Rotation numbers, eigenvalues and the Poincaré-Birkhoff theorem

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Using the relation between rotation numbers and eigenvalues, we prove the existence of nontrivial T-periodic solutions having precise oscillatory properties for a class of asymptotically linear second order differential equations.

Key Words: asymptotically linear second order differential equations, Poincaré-Birkhoff theorem, Hill's equation, characteristic values, T-periodic solutions.

1. INTRODUCTION

In this article, we study a few problems about the existence and multiplicity of periodic solutions for asymptotically linear equations. More precisely, consider the nonlinear equation

$$x'' + f(t, x) = 0, \tag{1}$$

where

$$f(t, 0) \equiv 0.$$

Furthermore, suppose that the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and is T-periodic in t. Just for simplicity, assume also that f is locally Lipschitz in x (in the Carathéodory sense), so that uniqueness is ensured for the solutions of Cauchy problems associated to Equation (1).

Afterwards, we'll restrict to the case of asymptotically linear equations, a situation in which the global existence of the solutions is guaranteed.

Our main tool is the Poincaré-Birkhoff fixed point theorem. To understand its statement, let us introduce the Poincaré map.

Rewrite Equation (1) in the phase plane as follows:

$$\begin{cases} x' = y \\ y' = -f(t, x) \end{cases} \tag{2}$$

and let $z = (x, y)$ be a point in $\mathbb{R}^2$. Under these assumptions, for each $z_0 \in \mathbb{R}^2$ there is a unique solution $z(\cdot) = z(\cdot; z_0)$ of (2) such that $z(0) = z_0$. Throughout the paper we assume that $z(\cdot)$ is defined in $\mathbb{R}$. This is not restrictive in view of the growth assumptions of f that we are going to consider (see (6) and (7)).

From the T-periodicity of f in t, it is clear that Equation (2) has a T-periodic solution $z(t; z_0)$ if and only if there is a point $z_0 \in \mathbb{R}^2$ such that $z(T; z_0) = z_0$. Thus, we may look for fixed points of the Poincaré's operator $P\colon z_0 \mapsto z(T; z_0)$, which is well defined as an area-preserving and orientation-preserving homeomorphism of $\mathbb{R}^2$ with $P(0) = 0$.

Now, to each initial point $z_0 \in \mathbb{R}^2 \setminus \{0\}$ and to each $t \in \mathbb{R}$, we can associate a t-rotation number $\operatorname{Rot}(t; z_0)$ which is a real number that measures the turns around the origin of the solution $z(t; z_0)$ in the time-interval $[0, t]$. Switching to the standard polar coordinates, (θ, ρ), we can define $\operatorname{Rot}(t; z_0)$ as follows:

$$\begin{aligned}\operatorname{Rot}(t; z_0) :&= \frac{\theta(0) - \theta(t)}{2\pi} = \\ &= \frac{1}{2\pi} \int_0^t \frac{y(s)^2 + f(s, x(s))x(s)}{x(s)^2 + y(s)^2}\, \mathrm{d}s.\end{aligned}$$

Notice that this rotation number counts positive the clockwise rotations around the origin.

Now we introduce a version of the Poincaré-Birkhoff theorem for the map P. First of all, using the polar coordinates (θ, ρ) we have that P may be expressed as a map

$$\tilde{P}\colon (\theta_0, \rho_0) \mapsto (\theta_1, \rho_1),$$

with respect the projection $\pi\colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}^2 \setminus \{0\}$, $(\theta, \rho) \mapsto (\rho\cos\theta, \rho\sin\theta)$,

$$\begin{cases} \rho_1 = R(\theta_0, \rho_0) = R(\theta_0 + 2\pi, \rho_0); \\ \theta_1 = \theta_0 + 2\pi\gamma(\theta_0, \rho_0); \quad \gamma(\theta_0 + 2\pi, \rho_0) = \gamma(\theta_0, \rho_0), \end{cases} \tag{3}$$

where R and γ are continuous functions defined on the set $\mathbb{R} \times \mathbb{R}_0^+$. In our case, there is a natural way to choose γ: we have, with $j \in \mathbb{Z}$

$$\gamma(\theta_0, \rho_0) = -2\pi \operatorname{Rot}(T; z_0) + 2j\pi \tag{4}$$

In the language of the lifting, any Jordan curve $\Gamma \subset \mathbb{R}^2$ (a homeomorphic image of S^1) surrounding 0, can be lifted to an arc $\tilde{\Gamma} \subset \mathbb{R} \times \mathbb{R}_0^+$ which is 2π-periodic in its θ-component. Let $\mathcal{A} \subset \mathbb{R}^2 \setminus \{0\}$ be a closed topological annulus, that is the part of the plane between two disjoint Jordan curves Γ_i, Γ_e, with 0 internal to Γ_i, and Γ_i internal to Γ_e. If

$$\begin{cases} \gamma(\theta, \rho) > 0, \quad \forall (\theta, \rho) \in \tilde{\Gamma}_i, \\ \gamma(\theta, \rho) < 0, \quad \forall (\theta, \rho) \in \tilde{\Gamma}_e \end{cases}$$

or, respectively, $\gamma(\theta,\rho) < 0$ on $\tilde{\Gamma}_i$, and $\gamma(\theta,\rho) > 0$ on $\tilde{\Gamma}_e$, then we say that P twists the boundaries of $\mathcal{A}$ in opposite angular directions. In this setting, under the technical assumption that Γ_i is star-shaped around the origin, we can conclude according to the generalized Poincaré-Birkhoff fixed point theorem of W. Y. Ding [9], that $\tilde{P}$ *has at least two geometrically distinct fixed points* $(\bar{\theta},\bar{\rho})$, $(\tilde{\theta},\tilde{\rho}) \in \tilde{\mathcal{A}} = \pi^{-1}(\mathcal{A})$, *such that* $\gamma(\bar{\theta},\bar{\rho}) = \gamma(\tilde{\theta},\tilde{\rho}) = 0$.

Rephrasing this result for Equation (1), we can state the following

Theorem 1. *Assume that there is an annulus* $\mathcal{A}$ *as above such that there is* $j \in \mathbb{Z}$ *such that*

$$\begin{cases} \mathrm{Rot}(T;z_0) > j \quad (\textit{resp.} \ < j) \quad \forall z_0 \in \Gamma_i, \\ \mathrm{Rot}(T;z_0) < j \quad (\textit{resp.} \ > j) \quad \forall z_0 \in \Gamma_e, \end{cases} \tag{5}$$

then (2) *has at least two* T*-periodic solutions* $\bar{z}(t)$ *and* $\tilde{z}(t)$ *with* $\bar{z}(0) \neq \tilde{z}(0) \in \mathring{\mathcal{A}}$. *The corresponding solutions* $\bar{x}$ *and* $\tilde{x}$ *of* (1) *have exactly* $2j$ *zeros in the interval* $[0,T[$.

Now we suppose that f is asymptotically linear in 0 and in ∞, that is, we assume that there exists

$$\lim_{x\to 0} \frac{f(t,x)}{x} = q_0(t), \ \text{uniformly a.e. in } t \tag{6}$$

and there exists

$$\lim_{x\to \infty} \frac{f(t,x)}{x} = q_\infty(t), \ \text{uniformly a.e. in } t. \tag{7}$$

In this way, we get two linear comparison equations for (1) in 0 and in ∞

$$\begin{aligned} x'' + q_0(t)x = 0; \\ x'' + q_\infty(t)x = 0. \end{aligned}$$

For these two equations we can define two T-rotation numbers at the same manner as for (1). These two rotation numbers will be denoted with Rot_0 and Rot_∞ respectively. Observe that Rot_0 and Rot_∞ have the same behavior of the rotation number for (1) for x small (resp. $|x|$ large), see [19, Lemma 3] and [24]. Note also that for rotation numbers associated to linear equations we have $\mathrm{Rot}(t;z_0) = \mathrm{Rot}(t;sz_0)$, $\forall s \neq 0$ and $\forall t \in \mathbb{R}$, $z_0 \in \mathbb{R}^2 \setminus \{0\}$.

At this point, we can apply the Poincaré-Birkhoff theorem as follows.

Corollary 1. *If* $\mathrm{Rot}_0(T;z_0) > j$ *and* $\mathrm{Rot}_\infty(T;z_0) < j$ *(or viceversa), for* $j \geqslant 1$ *and for all* z_0 *in* S^1*, then there exist two* T*-periodic solutions for Equation* (1)*, with exactly* $2j$ *zeros in* $[0,T[$. (For a proof see Section 3).

The study of rotation numbers associated to second order equations and to Hamiltonian systems in higher dimensions (see [10]) is a problem of great interest.

In some recent papers ([27] and [28]) M. Zhang, using the relationship between rotation numbers and eigenvalues ([12], [16], [21]), obtained some results of existence of at least one solution for Equation (1) and its generalizations under suitable assumptions of nonresonance at infinity.

From this point of view, this article can be considered as a contribute to map the road to finding nontrivial solutions. Indeed, using Zhang's approach combined with the Poincaré-Birkhoff theorem, we are able to obtain multiplicity results under suitable assumptions at zero and at infinity.

There are different ways to link T-rotation numbers to the eigenvalue problem, for example considering

$$x'' + (\lambda + q(t))x = 0$$

or

$$x'' + \lambda q(t)x = 0,\ q > 0.$$

In this work (see Section 2), we have developed a general theory for the equation

$$x'' + q_\lambda(t)x = 0, \tag{8}$$

under conditions for q_λ which contain as special cases the two equations considered above. A treatment of the periodic eigenvalues problem for Equation (8) was given in [14] but assuming stronger regularity conditions on q_λ. Moreover, in [14] the relationship between eigenvalues and rotation number is not ensured.

An immediate consequence of Corollary 1 and the main results developed in Section 2 is the following

Theorem 2. *Consider Equation* (1) *and suppose that the following limits hold uniformly in t*

$$(f_0) \qquad \frac{f(t,x)}{x} \to q_0(t), \qquad \text{for } x \to 0;$$

$$(f_\infty) \qquad \frac{f(t,x)}{x} \to q_\infty(t), \qquad \text{for } x \to \pm\infty,$$

where q_0 and q_∞ are two T-periodic, nonnegative and non identically zero functions, with q_0 and $q_\infty \in L^1([0,T])$. Let

$$-\infty < \overline{\lambda}_0^{\,0} < \underline{\lambda}_1^0 \leqslant \overline{\lambda}_1^{\,0} < \ldots < \underline{\lambda}_k^0 \leqslant \overline{\lambda}_k^{\,0} < \ldots$$

be the eigenvalues associated to the problem

$$\begin{cases} x'' + \lambda q_0(t)x = 0; \\ x\colon\ T\text{-periodic} \end{cases} \tag{9}$$

and let

$$-\infty < \overline{\lambda}_0^{\,\infty} < \underline{\lambda}_1^\infty \leqslant \overline{\lambda}_1^{\,\infty} < \ldots < \underline{\lambda}_k^\infty \leqslant \overline{\lambda}_k^{\,\infty} < \ldots$$

be the eigenvalues associated to the problem

$$\begin{cases} x'' + \lambda q_\infty(t)x = 0; \\ x\colon\ T\text{-periodic.} \end{cases} \tag{10}$$

Suppose that there is $j \in \mathbb{N}$ such that one of the following properties is satisfied:

$$\overline{\lambda}_j^{\,0} < 1 < \underline{\lambda}_j^{\infty},$$

or

$$\overline{\lambda}_j^{\,\infty} < 1 < \underline{\lambda}_j^{0}\,.$$

Then (1) *has at least two T-periodic solutions with exactly $2j$ zeros in* $[0, T[$.

We end the introduction with some definitions and remarks.

Instead of z_0, sometimes we prefer to use θ_0, the angle that defines z_0 in the phase plane.

Definition 1. Let the (asymptotic) rotation number of (1) be defined as follows

$$\varrho := \lim_{|t|\to\infty} \frac{T}{2\pi} \frac{\theta_0 - \theta(t,\theta_0)}{t}\,. \tag{11}$$

Notice that ϱ exists and is independent of θ_0 (see, for example, [4] and [13]). Moreover, ϱ depends continuously with respect to the L^1 norm in the space of the coefficients.

Now we give the relation between the two rotation numbers introduced above. The result is taken from [12].

Remark 1. Let j be an integer. Then,

i) $\varrho \geqslant j \Leftrightarrow \max_{\theta\in\mathbb{R}} \operatorname{Rot}(T;\theta) \geqslant j$;

ii) $\varrho \leqslant j \Leftrightarrow \min_{\theta\in\mathbb{R}} \operatorname{Rot}(T;\theta) \leqslant j$.

Throughout the paper, the following basic notation is employed. For a function q, $q^+ = \max\{0, q\}$, and $q^- = \min\{0, -q\}$.

2. EIGENVALUES AND CHARACTERISTIC VALUES

In this section we study a case of Hill's equation and in particular we study Equation (8) with the hypotheses:

- $\lambda \in \mathbb{R}$;
- $q_\lambda(\cdot)$ T-periodic;
- the mapping $\mathbb{R} \ni \lambda \mapsto q_\lambda \in L^1([0, T])$ is nondecreasing in λ;

- the mapping $\mathbb{R} \ni \lambda \mapsto q_\lambda \in L^1([0,T])$ is increasing in the mean (that is $\lambda_1 < \lambda_2 \Rightarrow \int_0^T q_{\lambda_1}(t)\mathrm{d}t < \int_0^T q_{\lambda_2}(t)\mathrm{d}t$);
- the mapping $\mathbb{R} \ni \lambda \mapsto q_\lambda \in L^1([0,T])$ depends continuously with respect to the L^1 norm $\|\cdot\|_1$ on $[0,T]$.

Following [14], we consider the boundary value problem

$$(P)\begin{cases} x'' + q_\lambda(t)x = 0; \\ x(0) = x(T); \\ x'(0) = x'(T). \end{cases}$$

The (real) values of λ for which (P) has a nontrivial solution are called *eigenvalues* and a nontrivial solution satisfying (P) for an eigenvalue is called an *eigenfunction*.

Switching to the polar coordinates (θ, ρ), we can write, for a nontrivial solution of (8)

$$-\dot{\theta} = \frac{y^2 + q_\lambda(t)x^2}{x^2 + y^2} = \sin^2\theta + q_\lambda(t)\cos^2\theta =: S(t,\theta,\lambda). \tag{12}$$

Afterwards, it will be convenient to set $\vartheta = -\theta$ in order to deal with the initial value problem

$$\begin{cases} \dot{\vartheta} = S(t,\vartheta,\lambda); \\ \vartheta(0) = \vartheta_0. \end{cases}$$

Following [13] or [14], we can define the (asymptotic) rotation number

$$\varrho = \varrho(\lambda) = \lim_{t\to\infty} T\frac{\vartheta(t,\vartheta_0,\lambda) - \vartheta_0}{2\pi t}. \tag{13}$$

Lemma 1. *The number $\varrho(\lambda)$ is well defined for each λ. Moreover, the mapping $\lambda \mapsto \varrho(\lambda)$ is a continuous, monotone and nondecreasing function.*

Proof. The well-posedness of $\varrho(\lambda)$ and its continuity with respect to λ can be proved following the same argument described in [13] in the case of a continuous coefficient. The proof for an L^1 term needs some more careful estimates (see [24] for the details). The proof of the monotonicity is based on a Sturm-type comparison result.

Let $\lambda_1 < \lambda_2$ be two real numbers. We need only to show that $\mathrm{Rot}_{\lambda_1}(t,\vartheta_0) \leqslant \mathrm{Rot}_{\lambda_2}(t,\vartheta_0)$. Let us consider the two problems:

$$\begin{cases} \vartheta_1' = S(t,\vartheta_1,\lambda_1); \\ \vartheta_1(0) = \vartheta_0, \end{cases} \quad \text{and} \quad \begin{cases} \vartheta_2' = S(t,\vartheta_2,\lambda_2); \\ \vartheta_2(0) = \vartheta_0. \end{cases}$$

Since $q_{\lambda_1} \leqslant q_{\lambda_2}$, with the strict inequality on a set of positive measure, we have $S(t,\vartheta_1,\lambda_1) \leqslant S(t,\vartheta_2,\lambda_2)$. Since,

$$\vartheta_1' = S(t,\vartheta_1,\lambda_1) \leqslant S(t,\vartheta_2,\lambda_2);$$
$$\vartheta_2' = S(t,\vartheta_2,\lambda_2);$$

and the two solutions ϑ_1 and ϑ_2 have the same initial point, then by a result on the differential inequalities (see [17]), we obtain $\vartheta_1(t) \leqslant \vartheta_2(t)$, $\forall t \geqslant 0$. Furthermore, for $t \geqslant T$, the strict inequality $\vartheta_1 < \vartheta_2$ holds. Indeed if, by contradiction, $\vartheta_1(t) = \vartheta_2(t)$ $\forall t \in [0,T]$ then,

$$0 = (\vartheta_2(t) - \vartheta_1(t))' = \int_0^T \left[q_{\lambda_2}(t) - q_{\lambda_1}(t)\right] \cos^2 \vartheta_1(t) \mathrm{d}t;$$

but this is impossible since $q_{\lambda_2} > q_{\lambda_1}$ on a set of positive measure and $\cos^2 \vartheta_1$ vanishes only in a finite number of points in $[0,T]$. Therefore, there exists a $t^* \in]0,T]$ such that $\vartheta_1(t^*) < \vartheta_2(t^*)$. Consider now the following problem:

$$\begin{cases} \hat{\vartheta}_2' = S(t, \hat{\vartheta}_2, \lambda); \\ \hat{\vartheta}_2(t^*) = \vartheta_1(t^*) < \vartheta_2(t^*). \end{cases}$$

By the uniqueness of the solutions for Cauchy problems, it follows that $\hat{\vartheta}_2(t) < \vartheta_2(t)$, $\forall t \geqslant t^*$. Hence,

$$\vartheta_1(t) \leqslant \hat{\vartheta}_2(t) < \vartheta_2(t), \quad \forall t \geqslant T.$$

In this manner we have proved that $\mathrm{Rot}_{\lambda_1}(t, \vartheta_0) < \mathrm{Rot}_{\lambda_2}(t, \vartheta_0)$, for each $t \geqslant T$ and then, for $t \to \infty$, we get the thesis, that is $\varrho(\lambda_1) \leqslant \varrho(\lambda_2)$. ∎

Next, we want to show that, under suitable weak assumptions, the following properties hold:

1. $\lim\limits_{\lambda \to -\infty} \varrho(\lambda) = 0$;
2. $\lim\limits_{\lambda \to +\infty} \varrho(\lambda) = +\infty$.

Let us begin with the first one.

Lemma 2. *Assume that*

1. $\lim\limits_{\lambda \to -\infty} \int_0^T q_\lambda^+(s)\mathrm{d}s = 0$;
2. $L := \lim\limits_{\lambda \to -\infty} \int_0^T q_\lambda^-(s)\mathrm{d}s > 0$.

Let x be a solution of Equation (8) *satisfying*

$$\begin{cases} x(t_0) \geqslant 0; \\ x'(t_0) \geqslant 0; \\ x(t_0) + x'(t_0) > 0. \end{cases}$$

Then,

i) $x(t) > 0 \quad \forall t \in]t_0, t_0 + T]$;

and

ii) for λ sufficiently large and negative, the following inequalities hold:

$$\begin{cases} x(t_0+T) \geqslant 0; \\ x'(t_0+T) \geqslant 0; \\ x(t_0+T) + x'(t_0+T) > 0. \end{cases}$$

Proof.

i) Suppose, by contradiction, that there exists a $\tilde{t} \in]t_0, t_0+T]$ such that $x(\tilde{t}) \leqslant 0$ and set $t_1 = \min\{t \in]t_0, t_0+T] \,|\, x(t) = 0\}$. The two conditions, $x(t_1) = 0$ and $x'(t_0) \geqslant 0$, imply necessarily that $x'(t_1) < 0$.

By the definition of t_1, we can deduce that $x(t) \geqslant 0$ for $t \in [t_0, t_1]$, with the strict inequality on the open interval $]t_0, t_1[$. Define $x^* = \max\{x(s) \,|\, s \in [t_0, t_1]\}$; now we give an estimate of this quantity. From $x(t) = x(t_1) + \int_{t_1}^{t} x'(s)\mathrm{d}s$, and the Cauchy-Schwartz inequality,

$$x(t) \leqslant \int_{t}^{t_1} |x'(s)|\mathrm{d}s \leqslant (t_1 - t)^{1/2} \left(\int_{t}^{t_1} x'(s)^2 \mathrm{d}s \right)^{1/2}.$$

Hence,

$$x^* \leqslant \sqrt{t_1 - t_0} \left(\int_{t_0}^{t_1} x'(s)^2 \mathrm{d}s \right)^{1/2}. \tag{14}$$

If we rewrite Equation (8) as

$$-x''(t) = q_\lambda(t)x(t),$$

and multiply by $x(t)$, then integrating the result between t_0 and t_1 we obtain:

$$\int_{t_0}^{t_1} -x''(t)x(t)\mathrm{d}t = \int_{t_0}^{t_1} q_\lambda(t)x^2(t)\mathrm{d}t.$$

Integrating by parts, we have:

$$\int_{t_0}^{t_1} q_\lambda(t)x^2(t)\mathrm{d}t = -x'(t_1)x(t_1) + x'(t_0)x(t_0) + \int_{t_0}^{t_1} x'(t)^2\mathrm{d}t \geqslant$$
$$\geqslant \int_{t_0}^{t_1} x'(t)^2\mathrm{d}t.$$

Splitting $q_\lambda = q_\lambda^+ - q_\lambda^-$, it follows:

$$\int_{t_0}^{t_1} x'(s)^2\mathrm{d}s \leqslant \int_{t_0}^{t_1} q_\lambda(s)x^2(s)\mathrm{d}s \leqslant \int_{t_0}^{t_1} q_\lambda^+(s)x^2(s)\mathrm{d}s \leqslant$$
$$\leqslant (x^*)^2 \int_{t_0}^{t_1} q_\lambda^+(s)\mathrm{d}s \leqslant (x^*)^2 \int_{t_0}^{t_0+T} q_\lambda^+(s)\mathrm{d}s = (x^*)^2 \int_{0}^{T} q_\lambda^+(s)\mathrm{d}s,$$

where the last equality follows from the hypothesis that q_λ is T-periodic. Now, from relation (14) we have:

$$\int_{t_0}^{t_1} x'(s)^2 \mathrm{d}s \leqslant (t_1 - t_0) \int_{t_0}^{t_1} x'(s)^2 \mathrm{d}s \int_0^T q_\lambda^+(s) \mathrm{d}s \leqslant$$
$$\leqslant T \int_0^T q_\lambda^+(s) \mathrm{d}s \int_{t_0}^{t_1} x'(s)^2 \mathrm{d}s.$$

We have thus obtained the following

$$1 \leqslant T \int_0^T q_\lambda^+(s) \mathrm{d}s.$$

By the hypothesis, there exists a real number η such that

$$\int_0^T q_\eta^+(s) \mathrm{d}s < \frac{1}{T}$$

and, by the monotonicity of $\lambda \mapsto q_\lambda$, we obtain also $q_\lambda^+ \leqslant q_\eta^+$, for $\lambda < \eta$. Hence we find the contradiction:

$$\frac{1}{T} \leqslant \int_0^T q_\lambda^+(s) \mathrm{d}s \leqslant \int_0^T q_\eta^+(s) \mathrm{d}s < \frac{1}{T}.$$

ii) In the previous step we verified that the solution $x(t)$ is always positive in $]t_0, t_0 + T[$ and, by continuity, it's obviously nonnegative in $t = t_0 + T$. By the uniqueness of the solutions for Equation (8) it would be sufficient to show that $x'(t_0 + T) \geqslant 0$ (in fact, if this is true, the third condition of (ii) follows immediately). Suppose, by contradiction, that $x'(t_0 + T) < 0$.

By the continuity, the solution $x(t)$ has in $[t_0, t_0 + T]$ a minimum point $x_{\min} = x(\tilde{t})$ and a maximum point $x_{\max} = x(\hat{t})$. We estimate $x_{\max}$ through $x_{\min}$, repeating in part the previous argument. Rewrite Equation (8) as follows: $-x''(t) = q_\lambda(t)x(t)$, and multiply both members by $x(t)$ and integrate between t_0 and $t_0 + T$:

$$\int_{t_0}^{t_0+T} -x''(t)x(t)\mathrm{d}t = \int_{t_0}^{t_0+T} q_\lambda(t)x^2(t)\mathrm{d}t.$$

Integrating by parts,

$$-x'(t_0+T)x(t_0+T)+x'(t_0)x(t_0)+\int_{t_0}^{t_0+T} x'(t)^2\mathrm{d}t = \int_{t_0}^{t_0+T} q_\lambda(t)x^2(t)\mathrm{d}t.$$

By hypothesis, we can deduce

$$\int_{t_0}^{t_0+T} x'(t)^2\mathrm{d}t \leqslant \int_{t_0}^{t_0+T} x'(t)^2\mathrm{d}t - x'(t_0+T)x(t_0+T) + x'(t_0)x(t_0) =$$
$$= \int_{t_0}^{t_0+T} q_\lambda(t)x^2(t)\mathrm{d}t \leqslant \left(\int_0^T q_\lambda^+(t)\mathrm{d}t\right) x_{\max}^2,$$

and, taking the square roots, we obtain:

$$\sqrt{\int_{t_0}^{t_0+T} x'(t)^2 \mathrm{d}t} \leqslant x_{\max} \sqrt{\int_0^T q_\lambda^+(t)\mathrm{d}t}.$$

On the other hand,

$$x_{\max} - x_{\min} \leqslant \sqrt{|\hat{t}-\tilde{t}|} \left| \int_{\tilde{t}}^{\hat{t}} x'(s)^2 \mathrm{d}s \right|^{1/2} \leqslant \sqrt{T} \left(\int_{t_0}^{t_0+T} x'(s)^2 \mathrm{d}s \right)^{1/2} \leqslant$$

$$\leqslant \sqrt{T}\, x_{\max} \sqrt{\int_0^T q_\lambda^+(t)\mathrm{d}t}.$$

From that,

$$x_{\max} - x_{\min} \leqslant x_{\max} \sqrt{T \int_0^T q_\lambda^+(t)\mathrm{d}t}$$

and

$$x_{\max}\Big(1 - \sqrt{T \int_0^T q_\lambda^+(t)\mathrm{d}t}\Big) \leqslant x_{\min}.$$

The term in the parenthesis is positive for λ sufficiently large and negative and therefore we have

$$x_{\max} \leqslant \frac{x_{\min}}{1 - \sqrt{T \int_0^T q_\lambda^+(t)\mathrm{d}t}}. \tag{15}$$

Using once again Equation (8), we can write:

$$x'' = -q_\lambda(t)x(t) = -\Big(q_\lambda^+(t) - q_\lambda^-(t)\Big)x(t) \geqslant q_\lambda^-(t)x_{\min} - q_\lambda^+(t)x_{\max};$$

integrating between t_0 and t_0+T, and recalling the hypothesis, we obtain

$$x'(t_0+T) \geqslant x'(t_0+T) - x'(t_0) \geqslant \left(\int_{t_0}^{t_0+T} q_\lambda^-(t)\mathrm{d}t \right) x_{\min} +$$

$$- \left(\int_{t_0}^{t_0+T} q_\lambda^+(t)\mathrm{d}t \right) x_{\max} =$$

$$= x_{\min} \int_0^T q_\lambda^-(t)\mathrm{d}t - x_{\max} \int_0^T q_\lambda^+(t)\mathrm{d}t.$$

Finally, using the relation (15), we can find

$$x'(t_0+T) \geqslant x_{\min} \left[\int_0^T q_\lambda^-(t)\mathrm{d}t - \frac{\int_0^T q_\lambda^+(t)\mathrm{d}t}{1 - \sqrt{T \int_0^T q_\lambda^+(t)\mathrm{d}t}} \right].$$

This yields to a contradiction, because the limit as $\lambda \to \infty$ of the term inside the square brackets is positive. ∎

Corollary 2. *Under the assumptions of Lemma 2, $\varrho(\lambda) = 0$ for $\lambda \ll 0$.*

Proof. The easy proof is omitted. ∎

Remark 2. If one of the following two conditions hold

(A) $q_\lambda(t) = \lambda q(t)$, with $q \in L^1([0,T])$, $q \geqslant 0$, $q \neq 0$ and T-periodic;

(B) $q_\lambda(t) = \lambda + q(t)$, with $q \in L^1([0,T])$ and T-periodic;

then conditions (1) and (2) of Lemma 2 are satisfied with $L = +\infty$.

Lemma 3. *For Equation (8) suppose that*

a) $\lim\limits_{\lambda\to+\infty} \int_0^T q_\lambda^-(t)\mathrm{d}t = 0$*;*

b) there exists $\eta \in \mathbb{R}$ such that for any interval $I \subset [0,T]$ we have:

$$\int_{\mathrm{I}} q_\eta(t)\mathrm{d}t > 0 \Rightarrow \lim_{\lambda\to+\infty} \int_{\mathrm{I}} q_\lambda(t)\mathrm{d}t = +\infty.$$

Then

$$\lim_{\lambda\to+\infty} \vartheta(T;\vartheta_0,\lambda) = +\infty.$$

Proof. Idea of the proof: we divide the phase plane into four parts and we prove that the solution crosses through all of them in time-intervals that become shorter and shorter as λ increases. Accordingly, we prove the following claims.

Claim 1. Fix an arbitrary $\epsilon_1 \in]0,\pi/4[$ and let, for $k \in \mathbb{Z}$, $\mathrm{J}_{1,k}$ be an interval contained in

$$\left\{t \in [0,T] : \left|\vartheta(t) - \frac{\pi}{2} + k\pi\right| \leqslant \epsilon_1\right\}.$$

Then, uniformly with respect to $k \in \mathbb{Z}$,

$$\mu(\mathrm{J}_{1,k}) \leqslant 2\tan\epsilon_1 + \int_0^T q_\lambda^-(t)\mathrm{d}t,$$

where $\mu(\mathrm{J}_{1,k})$ is the Lebesgue measure of $\mathrm{J}_{1,k}$.

Proof of Claim 1. Since the proof is the same for every $k \in \mathbb{Z}$, for sake of simplicity, we fix k and define $\mathrm{J}_{1,k} =: \mathrm{J}_1$.

Switching to the polar coordinates, using Equation (12) and the fact that if $t \in \mathrm{J}_1$, then $\cos^2\vartheta(t) < \sin^2\vartheta(t)$, for $\epsilon_1 \in]0, \pi/4[$, $\forall k \in \mathbb{Z}$, we obtain that

$$\begin{aligned}
\dot{\vartheta} &= \sin^2\vartheta(t) + q_\lambda(t)\cos^2\vartheta(t) \geqslant \sin^2\vartheta(t) - q_\lambda^-(t)\cos^2\vartheta(t) \geqslant \\
&\geqslant \sin^2\vartheta(t) - q_\lambda^-(t)\sin^2\vartheta(t) = \big(1 - q_\lambda^-(t)\big)\sin^2\vartheta(t) \\
&\Rightarrow \frac{\dot{\vartheta}}{\sin^2\vartheta(t)} \geqslant 1 - q_\lambda^-(t),
\end{aligned}$$

for $t \in \mathrm{J}_1$. Assume $\mathrm{J}_1 = [\sigma, \tau]$, then

$$\int_\sigma^\tau \frac{\dot{\vartheta}}{\sin^2\vartheta(t)}\,\mathrm{d}t \geqslant \int_\sigma^\tau \big(1 - q_\lambda^-(t)\big)\mathrm{d}t$$

and, hence

$$-\mathrm{cotan}\,\vartheta(\tau) + \mathrm{cotan}\,\vartheta(\sigma) \geqslant (\tau - \sigma) - \int_\sigma^\tau q_\lambda^-(t)\mathrm{d}t.$$

Since,

$$\begin{aligned}
&-\mathrm{cotan}\,\vartheta(\tau) + \mathrm{cotan}\,\vartheta(\sigma) \leqslant |\mathrm{cotan}\,\vartheta(\tau)| + |\mathrm{cotan}\,\vartheta(\sigma)|, \\
&\text{and } \vartheta(\tau),\ \vartheta(\sigma) \in \left[\frac{\pi}{2} - \epsilon_1,\ \frac{\pi}{2} + \epsilon_1\right],
\end{aligned}$$

we can deduce

$$-\mathrm{cotan}\,\vartheta(\tau) + \mathrm{cotan}\,\vartheta(\sigma) \leqslant \left|\frac{1}{\tan(\frac{\pi}{2} - \epsilon_1)}\right| + \left|\frac{1}{\tan(\frac{\pi}{2} + \epsilon_1)}\right| = 2\tan\epsilon_1.$$

Since $\tau - \sigma = \mu(\mathrm{J}_1)$, we have

$$2\tan\epsilon_1 \geqslant \mu(\mathrm{J}_1) - \int_0^T q_\lambda^-(t)\mathrm{d}t$$

and then

$$\mu(\mathrm{J}_1) \leqslant 2\tan\epsilon_1 + \int_0^T q_\lambda^-(t)\mathrm{d}t,$$

therefore, Claim 1 is proved.

We proceed with the proof of Lemma 3. The next claim is

Claim 2. Fix an arbitrary $\epsilon_2 \in]0, \pi/4[$ and let, for $h \in \mathbb{Z}$, $\mathrm{J}_{2,h}$ be an interval contained in

$$\left\{t \in [0, T] : \frac{\pi}{2} + h\pi + \epsilon_2 \leqslant \vartheta(t) \leqslant \frac{\pi}{2} + (h+1)\pi - \epsilon_2\right\}.$$

Then, uniformly with respect to $h \in \mathbb{Z}$,

$$\int_{\mathrm{J}_{2,h}} q_\lambda(t)\mathrm{d}t \leqslant \frac{2}{|\tan \epsilon_2|}.$$

Proof of Claim 2. Since the proof is the same for each $h \in \mathbb{Z}$, for sake of simplicity, we fix an h and assume $\mathrm{J}_{2,h} =: \mathrm{J}_2$.

Using Equation (12), we can get the following estimate, for $\lambda \geqslant \lambda_2$:

$$\dot{\vartheta} = \sin^2 \vartheta(t) + q_\lambda(t) \cos^2 \vartheta(t) \geqslant q_\lambda(t) \cos^2 \vartheta(t)$$

Hence, $\dot{\vartheta}/(\cos^2 \vartheta(t)) \geqslant q_\lambda(t)$ a.e. in $[0, T]$.
Integrating on the interval $\mathrm{J}_2 := [\sigma, \tau]$,

$$\int_{\mathrm{J}_2} \frac{\dot{\vartheta}}{\cos^2 \vartheta(t)}\mathrm{d}t \geqslant \int_{\mathrm{J}_2} q_\lambda(t)\mathrm{d}t.$$

Hence,

$$\tan \vartheta(\tau) - \tan \vartheta(\sigma) \geqslant \int_{\mathrm{J}_2} q_\lambda(t)\mathrm{d}t.$$

Finally, the following inequalities are satisfied:

$$\tan \vartheta(\tau) - \tan \vartheta(\sigma) \leqslant |\tan \vartheta(\tau)| + |\tan \vartheta(\sigma)| \leqslant 2\Big|\tan\Big(\frac{\pi}{2} - \epsilon_2\Big)\Big| = \frac{2}{|\tan \epsilon_2|}.$$

Therefore Claim 2 is proved too.

We proceed with the proof of Lemma 3. The next claim is
Claim 3. For Equation (8), for any $\epsilon \in]0, \pi/4[$, and for λ such that

$$\int_0^T q_\lambda^-(t)\mathrm{d}t < \frac{\epsilon}{2}$$

we have:

i) if there exist $t_0 \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\vartheta(t_0) = \frac{\pi}{2} + k\pi + 2\epsilon$ then

$$\vartheta(t) > \frac{\pi}{2} + k\pi + \epsilon, \ \forall t \geqslant t_0, \text{ where } \vartheta \text{ is defined;}$$

ii) if there exist $t_0 \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\vartheta(t_0) = \frac{\pi}{2} + k\pi - 2\epsilon$ then

$$\vartheta(t) < \frac{\pi}{2} + k\pi - \epsilon, \ \forall t \leqslant t_0, \text{ where } \vartheta \text{ is defined.}$$

Proof of Claim 3.

i) In Claim 2 we got

$$\dot{\vartheta}(t) \geqslant \sin^2 \vartheta(t) - q_\lambda^-(t) \cos^2 \vartheta(t).$$

Suppose, by contradiction, that there exists a $\tilde{t} > t_0$ such that $\vartheta(\tilde{t}) \leqslant \pi/2 + k\pi + \epsilon$ and define $t_2 := \min\{t | \vartheta(t) = \pi/2 + k\pi + \epsilon\}$. Hence, $\vartheta(t) > \pi/2 + k\pi + \epsilon$, for $t \in]t_0, t_2]$. Define $t_1 := \max\{t \in [t_0, t_2[| \vartheta(t) = \pi/2 + k\pi + 2\epsilon\}$. If $t \in [t_1, t_2]$, then $\vartheta(t) \in]\pi/2 + k\pi + \epsilon, \pi/2 + k\pi + 2\epsilon[$. Therefore, for each $t \in [t_1, t_2]$,

$$\begin{aligned}
\dot{\vartheta}(t) &\geqslant \sin^2\left(\frac{\pi}{2} + k\pi + 2\epsilon\right) - q_\lambda^-(t) \cos^2\left(\frac{\pi}{2} + k\pi + 2\epsilon\right) = \\
&= \cos^2 2\epsilon - q_\lambda^-(t) \sin^2 2\epsilon \geqslant \cos^2 \epsilon - q_\lambda^-(t) \sin^2 2\epsilon = \\
&= \cos^2 \epsilon - q_\lambda^-(t) 4 \sin^2 \epsilon \cos^2 \epsilon = \\
&= \cos^2 \epsilon \big(1 - q_\lambda^-(t) 4 \sin^2 \epsilon\big) > \\
&> \cos^2 \epsilon \big(1 - 2q_\lambda^-(t)\big).
\end{aligned}$$

Integrating between t_1 and t_2,

$$-\epsilon = \vartheta(t_2) - \vartheta(t_1) \geqslant \cos^2 \epsilon \left(\int_{t_1}^{t_2} (1 - 2q_\lambda^-(t)) \mathrm{d}t\right) \geqslant -2 \int_0^T q_\lambda^-(t) \mathrm{d}t.$$

Hence

$$\frac{\epsilon}{2} \leqslant \int_0^T q_\lambda^-(t) \mathrm{d}t$$

in contradiction with the hypothesis.

ii) the proof follows an analogous argument as in (i).

Now we are in position to complete the proof of Lemma 3.

Using the definition of limit, we can rewrite the thesis as follows: $\forall \mathrm{N} > 0$ there exists λ_N such that $\vartheta(T; \vartheta_0, \lambda) - \vartheta_0 > \mathrm{N}$ holds for $\lambda > \lambda_N$.

Loosely speaking, the idea is the following: for each k in $\mathbb{Z}$, we divide the (t, ϑ) plane in some strips defined as:

$$S_k := \{(t, \vartheta) \in \mathbb{R}^2 | \frac{\pi}{2} + (k-1)\pi \leqslant \vartheta \leqslant \frac{\pi}{2} + k\pi\}.$$

Fix a positive integer N. The aim is to prove that it's impossible for the solution to remain in $N+1$ strips, when λ is sufficiently large.

Therefore, fix N and suppose, by contradiction, that the solution goes through at most $N+1$ strips. Without losing generality we can enumerate these strips giving them an index from 1 to $N+1$.

Fix an arbitrary $\epsilon \in]0, \pi/8[$ and define $I_k := \{t \in [0, T] | (t, \vartheta(t)) \in S_k\}$ with k varying in $\mathbb{Z}$. Since the first derivative of the solution $\vartheta(t)$ is positive at points $\pi/2 + k\pi$ (its value is 1), we can deduce that I_k is an interval, that we divide in the following convenient manner:

$$
\begin{aligned}
I_k^{(1)} &:= \left\{ t | \frac{\pi}{2} + (k-1)\pi \leqslant \vartheta(t) \leqslant \frac{\pi}{2} + (k-1)\pi + 2\epsilon \right\} ; \\
I_k^{(2)} &:= \left\{ t | \frac{\pi}{2} + (k-1)\pi + \epsilon \leqslant \vartheta(t) \leqslant \frac{\pi}{2} + k\pi - \epsilon \right\} ; \\
I_k^{(3)} &:= \left\{ t | \frac{\pi}{2} + k\pi - 2\epsilon \leqslant \vartheta(t) \leqslant \frac{\pi}{2} + k\pi \right\} .
\end{aligned}
$$

Observe that the subintervals satisfy the following two properties:

1. $I_{k_1}^{(i)} \cap I_{k_2}^{(j)} = \emptyset$ for $k_1 \neq k_2$ and for each i, $j = 1, 2, 3$;
2. $I_k^{(1)} \cap I_k^{(3)} = \emptyset \ \forall k$.

Indeed, the first property follows from the fact that the strips S_k are pairwise disjoint and from the definition of the subintervals, while the second one is the thesis of the Claim 3, which we have already proved. Notice that dividing I_k into the three subintervals corresponds to dividing the strip S_k into three substrips, that will be called "of type A, B and C". More precisely, for $t \in I_k^{(1)}$ we say that the point $(t, \vartheta(t))$ belongs to a substrip of type A, for $t \in I_k^{(2)}$ we say that the point $(t, \vartheta(t))$ belongs to a supstrip of type B, finally, for $t \in I_k^{(3)}$, we say that the point $(t, \vartheta(t))$ belongs to a substrip of type C.

Note that if the solution $\vartheta(t)$ leaves the A-strip, then there exists

$$
t^* := \min \left\{ t \in]0, T] | \vartheta(t^*) = \frac{\pi}{2} + (k-1)\pi + 2\epsilon \right\}
$$

and, from Claim 3, it follows that $\vartheta(t) \geqslant \pi/2 + (k-1)\pi + \epsilon$, $\forall t \geqslant t^*$, e.g. the solution will leave the B-substrip only upwards, coming into a C-strip, for $t \geqslant t^*$, and not downwards into an A-strip. If the solution leaves the B-strip too, then there exists

$$
t^{**} := \min \left\{ t \in]t^*, T] | \vartheta(t^{**}) = \frac{\pi}{2} + k\pi - \epsilon \right\}
$$

and $\vartheta(t) \geqslant \pi/2 + k\pi - 2\epsilon$, $\forall t \geqslant t^{**}$, e.g. the solution can leave the C-strip only coming into the A-strip of the superior level, the $k+1$ level, for $t \geqslant t^{**}$. If $\vartheta(t)$ leaves the strip S_k, in S_{k+1} the situation is the same.

In this way, for each strip we can count at most three time intervals. Now we want to give an estimate of these time intervals, for sufficiently large values of λ. By Claim 1, we can say that the following estimate for the measure of $I_k^{(1)}$ holds, for $k \in \mathbb{Z}$:

$$
\mu(I_k^{(1)}) \leqslant 2\tan(2\epsilon) + \int_0^T q_\lambda^-(t) \mathrm{d}t. \tag{16}
$$

Hence, recalling assumption (a)

$$\mu(I_k^{(1)}) < 3\tan(2\epsilon).$$

This inequality holds for each fixed $\epsilon > 0$, with a sufficiently large λ. An analogous estimate holds for the subinterval $I_k^{(3)}$. Therefore we find that the global measure of all the $\{I_k^{(1)} \cup I_k^{(3)}\}$, for $k \in \{1, \dots, N+1\}$, is less than $6(N+1)\tan(2\epsilon)$.

The aim is now to give an estimate of $\{I_k^{(2)}\}$ for $k \in \{1, \dots, N+1\}$. Choose a sufficiently large natural number K. The precise value of K will be determined later, depending on N.

For hypothesis (b), we can say that there exists an $\eta > 0$ such that $\int_0^T q_\eta(t)\mathrm{d}t > 2K+1$.

Define now the following auxiliary function

$$Q(t) := \int_0^t q_\eta(\xi)\mathrm{d}\xi;$$

and consider the straight lines $Q(t) = i$ for $i \in \{1, \dots, 2K+1\}$ crossed by the auxiliary function. By the graph of Q, it's possible to determine on the t-axis an analogous number of intervals. We denote them with $[a_i, b_i] \subset [0, T]$. They are such that for each $t \in [a_i, b_i]$, $i-1 = Q(a_i) \leqslant Q(t) \leqslant Q(b_i) = i$ holds for i varying in $\{1, \dots, 2K+1\}$. The a_i, b_i can be defined as follows

$$\begin{aligned}
&a_1 := \max\{t \in [0,T] | Q(t) = 0\};\\
&b_1 := \min\{t \in [a_1,T] | Q(t) = 1\};\\
&\dots\\
&a_i := \max\{t \in [b_{i-1},T] | Q(t) = i-1\};\\
&b_i := \min\{t \in [a_i,T] | Q(t) = i\};\\
&\dots
\end{aligned}$$

Note that the $[a_i, b_i]$ interval precedes the $[a_{i+1}, b_{i+1}]$ interval on the t-axis. They may be adjacent if $Q(t)$ is a strictly monotone function.

We need separate intervals, and, to this aim, we now consider only those with an odd index, that is

$$2j-1 = Q(b_{2j-1}) < Q(a_{2j+1}) = 2j, \quad j = 1, \dots, K.$$

Moreover,

$$\int_{a_{2j+1}}^{b_{2j+1}} q_\eta(t)\mathrm{d}t = Q(b_{2j+1}) - Q(a_{2j+1}) = 1.$$

Define $\delta := \min\{-b_{2j-1} + a_{2j+1} | j = 1, \dots, K\} > 0$.
Let us summarize the steps that we have done until now:

1. there exists an $\eta > 0$ such that $\int_0^T q_\eta(t)\mathrm{d}t > 2K+1$;

2. there are $K+1$ intervals, still denoted with $[a_i, b_i]$ (renaming those with an odd index chosen before), mutually disjoint, such that the inequality $\int_{a_i}^{b_i} q_\eta(t)\mathrm{d}t > 0$ holds;

3. $\delta > 0$ is determined as the least distance between these intervals.

We can now fix $\epsilon > 0$. Our choice is such that

$$3 \tan 2\epsilon < \delta.$$

Retaking the previous list of properties, we can add the following items:

4. there is a λ_ϵ^1 such that the inequality $\int_0^T q_\lambda^-(t)\mathrm{d}t < \tan 2\epsilon$ holds for $\lambda > \max\{\lambda_\epsilon^1, \eta\}$;

5. for $\lambda > \eta$,
$$1 = \int_{a_i}^{b_i} q_\eta(t)\mathrm{d}t \leqslant \int_{a_i}^{b_i} q_\lambda(t)\mathrm{d}t.$$

Our aim is now to prove that there are a k and an i such that

$$[a_i, b_i] \subset I_k^{(2)}. \tag{17}$$

It is sufficient to prove that there exists an $i \in \{1, \ldots, K+1\}$ such that

$$J := \bigcup_{k=1}^{N+1} I_k^{(2)} \supset [a_i, b_i]. \tag{18}$$

In fact, in this case, it would be guaranteed the existence of at least one $k \in \{1, \ldots, N+1\}$ such that $I_k^{(2)} \cap [a_i, b_i] = \emptyset$. $\big($If, by contradiction, there are two distinct values, $k \neq j$, such that

$$\begin{array}{l} I_k^{(2)} \cap [a_i, b_i] \neq \emptyset \\ I_j^{(2)} \cap [a_i, b_i] \neq \emptyset, \end{array}$$

then, if t_1 belongs to $I_k^{(2)} \cap [a_i, b_i]$ and t_2 belongs to $I_j^{(2)} \cap [a_i, b_i]$, we can deduce that the entire interval, whose extreme points are t_1 and t_2, is included in $[a_i, b_i]$ and hence in J. This is impossible because $I_k^{(2)}$ and $I_j^{(2)}$ are disjoint$\big)$.

Let us prove now inclusion (18): if, by contradiction, it were false, then we could deduce the following relation:

$$\forall i = 1, \ldots, K, \quad [a_i, b_i] \cap \bigcup_{k=1}^{N+1} \left(I_k^{(1)} \cup I_k^{(3)} \right) \neq \emptyset. \tag{19}$$

For simplicity, we rename the $I_k^{(1)}$ and $I_k^{(3)}$ intervals as J_l, with $l = 1, \dots, 2N+2$. Let us consider the set

$$\mathcal{L} := \{l | \exists i = 1, \dots, K+1 : J_l \cap [a_i, b_i] \neq \emptyset\}.$$

Then $\mathcal{L}$ has at most $2N+2$ elements. Let us observe that for each $l \in \mathcal{L}$ there exists an unique index $i = i_l$ such that $J_l \cap [a_i, b_i] \neq \emptyset$. In this way we have obtained an application $\mathcal{L} \to \{1, \dots, K\}$ such that $l \mapsto i_l$. From this, we can deduce that the target set has at most $2N+2$ elements and, therefore, if at the beginning we have chosen $K > 2N+2$, the mapping can't be surjective. Hence, we can be sure that there exists $i^* \in \{1, \dots, K+1\}$ such that $[a_{i^*}, b_{i^*}] \cap J_l = \emptyset$, $\forall l = 1, \dots, 2N+2$, in contradiction with (19).

It follows that there exists at least one $i \in \{1, \dots, K+1\}$ such that $[a_i, b_i] \subset I_k^{(2)}$ and for this i, remembering that $q_\lambda = q_\lambda^+ - q_\lambda^-$, we get

$$\begin{aligned}\int_{a_i}^{b_i} q_\lambda(s)\mathrm{d}s &\leqslant \int_{a_i}^{b_i} q_\lambda^+(s)\mathrm{d}s \leqslant \int_{I_k^{(2)}} q_\lambda^+(s)\mathrm{d}s = \int_{I_k^{(2)}} q_\lambda(s)\mathrm{d}s + \int_{I_k^{(2)}} q_\lambda^-(s)\mathrm{d}s \\ &\leqslant \int_{I_k^{(2)}} q_\lambda(s)\mathrm{d}s + \int_0^T q_\lambda^-(s)\mathrm{d}s.\end{aligned}$$

By Claim 2,

$$\int_{I_k^{(2)}} q_\lambda(t)\mathrm{d}t \leqslant \frac{2}{\tan 2\epsilon}$$

and, since there exists (by hypothesis (a)) λ'_ϵ such that for $\lambda > \lambda'_\epsilon$ the inequality $\int_0^T q_\lambda^-(t)\mathrm{d}t < \epsilon/2$ holds, we can deduce that

$$\int_{a_i}^{b_i} q_\lambda(s)\mathrm{d}s \leqslant \frac{2}{\tan 2\epsilon} + \frac{\epsilon}{2}.$$

Finally, passing to the limit for $\lambda \to +\infty$, we get the contradiction

$$+\infty = \lim_{\lambda \to \beta^-} \int_{a_i}^{b_i} q_\lambda(s)\mathrm{d}s \leqslant \frac{2}{\tan \epsilon} + \frac{\epsilon}{2}. \quad \blacksquare$$

Remark 3. It is possible to obtain the same result by replacing hypothesis (b) of Lemma 3 with the following one:

$$\frac{\partial q_\lambda(t)}{\partial \lambda} \geqslant m(t); \tag{20}$$

where $m(t)$ is a nonnegative, integrable and not identically zero function. Actually, by integrating Equation (20) between 0 and λ, we obtain $q_\lambda(t) - q_0(t) \geqslant \lambda m(t)$ and hence $q_\lambda(t) \geqslant q_0(t) + \lambda m(t)$. Going on with the proof of the final part of Lemma 3, we obtain

$$\int_{a_i}^{b_i} m(s)\mathrm{d}s = \frac{1}{2K+1} \int_0^T m(s)\mathrm{d}s > 0,$$

and, by the following inequality, a contradiction is achieved letting $\lambda \to +\infty$:

$$\int_{a_i}^{b_i} q_\lambda(s)\mathrm{d}s \geqslant \int_{a_i}^{b_i} q_0(s)\mathrm{d}s + \lambda \int_{a_i}^{b_i} m(s)\mathrm{d}s.$$

Remark 4. The conditions of the Lemma 3 and Remark 3 are automatically satisfied in the cases

i) $q_\lambda(t) = \lambda + q(t)$, with $q \in L^1([0,T])$, T-periodic;

ii) $q_\lambda(t) = \lambda q(t)$ with q non negative and not identically zero, T-periodic.

Following [12], we can now introduce the concept of characteristic values.

Definition 2. With respect to Equation (8), we define as characteristic values:

$$\underline{\lambda}_j(q) = \underline{\lambda}_j := \min\left\{\lambda \in \mathbb{R} | \varrho(\lambda) = \frac{j}{2}\right\}, \qquad \forall j \in \mathbb{N}^+;$$

$$\overline{\lambda}_j(q) = \overline{\lambda}_j := \max\left\{\lambda \in \mathbb{R} | \varrho(\lambda) = \frac{j}{2}\right\}, \qquad \forall j \in \mathbb{N}.$$

Theorem 3. *Under the assumptions of Lemma 2 and Lemma 3, the characteristic values $\underline{\lambda}_j$ and $\overline{\lambda}_j$, for Equation (8) are defined for each $j \in \mathbb{N}^+$, and satisfy the following properties:*

i) they form a sequence such that

$$-\infty < \overline{\lambda}_0 < \underline{\lambda}_1 \leqslant \overline{\lambda}_1 < \ldots < \underline{\lambda}_k \leqslant \overline{\lambda}_k < \ldots$$

ii) they are eigenvalues for T-periodic and antiperiodic problems associated to Equation (8); in particular, for j even, they are eigenvalues for the periodic problem; for j odd, they are eigenvalues for the antiperiodic problem;

iii) in the periodic case, if $\underline{\lambda}_{2j} = \overline{\lambda}_{2j}$ then the corresponding eigenfunctions are T-periodic with $2j$ zeros.

Proof. We prove only the periodic case. The proof is divided into four steps.

1. The result is the same as in [12], but here it's obtained in a different order. Rewrite Equation (8) as

$$-\dot{\theta} = \frac{y^2 + q_\lambda(t)x^2}{x^2 + y^2} = S(t, \theta, \lambda).$$

From simple calculations we can find the following expression for $\dot\rho$:

$$\frac{\dot\rho}{\rho} = (1 - q_\lambda)\cos\theta\sin\theta.$$

Applying the theorem of differentiable dependence of the solutions from initial data we have

$$P'(\vartheta_0) = v(T),$$

where v is the solution of the system

$$\begin{cases} \dot v = \dfrac{\partial S(t, \vartheta(t; 0, \vartheta_0), \lambda)v}{\partial\vartheta} \\ v(0) = 1. \end{cases} \tag{21}$$

On the other hand, evaluating $\partial S/\partial\vartheta$ we obtain

$$\dot v = -\frac{2\dot\rho}{\rho}\, v$$

By an elementary integration, we can calculate the expression of v, which is

$$v(t) = \frac{\rho^2(0)}{\rho^2(t)}.$$

We can always suppose that $\rho(0) = 1$, obtaining $P'(\theta_0) = \rho^{-2}(T)$.

2. From [12], the following claim holds:

   Claim. Let k be an integer. Then the following statements hold.

   i) $\lambda = \underline{\lambda}_{2k} \Leftrightarrow \max\limits_{\vartheta_0\in\mathbb{R}} \mathrm{Rot}_\lambda(T; \vartheta_0) = k$;

   ii) $\lambda = \overline{\lambda}_{2k} \Leftrightarrow \min\limits_{\vartheta_0\in\mathbb{R}} \mathrm{Rot}_\lambda(T; \vartheta_0) = k$.

3. The previous step is the same as

$$\lambda = \underline{\lambda}_{2k} \Leftrightarrow \max_{\vartheta\in\mathbb{R}}(P(\vartheta) - \vartheta - 2k\pi) = 0 \Leftrightarrow \max_{\vartheta\in\mathbb{R}}(P(\vartheta) - \vartheta) = 2k\pi.$$

   If $\lambda = \underline{\lambda}_{2k}$, then there exists ϑ_0 such that $P'(\vartheta_0) - 1 = 0$ and from step 1 it's the same as $\rho^2(T) = 1$. In this way we have got a T-periodic nontrivial solution with $2k$ zeros in $[0, T[$:

$$\begin{cases} \rho(T; 0, \vartheta_0) = \rho(0) = 1 \\ \vartheta(T; 0, \vartheta_0) - \vartheta_0 = 2k\pi \Leftrightarrow \vartheta(T; 0, \vartheta_0) = \vartheta_0 + 2k\pi. \end{cases}$$

4. Since we have found an eigenfunction, we can say that $\underline{\lambda}_{2k}$ and $\overline{\lambda}_{2k}$ are eigenvalues for the periodic problem studied. If $\underline{\lambda}_{2j} = \overline{\lambda}_{2j}$ then, using the Claim of step 2, we can say that

$$\max_{\vartheta_0 \in \mathbb{R}} \mathrm{Rot}_\lambda(T;\vartheta_0) = \min_{\vartheta_0 \in \mathbb{R}} \mathrm{Rot}_\lambda(T;\vartheta_0)$$

and it's equivalent to $P(\vartheta) - \vartheta =$constant. At this point, the proof proceeds as in step 3. ∎

We reconsider now problem (P) and the corresponding antiperiodic problem (AP)

$$(AP) \begin{cases} x'' + q_\lambda(t)x = 0; \\ x(0) = -x(T); \\ x'(0) = -x'(T). \end{cases}$$

Then, we have

Theorem 4. *Under the assumptions of Lemma 2 and Lemma 3, the eigenvalues for (P) $\hat{\lambda}_i$ and $\check{\lambda}_i$, $i \geqslant 0$, and for (AP) $\tilde{\lambda}_i$ and $\utilde{\lambda}_i$, $i \geqslant 1$, form such sequences that*

$$-\infty < \check{\lambda}_0 < \tilde{\lambda}_1 \leqslant \utilde{\lambda}_1 < \hat{\lambda}_1 \leqslant \check{\lambda}_1 < \tilde{\lambda}_2 \leqslant \utilde{\lambda}_2 < \hat{\lambda}_2 \leqslant \check{\lambda}_2 < \dots \tag{22}$$

For $\lambda = \check{\lambda}_0$ there exists a unique eigenfunction, φ_0. This function φ_0 has no zeros in $[0,T[$. For $j > 0$

a) if $\hat{\lambda}_j < \check{\lambda}_j$, then there is a unique eigenfunction $\hat{\varphi}_j$ at $\lambda = \hat{\lambda}_j$ and a unique eigenfunction $\check{\varphi}_j$ at $\lambda = \check{\lambda}_j$; furthermore, each of $\hat{\varphi}_j$ and $\check{\varphi}_j$ has exactly $2j$ zeros in $[0,T[$;

b) if $\hat{\lambda}_j = \check{\lambda}_j$, then there are two independent eigenfunctions $\varphi_j^{(1)}$ and $\varphi_j^{(2)}$, both having exactly $2j$ zeros in $[0,T[$. (In this case it follows that all solutions of Equation (8) are T-periodic with an even number of zeros in $[0,T[$).

Similar results hold for the eigenvalues of the antiperiodic problem.

Proof. The proof follows the classical argument for the eigenvalues problems for Hill's equation (see [4] or [14]). Usually in literature proofs are given in the case of a continuous coefficient and for special dependence of q_λ on λ (for example $q_\lambda = \lambda + q$). The key steps in those proofs make use of the study of the auxiliary function

$$f(\lambda) = \varphi(T,\lambda) + \psi'(T,\lambda)$$

where $\varphi(t) = \varphi(t, \lambda)$ and $\psi(t) = \psi(t, \lambda)$ are the fundamental solutions of Equation (8) satisfying

$$\begin{cases} \varphi(0) = \psi'(0) = 1 \\ \varphi'(0) = \psi(0) = 0 \end{cases}$$

and of a comparison to the eigenvalues μ_i for the two-point problem (D)

$$(D) \quad \begin{cases} x'' + q_\lambda(t)x = 0; \\ x(0) = x(T) = 0. \end{cases}$$

In particular a crucial property which allows to study the behavior of the function f is given in the following

Claim. There exists a ν_0 such that

$$\nu_0 < \mu_1 < \mu_2 < \dots < \mu_k < \dots$$

and such that

i) $f(\nu_0) \geqslant 2, \qquad f(\mu_{2i-1}) \leqslant -2, \qquad f(\mu_{2i}) \geqslant 2 \quad i = 1, 2, \dots.$

If $f(\hat{\lambda}) = 2$ for some $\hat{\lambda} \neq \mu_i$ then such a $\hat{\lambda}$ is a simple eigenvalue for (P) and for such a $\hat{\lambda}$,

ii) if $\hat{\lambda} < \mu_1$, then $\dfrac{\mathrm{d}f}{\mathrm{d}\lambda}(\hat{\lambda}) < 0$, else if $i \geqslant 1$ then

$$\begin{cases} \dfrac{\mathrm{d}f}{\mathrm{d}\lambda}(\hat{\lambda}) > 0, & \text{if } \mu_{2i-1} < \hat{\lambda} < \mu_{2i}; \\ \dfrac{\mathrm{d}f}{\mathrm{d}\lambda}(\hat{\lambda}) < 0, & \text{if } \mu_{2i} < \hat{\lambda} < \mu_{2i+1}. \end{cases}$$

If $f(\mu_{2i}) = 2$ and $\dfrac{\mathrm{d}f}{\mathrm{d}\lambda} \neq 0$ at $\lambda = \mu_{2i}$, then μ_{2i} is a simple eigenvalue for problem (P).

The proof of the claim follows the same argument as in [4] using some careful estimates due to our more general setting and therefore it is omitted. (For the missing details see [24]).

An immediate consequence of this claim is the existence of the eigenvalues. Actually, we can define them as follows:

$$\begin{aligned} \check{\lambda}_0 &:= \max\{\lambda | f(s) \geqslant 2 \; \forall s \leqslant \lambda\} \\ \underset{\sim}{\lambda}_1 &:= \min\{\lambda \geqslant \lambda_0 | f(\lambda) \leqslant -2\} \\ \tilde{\lambda}_1 &:= \max\{\lambda \leqslant \mu_2 | f(\lambda) \leqslant -2\} \\ \hat{\lambda}_1 &:= \min\{\lambda \geqslant \tilde{\lambda}_1 | f(\lambda) \geqslant 2\} \\ &\dots \qquad \blacksquare \end{aligned}$$

At this point, we are in position to prove the following important result.

THEOREM 5. *The eigenvalues for the periodic and antiperiodic problems defined in Theorem 4, and the characteristic values defined in Definition 2, for Equation* (8) *are the same.*

Proof. In Theorem 3 we have proved that the characteristic values are eigenvalues. By an analysis of the respective eigenfunctions, we can state, using the same notation as in Theorem 4, that

i) $\overline{\lambda}_0 = \check{\lambda}_0$;

ii) $\{\underline{\lambda}_{2j}, \overline{\lambda}_{2j}\} \subset \{\hat{\lambda}_j, \check{\lambda}_j\}$, for each $j \geqslant 1$;

iii) $\{\underline{\lambda}_{2j-1}, \overline{\lambda}_{2j-1}\} \subset \{\underset{\sim}{\lambda}_j, \tilde{\lambda}_j\}$, for each $j \geqslant 1$.

It's sufficient to verify that for $j \geqslant 1$

$$\underline{\lambda}_{2j} = \overline{\lambda}_{2j} \Rightarrow \hat{\lambda}_j = \check{\lambda}_j \quad \text{and} \quad \underline{\lambda}_{2j-1} = \overline{\lambda}_{2j-1} \Rightarrow \underset{\sim}{\lambda}_j = \tilde{\lambda}_j.$$

Let's see the first relation. If $\underline{\lambda}_{2j} = \overline{\lambda}_{2j} = \lambda$, then for the claim stated in proof of Theorem 3, we get that the function $\vartheta_0 \mapsto \vartheta(T; \vartheta_0, \lambda) - \vartheta_0$ is constant. Then $1/\rho(T; \vartheta_0, \lambda) \equiv 1$, $\forall \vartheta_0 \in \mathbb{R}$. It means that all the solutions are periodic and we prove the thesis. ∎

3. PROOF OF THEOREM 2 AND SOME APPLICATIONS

As a corollary of the results in the previous section, we have

LEMMA 4. *Let us consider the following problems:*

$$(P_\lambda) \begin{cases} x'' + \lambda q(t) x = 0 \\ x : T\text{-periodic}; \end{cases} \qquad (P)\ x'' + q(t)x = 0,$$

where $q(t) \geqslant 0$ *a.e. and* $\int_0^T q(t)\mathrm{d}t > 0$.
Then, $\forall j \in \mathbb{N}$ *and* $\forall z_0 \neq 0$,

i) $1 < \hat{\lambda}_j \Rightarrow \mathrm{Rot}^{(P)}(T; z_0) < j$;

ii) $1 > \check{\lambda}_j \Rightarrow \mathrm{Rot}^{(P)}(T; z_0) > j$.

where $\mathrm{Rot}^{(P)}(T; z_0)$ *denotes the rotation number associated to Equation* (P).

Proof.

i) From Theorem 5 we can state that $\hat{\lambda}_j = \underline{\lambda}_{2j}$. By Claim in step 2 of the proof of Theorem 3, $\mathrm{Rot}^{(P_{\lambda,2j})}(T; z_0) \leqslant j$, where $\mathrm{Rot}^{(P_{\lambda,2j})}(T; z_0)$ is the rotation number associated to the equation of problem $(P_{\underline{\lambda}_{2j}})$. Furthermore, by a comparison result, we can deduce that

$$\mathrm{Rot}^{(P)}(T; z_0) < \mathrm{Rot}^{(P_{\lambda,2j})}(T; z_0).$$

ii) It follows an argument analogous to the previous one. ∎

We are now in position to prove Theorem 2.

Proof of Theorem 2. Using the previous lemma, we have that $\mathrm{Rot}_\infty(T; z_0) < j < \mathrm{Rot}_0(T; z_0)$ for all $z_0 \neq 0$. Let

$$\begin{aligned} \eta_0 &= \min_{z_0 \in S^1} \mathrm{Rot}_0(T; z_0) \quad &> j \\ \eta_\infty &= \max_{z_0 \in S^1} \mathrm{Rot}_\infty(T; z_0) \quad &< j \end{aligned}$$

and consider $0 < \delta < \min\{j - \eta_\infty, \eta_0 - j\}$.

From [19] and [24] we can find $R > 0$ sufficiently large and $\epsilon > 0$ sufficiently small such that

$$|\mathrm{Rot}_\infty(T; z_0) - \mathrm{Rot}(T; z_0)| \leq \delta \quad \text{for all } \|z_0\| \geq R$$

and

$$|\mathrm{Rot}_0(T; z_0) - \mathrm{Rot}(T; z_0)| \leq \delta \quad \text{for all } 0 < \|z_0\| \leq \epsilon.$$

Therefore

$$\begin{aligned} \mathrm{Rot}(T; z_0) &< j \qquad \text{for all } \|z_0\| \geq R \\ \mathrm{Rot}(T; z_0) &> j \qquad \text{for all } 0 < \|z_0\| \leq \epsilon. \end{aligned}$$

Now we can apply Theorem 1 taking $\Gamma_i = \{z | \, \|z\| = \epsilon\}$ and $\Gamma_e = \{z | \, \|z\| = R\}$, since the twist condition (5) is satisfied and the result follows.

Note that along this argument we also proved Corollary 1. ∎

Example 1. Let's study Equation (1) with the following two conditions:

1. $\dfrac{f(t,x)}{x} \to 0$ for $|x| \to +\infty$;
2. $\dfrac{f(t,x)}{x} \to q_0(t)$ for $x \to 0$.

Condition (1) implies, by definition, that there exists $k > 0$ such that

$$\left| \frac{f(t,x)}{x} \right| \leq 1, \ \ \forall |x| \geq k.$$

If f satisfies the Caratheódory conditions, then there exists $l(t) \in L^1$, $l(t) \geq 0$ such that $|f(t,x)| \leq l(t)$, $\forall |x| \leq k$ and for a.e. t. We have obtained that

$$|f(t,x)| \leq |x| + l(t) \ \forall x \in \mathbb{R},$$

which means that f grows at most linearly. Using the Gronwall inequality, we can deduce the continuity of solutions, also in $[0, T]$. By condition (1), we know that for rays sufficiently large the T-rotation number is less than 1 in the phase plane (it doesn't complete a full turn). (In [24] we proved that $\mathrm{Rot}(T; z_0)$ is as little as we want, for suitable z_0.) Hence, if the initial point is sufficiently large, then the solution is uniformly greater than a suitable ray. Indeed in [25] is proved that for each R^* there exists $\hat{R}$ such that if $\|z_0\| \geq \hat{R}$, then $\|z(t; z_0)\| \geq R^*$, $\forall t \in [t_0, T]$. Afterwards, we can deduce that $\mathrm{Rot}(T; z_0) < 1$, $\forall \|z_0\| \geq \hat{R}$.

Let us consider now condition (2). If $q_0(t) \geq 0$ and not identically zero, then we can study the eigenvalues ($\lambda_0 < \underline{\lambda}_1 \leq \overline{\lambda}_1 < \ldots$) associated to the linear equation $x'' + \lambda q_0(t)x = 0$. In [24] we proved that for little trajectories, if $\overline{\lambda}_j < 1$, then the T-rotation number is greater than j.

Using together the two conditions we obtain that there are $2j$ T-periodic solutions.

We can apply this result to the following equation studied in [6], [20]:

$$x'' + \frac{x}{(x^2 + r(t)^2)^{3/2}} = 0.$$

The equation is like Equation (1), with f satisfying the conditions (1) and (2), and $q_0(t) = r(t)^{-3}$. In [6], the authors proved the existence of a chaotic like dynamics under the condition that $r(t) = 1 + \epsilon \cos t + \mathcal{O}(\epsilon^2)$, where ϵ denotes a small parameter closely related to the ellipticity. Here we can obtain a result of multiplicity of T-periodic solutions just analyzing the j-th eigenvalue of

$$x'' + \lambda \frac{x}{r(t)^3} = 0.$$

Remark 5. Example 1 can be viewed as a generalization of [5, Propositions 4.1 and 4.2, Ch. 4].

EXAMPLE 2. Let us consider $x'' + \lambda q(t)x = 0$ with $q(t) = 1 + p(t)$ and $p \geq 0$. Since the eigenvalues depend with continuity respect to the L^1-norm, we study the 2π-periodic problem $x'' + \lambda x = 0$. It's well known that for this last problem $\lambda_j = j^2$. Hence, $\overline{\lambda}_j \approx j^2$ if $\|p\|_{L^1} < \epsilon$ with ϵ sufficiently small. On the other hand, $\|p\|_\infty$ may be large so that the range of $q(t)$ can cross a large number of eigenvalues of $x'' + \lambda x = 0$.

Remark 6. This example shows the possibility of a generalization of some results given in [5], where it was assumed that for asymptotically linear problems, the limits are different from the eigenvalues. Some developments in this direction will be considered in a successive work.

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REFERENCES

[1] G. D. Birkhoff, *Proof of Poincaré's Geometric Theorem*, Trans. Amer. Math. Soc., Vol. 14 (1913), 14-22, (Collected Mathematical Papers, 1, pp. 673-681, Dover, New York, 1968).

[2] G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc. Colloquium Publications, Vol. IX (1927).

[3] M. Brown and W. D. Neumann, *Proof of the Poincaré-Birkhoff Fixed Point Theorem*, Michigan Math. J., 24 (1977), 21-31.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill book Company, New York Toronto London, 1955.

[5] M. A. Del Pino, R. F. Manásevich and A. Murua, *On the Number of* 2π *Periodic Solutions for* $u'' + g(u) = s(1 + h(t))$ *Using the Poincaré-Birkhoff Theorem*, J. Differential Equations, 95 (1992), 240-258.

[6] H. Dankowicz and P. Holmes, *The Existence of Transverse Homoclinic Points in the Sitnikov Problem*, J. Differential Equations, 116 (1995), 468-483.

[7] T. Ding, *An Infinite Class of Periodic Solutions of Periodically Perturbed Duffing Equations at Resonance*, Proc. Amer. Soc., 86 (1982), 47-54.

[8] T. Ding and F. Zanolin, *Subharmonic Solutions of Second Order Nonlinear Equations: a Time-Map Approach*, Nonlinear Analysis, Theory, Methods and Applications, 20 (1993), 509-532.

[9] W. Y. Ding, *A Generalization of the Poincaré-Birkhoff Theorem*, Proc. Amer. Math. Soc., 88 (1983), 341-346.

[10] R. Fabbri, R. Johnson and C. Núñez, Rotation Number for Non-Autonomous Linear Hamiltonian Systems I: Basic Properties, Preprint (2001).

[11] J. Franks, *Generalizations of the Poincaré-Birkhoff Theorem*, Annals of Math., 128 (1988), 139-151.

[12] S. Gan and M. Zhang, *Resonance pockets of Hill's equations with two-step potentials*, SIAM J. Math. Anal., 32 (2000), 651-664.

[13] J. K. Hale, *Ordinary differential equations*, Robert E. Krieger Publishing Company Huntington, New York, 1980.

[14] E. L. Ince, *Ordinary differential equations*, Dover Publications, Inc., New York, 1956.

[15] H. Jacobowitz, *Periodic Solutions of* $x'' + f(t,x) = 0$ *via the Poincaré-Birkhoff Theorem*, J. Differential Equations, 20 (1976), 37-52.

[16] R. Johnson and J. Moser, *The Rotation Number for Almost Periodic Potentials*, Comm. Math. Phys, 84 (1982), 403-438.

[17] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities. Theory and Applications*, Academic Press, New York and London, Vol. I, ODE (1969).

[18] W. Magnus and S. Winkler, *Hill's Equations*, Dover Publications, Inc., New York, 1966.

[19] A. Margheri, C. Rebelo and F. Zanolin, *Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, J. Differential Equations (to appear).

[20] J. Moser, *Stable and Random Motions in Dynamical Systems*, with Special Emphasis on Celestial Mechanics, Princeton, New Jersey, 1973.

[21] J. Moser, *Integrable Hamiltonian Systems and Spectral Theory*, Lezioni Fermiane, Accademia Nazionale dei Lincei, Pisa, 1981.

[22] M. H. Poincaré, *Sur un Théorème de Géométrie*, Rend. Circ. Mat. Palermo, 33 (1912), 375-407.

[23] C. Rebelo, *A Note on the Poincaré-Birkhoff Fixed Point Theorem and Periodic Solutions of Planar Systems*, Nonlinear Analysis Theory, Methods and Applications, 29 (1997), 291-311.

[24] C. Zanini, *Periodic Solutions of planar systems using the Poincaré-Birkhoff theorem*, degree thesis, (2001) (Italian).

[25] F. Zanolin, *Continuation Theorems for the Periodic Problem via the Translation Operator*, Rend. Sem. Mat. Univ. Pol. Torino, 54 (1996), 1-23.

[26] F. Zanolin, *Time-Maps and Boundary Value Problems for Ordinary Differential Equations*, Tricomi's ideas and contemporary applied mathematics (Rome/Turin, 1997), Atti Convegni Lincei, 147, Accad. Naz. Lincei, Rome, 1998, 281-302.

[27] M. Zhang, *Nonuniform Nonresonance of Semilinear Differential equations*, J. Differential Equations, 166 (2000), 33-50.

[28] M. Zhang, *The Rotation Number Approach to Eigenvalues of the One-dimensional p-Laplacian with Periodic Potentials*, J. London Math. Soc., 64 (2001), 125-143.