathbb{H}^2(\hat{\Omega})$, thus, $\hat{u} \in C([0, 1])$.

$\hat{\Pi}$ is $P_1([0, 1])$ invariant: $\hat{\Pi}\hat{w} = \hat{w}$, $\forall \hat{w} \in P_1([0, 1])$, thus,

$$\left\| (I - \hat{\Pi})\hat{u} \right\|_{0,\hat{\Omega}} = \left\| (I - \hat{\Pi})(\hat{u} + \hat{w}) \right\|_{0,\hat{\Omega}} \leq \left\| I - \hat{\Pi} \right\| \left\| \hat{u} + \hat{w} \right\|_{2,\hat{\Omega}},$$

where $\left\| I - \hat{\Pi} \right\|$ is the norm of $I - \hat{\Pi} : \mathbb{H}^2(\hat{\Omega}) \to L^2(\hat{\Omega})$.

★ This shows that $I - \hat{\Pi} \in \mathcal{L}(\mathbb{H}^2(0, 1)/P_1([0, 1]); L^2(0, 1))$, and

$$\left(1\right) \left\| \hat{u} - \hat{\Pi}\hat{u} \right\|_{0,\hat{\Omega}} \leq \left\| I - \hat{\Pi} \right\| \inf_{\hat{\Pi} \in P_1(\hat{\Omega})} \left\| \hat{u} + \hat{w} \right\|_{2,\hat{\Omega}},$$

where $\inf_{\hat{w} \in P_1(\hat{\Omega})} \left\| \hat{u} + \hat{w} \right\|_{2,\hat{\Omega}}$ is the norm of $\hat{u}$ in the quotient space $\mathbb{H}^2(0, 1)/P_1([0, 1])$. 
1-D Example on Linear Interpolation Error Estimation for $H^2$ Functions

★ It can be shown that, $\exists \text{ const. } C(\hat{\Omega}) > 0$ s.t.
\begin{equation}
|\hat{u}|_{2,\hat{\Omega}} \leq \inf_{\hat{w} \in P_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}} \leq C(\hat{\Omega})|\hat{u}|_{2,\hat{\Omega}}.
\end{equation}

★ It follows from the chain rule that $\hat{u}''(\hat{x}) = h^2 u''(x)$.

★ By a change of the integral variable, and $dx = h d\hat{x}$, we obtain
\begin{equation}
\hat{u} \in H^2(\hat{\Omega}), \text{ and } |\hat{u}|_{2,\hat{\Omega}}^2 = h^3|u|_{2,\Omega}^2;
\end{equation}
\begin{equation}
\|u - \Pi u\|_{0,\Omega}^2 = h\|\hat{u} - \hat{\Pi}\hat{u}\|_{0,\hat{\Omega}}^2.
\end{equation}
1-D Example on Linear Interpolation Error Estimation for $H^2$ Functions

- The conclusion (1) says that the $L^2$ norm of the error of a $P_1$ invariant interpolation can be bounded by the quotient norm of the function in $H^2(0, 1)/P_1([0, 1])$.

- The conclusion (2) says that the semi norm $| \cdot |_{2,(0,1)}$ is an equivalent norm of the quotient space $H^2(0, 1)/P_1([0, 1])$.

- The conclusions (3) and (4) present the relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets.
The combination of (4) and (1) yields
\[
\| u - \Pi u \|_{0, \Omega} \leq h^{\frac{1}{2}} \| I - \hat{\Pi} \| \inf_{\hat{w} \in P_1(\hat{\Omega})} \| \hat{u} + \hat{w} \|_{2, \hat{\Omega}}
\]

This together with (2) and (3) lead to the expected interpolation error estimate:
\[
\| u - \Pi u \|_{0, \Omega} \leq \| I - \hat{\Pi} \| C(\hat{\Omega}) |u|_{2, \Omega} h^2, \quad \forall u \in H^2(\Omega).
\]
A Framework for Interpolation Error Estimation of Affine Equivalent FEs

1. The polynomial quotient spaces of a Sobolev space and their equivalent quotient norms ((2) in the example);

2. The relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets ((3), (4) in the example);

3. The abstract error estimates for the polynomial invariant operators ((1) in the example);

4. To estimate the constants appeared in the relations of the Sobolev semi-norms by means of the geometric parameters of the corresponding affine-equivalent open sets.
The change of integral variable will introduce the Jacobi determinant $\det \left( \frac{\partial F(\hat{x})}{\partial \hat{x}} \right)$;

in high dimensions, the Jacobi determinant represents the ratio of the volumes $|\Omega|/|\hat{\Omega}|$;

the chain rule for the $m$th derivative will produce $h^m$.

$h$ actually represents the ratio of the lengths in the directions of corresponding directional derivatives of the regions $\Omega = F(\hat{\Omega})$ and $\hat{\Omega}$.

The related technique is often referred to as the scaling technique.
Polynomial Quotient Spaces

1. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$, in which a function $\hat{v}$ is the equivalent class of $v \in \mathbb{W}^{k+1,p}(\Omega)$ in the sense that

$$\hat{v} = \{ w \in \mathbb{W}^{k+1,p}(\Omega) : (w - v) \in \mathcal{P}_k(\Omega) \}.$$

2. The quotient norm of a function $\hat{v}$ is defined by

$$\hat{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega) \rightarrow \| \hat{v} \|_{k+1,p,\Omega} := \inf_{w \in \mathcal{P}_k(\Omega)} \| v + w \|_{k+1,p,\Omega}.$$
Polynomial Quotient Spaces

3 The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ is a Banach space.

4 $\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \rightarrow |\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is a semi-norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$, and obviously $|\dot{v}|_{k+1,p,\Omega} \leq \|\dot{v}\|_{k+1,p,\Omega}$.

5 In fact, $|\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is an equivalent norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$. 
The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

**Theorem**

There exists a constant $C(\Omega)$ such that

$$\|\mathring{v}\|_{k+1,p,\Omega} \leq C(\Omega)|\mathring{v}|_{k+1,p,\Omega}, \quad \forall \mathring{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega).$$

**Proof:**

1. Let $\{p_i\}_{i=1}^N$ be a basis of $\mathbb{P}_k(\Omega)$, and let $f_i$, $i = 1, \ldots, N$, be the corresponding dual basis, meaning $f_i(p_j) = \delta_{ij}$.

2. For any $w \in \mathbb{P}_k(\Omega)$, $f_i(w) = 0$, $i = 1, \ldots, N \iff w = 0$.

3. Extend $f_i$, $i = 1, \ldots, N$, to a set of bounded linear functionals defined on $\mathbb{W}^{k+1,p}(\Omega)$. 
The Semi-norm $|v|_{k+1,p,\Omega}$ is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

4. We claim that there exists a constant $C(\Omega)$ such that
$$\|v\|_{k+1,p,\Omega} \leq C(\Omega)(|v|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v)|), \forall v \in \mathbb{W}^{k+1,p}(\Omega).$$

5. For $v \in \mathbb{W}^{k+1,p}(\Omega)$, define $\tilde{w} = -\sum_{j=1}^{N} f_j(v) p_j$, then,
$$f_i(v + \tilde{w}) = 0, \ i = 1, \ldots, N,$$ consequently,
$$\inf_{w \in \mathbb{P}_k(\Omega)} \|v + w\|_{k+1,p,\Omega} \leq \|v + \tilde{w}\|_{k+1,p,\Omega} \leq C(\Omega)|v|_{k+1,p,\Omega}.$$

What remains to show is 4. Suppose 4 doesn’t hold.

6. Then, there exists a sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathbb{W}^{k+1,p}(\Omega)$ such that
$$\|v_j\|_{k+1,p,\Omega} = 1, \forall j \geq 1$$ and
$$\lim_{j \to \infty}(|v_j|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v_j)|) = 0.$$

7. $\mathbb{W}^{k+1,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,p}(\Omega), 1 \leq p < \infty$; $\mathbb{W}^{k+1,\infty}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\bar{\Omega})$. 
The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $W^{k+1,p}(\Omega)/P_k(\Omega)$

8 So, there exist a subsequence of $\{v_j\}_{j=1}^{\infty}$, denoted again as $\{v_j\}_{j=1}^{\infty}$, and a function $v \in W^{k,p}(\Omega)$, such that
$$\lim_{j \to \infty} \|v_j - v\|_{k,p,\Omega} = 0.$$ 

9 6 and 8 imply $\{v_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $W^{k+1,p}(\Omega)$.

10 Therefore, $v$ in 8 is actually a function in $W^{k+1,p}(\Omega)$.

11 Thus, it follows from 6 that
$$|\partial^\alpha v|_{0,p,\Omega} = \lim_{j \to \infty} |\partial^\alpha v_j|_{0,p,\Omega} = 0, \quad \forall \alpha, \ |\alpha| = k + 1,$$
The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

12. By Theorem 5.2, (11) implies $v \in \mathbb{P}_k(\Omega)$.

13. On the other hand, it follows from (6) that
   
   $$f_i(v) = \lim_{j \to \infty} f_i(v_j) = 0, \quad i = 1, \ldots, N,$$

14. Therefore, by (2), we have $v = 0$.

15. On the other hand, since $v_j$ converges to $v$ in $\mathbb{W}^{k+1,p}(\Omega)$, by (6), we have $\|v\|_{k+1,p,\Omega} = \lim_{j \to \infty} \|v_j\|_{k+1,p,\Omega} = 1$.

16. The contradiction of (14) and (15) completes the proof. \qed
Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = h\hat{x} + b \in \mathbb{R}^n$

1. Let $F : \hat{x} \in \mathbb{R}^n \rightarrow F(\hat{x}) = h\hat{x} + b \in \mathbb{R}^n$, and $\Omega = F(\hat{\Omega})$, \[ \Rightarrow \text{diam}(\Omega)/\text{diam}(\hat{\Omega}) = h. \]

2. Then, $\partial^\alpha v(x) = h^{-|\alpha|} \partial^\alpha \hat{v}(\hat{x})$, and \[ dx = |\det(B)| \, d\hat{x} = h^n \, d\hat{x}. \]

3. Therefore, by a change of integral variable, we have \[ |v|_{m,p,\Omega} = h^{-m} |\det(B)|^{1/p} |\hat{v}|_{m,p,\hat{\Omega}} = h^{-m+n/p} |\hat{v}|_{m,p,\hat{\Omega}}. \]

4. \[ |v|_{m,p,\Omega}/|\hat{v}|_{m,p,\hat{\Omega}} \propto h^{-m+n/p}, \text{ with } h = \text{diam}(\Omega)/\text{diam}(\hat{\Omega}). \]
Affine Equivalent Open Sets Related by $F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n$

Let $\Omega = F(\hat{\Omega})$ be affine equivalent open set in $\mathbb{R}^n$ with

$$F : \hat{x} \in \mathbb{R}^n \to F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n,$$

For $v \in W^{m,p}(\Omega)$ and $\hat{v}(\hat{x}) = v(F(\hat{x}))$, the Sobolev semi-norms $|v|_{m,p,\Omega}$ and $|\hat{v}|_{m,p,\hat{\Omega}}$ have a similar relation for general $B$, i.e.

$$|v|_{m,p,\Omega}/|\hat{v}|_{m,p,\hat{\Omega}} \propto h^{-m+n/p},$$

where $h = \text{diam}(\Omega)/\text{diam}(\hat{\Omega})$. 
Theorem

Let $\Omega$ and $\hat{\Omega}$ be two affine equivalent open sets in $\mathbb{R}^n$. Let $v \in \mathbb{W}^{m,p}(\Omega)$ for some $p \in [1, \infty]$ and nonnegative integer $m$. Then, $\hat{v} = v \circ F \in \mathbb{W}^{m,p}(\hat{\Omega})$, and there exists a constant $C = C(m,n)$ such that

$$|\hat{v}|_{m,p,\hat{\Omega}} \leq C \|B\|^m |\det(B)|^{-1/p} |v|_{m,p,\Omega},$$

where $B$ is the matrix in the affine mapping $F$, $\| \cdot \|$ represents the operator norms induced from the Euclidian norm of $\mathbb{R}^n$. Similarly, we also have

$$|v|_{m,p,\Omega} \leq C \|B^{-1}\|^m |\det(B)|^{1/p} |\hat{v}|_{m,p,\hat{\Omega}}.$$
Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n, m)\|B\|^m |\text{det}(B)|^{-1/p} |v|_{m,p,\Omega}$

1. Let $\xi_i = (\xi_{i1}, \ldots, \xi_{in})^T \in \mathbb{R}^n$, $i = 1, \ldots, m$, be unit vectors, $D = (\partial_1, \ldots, \partial_n)$, $D^m \hat{v}(\hat{x})(\xi_1, \ldots, \xi_m) = (\prod_{i=1}^m D \cdot \xi_i) \hat{v}(\hat{x})$.

2. Assume $v \in C^m(\bar{\Omega})$, therefore, $\hat{v} \in C^m(\bar{\Omega})$ also. We have

$$|\partial^\alpha \hat{v}(\hat{x})| \leq \|D^m \hat{v}(\hat{x})\| := \sup_{\|\xi_i\| = 1, 1 \leq i \leq m} |D^m \hat{v}(\hat{x})(\xi_1, \ldots, \xi_m)|, \forall |\alpha| = m.$$

3. Let $C_1(m, n)$ be the cardinal number of $\alpha$, then

$$|\hat{v}|_{m,p,\hat{\Omega}} = \left( \int_{\hat{\Omega}} \sum_{|\alpha| = m} |\partial^\alpha \hat{v}(\hat{x})|^p d\hat{x} \right)^{1/p} \leq C_1(m, n) \left( \int_{\hat{\Omega}} \|D^m \hat{v}(\hat{x})\|^p d\hat{x} \right)^{1/p}.$$
Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m |\text{det}(B)|^{-1/p} |v|_{m,p,\Omega}$

4. On the other hand, by the chain rule of differentiations for composition of functions,

$$(D \cdot \xi)\hat{v}(\hat{x}) = D(v \circ F(\hat{x})) \xi = Dv(x) \frac{\partial F(\hat{x})}{\partial \hat{x}} \xi = (D \cdot B\xi)v(x).$$

5. Therefore, $(\prod_{i=1}^m D \cdot \xi_i)\hat{v}(\hat{x}) = (\prod_{i=1}^m D \cdot B\xi_i) v(x)$, i.e.

$$D^m\hat{v}(\hat{x})(\xi_1, \ldots, \xi_m) = D^m v(x)(B\xi_1, \ldots, B\xi_m).$$

6. Consequently, $\|D^m\hat{v}(\hat{x})\| \leq \|B\|^m \|D^m v(x)\|$.

7. Thus, by a change of integral variable, we obtain

$$\int_{\hat{\Omega}} \|D^m\hat{v}(\hat{x})\|^p d\hat{x} \leq \|B\|^{mp} |\text{det}(B^{-1})| \int_{\Omega} \|D^m v(x)\|^p dx.$$
Proof of \( |\hat{v}|_{m,p,\hat{\Omega}} \leq C(n, m) \| B \|^m |\det(B)|^{-1/p} |v|_{m,p,\Omega} \)

8 For any given \( \eta_i \in \mathbb{R}^n \) with \( \| \eta_i \| = 1, 1 \leq i \leq m \), we have

\[
D^m v(x)(\eta_1, \ldots, \eta_m) = \left[ \prod_{i=1}^{m} \sum_{j=1}^{n} \eta_{ij} \partial_j \right] v(x) = \sum_{j_1, \ldots, j_m=1}^{n} \left[ \prod_{i=1}^{m} \eta_{ij_i} \partial_{j_i} \right] v(x).
\]

9 Since, \( |\eta_{ij}| \leq 1, 1 \leq i \leq m, 1 \leq j \leq n \), we are lead to

\[
\| D^m v(x) \| \leq n^m \max_{|\alpha|=m} |\partial^\alpha v(x)| \leq n^m \left( \sum_{|\alpha|=m} |\partial^\alpha v(x)|^p \right)^{1/p}.
\]
Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n, m) \|B\|^m |\det(B)|^{-1/p} |v|_{m,p,\Omega}$

10. By (3), (7) and (9), the inequality hold for $v \in C^m(\bar{\Omega})$.

11. For $1 \leq p < \infty$, $C^m(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$, so the inequality also holds for all $v \in W^{m,p}(\Omega)$.

12. If $p = \infty$, since the inequality holds uniformly for $1 \leq q < \infty$, and for the bounded domain $\Omega$

$$\|w\|_{0,\infty,\Omega} = \lim_{q \to \infty} \|w\|_{0,q,\Omega}, \quad \forall w \in L^{\infty}(\Omega),$$

therefore, the inequality holds also for $v \in W^{m,\infty}(\Omega)$. ■
Denote the exterior and interior diameters of a region $\Omega$ as

\[
\begin{align*}
    h_\Omega &:= \text{diam} (\Omega), \\
    \rho_\Omega &:= \sup \{ \text{diam} (S) : S \subset \Omega \text{ is a } n\text{-dimensional ball} \}.
\end{align*}
\]

**Theorem**

Let $\Omega$ and $\hat{\Omega}$ be two affine-equivalent open sets in $\mathbb{R}^n$, let $F(\hat{x}) = B\hat{x} + b$ be the invertible affine mapping, and $\Omega = F(\hat{\Omega})$. Then,

\[
\|B\| \leq \frac{h}{\hat{\rho}}, \quad \text{and} \quad \|B^{-1}\| \leq \frac{\hat{h}}{\rho},
\]

where $h = h_\Omega$, $\hat{h} = h_{\hat{\Omega}}$, $\rho = \rho_\Omega$, and $\hat{\rho} = \rho_{\hat{\Omega}}$. 

Proof of $\|B\| \leq \frac{h}{\hat{\rho}}$ and the Geometric Meaning of $\det(B)$

1. By the definition of $\|B\|$, we have
   $$\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\xi\|=\hat{\rho}} \|B\xi\|.$$

2. Let the vectors $\hat{x}, \hat{y} \in \hat{\Omega}$ be such that $\|\hat{y} - \hat{x}\| = \hat{\rho}$, then, we have $x = F(\hat{x}) \in \bar{\Omega}$, $y = F(\hat{y}) \in \bar{\Omega}$.

3. Therefore, $\|B(\hat{y} - \hat{x})\| = \|F(\hat{y}) - F(\hat{x})\| \leq h \Rightarrow \|B\| \leq \frac{h}{\hat{\rho}}$. ■

The determinant $\det(B)$ also has an obvious geometric meaning:

$$|\det(B)| = \frac{\text{meas}(\Omega)}{\text{meas}(\hat{\Omega})} \quad \text{and} \quad |\det(B^{-1})| = \frac{\text{meas}(\hat{\Omega})}{\text{meas}(\Omega)}.$$
Thank You!

习题 7：1, 3, 4