Lecture 15 Multiscale Analysis of SDEs *

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1 Introduction

The multiscale is very common in different fields of science and engineering. Consider the toy model

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = \frac{1}{\epsilon}(g(x) - y), \quad \epsilon \ll 1, \epsilon > 0.$$

We call x the slow variable and y the fast variable.

The exact solution of y given x has the form

$$y(t) = e^{-t/\epsilon}y_0 + (1 - e^{-t/\epsilon})g(x) \to g(x)$$

as $t \to \infty$. That is, y will relax to $y = g(x)$ fast in $O(\epsilon)$ timescale. $y = g(x)$ is called the slow manifold. Finally we get the adiabatic approximation:

$$\frac{dx}{dt} = f(x, g(x))$$

as $\epsilon \to 0$.

A slight generalization is

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = \frac{1}{\epsilon}(g(x) - y) + \sqrt{\frac{2}{\epsilon}}\dot{W}, \quad \epsilon \ll 1, \epsilon > 0.$$

Given x, $y(t)$ has an invariant distribution

$$y(t) \sim N(g(x), 1) := \mu_{g(x)}(y)dy$$

The effective dynamics is

$$\frac{dx}{dt} = \langle f(x, y)\rangle_{\mu_{g(x)}} = \int_{\mathbb{R}} f(x, y)\mu_{g(x)}(y)dy$$

as $\epsilon \to 0$.

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2 Asymptotic Analysis of SDEs

As an example of the application of PDE methods to the study of diffusion processes, we discuss briefly some results on asymptotic analysis. For SDEs or ODEs, the presence of a small parameter usually means that the system has some disparate time scales. Our task is to eliminate the fast time scales in the system and derive effective equations that govern the dynamics on the slow time scale.

Let us start with a simple example. Let $Y_t = y(t)$ be a stationary two-state Markov jump process taking values $\pm\alpha$ with jump rate β between these two states. With matrix notation, the infinitesimal generator for Y has the form

$$A = \begin{pmatrix} -\beta & \beta \\ \beta & -\beta \end{pmatrix}.$$

Let $y^\epsilon(t) = y(t/\epsilon^2)$ where ϵ is a small parameter. Consider the SDE

$$\frac{dx^\epsilon(t)}{dt} = \frac{1}{\epsilon} y^\epsilon(t), \quad x^\epsilon(0) = x. \tag{2.1}$$

Let

$$u^\epsilon(x, y, t) = \mathbb{E}^{(x,y)}\Big(f(x^\epsilon(t), y^\epsilon(t))\Big),$$

where f is any given smooth function. Then u^ϵ satisfies the backward Kolmoogorov equation:

$$\frac{\partial u^\epsilon}{\partial t} = \frac{1}{\epsilon} y \frac{\partial u^\epsilon}{\partial x} + \frac{1}{\epsilon^2} A u^\epsilon, \quad u^\epsilon(x, y, 0) = f(x, y). \tag{2.2}$$

Since y can only take two values, by defining

$$u_\pm(x, t) = u^\epsilon(x, \pm\alpha, t), \;\; f_\pm(x, t) = f(x, \pm\alpha),$$

we can rewrite (2.2) as

$$\frac{\partial}{\partial t}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \frac{1}{\epsilon}\begin{pmatrix} +\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} + \frac{1}{\epsilon^2}\begin{pmatrix} -\beta & \beta \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

with initial condition $u_\pm(x, 0) = f_\pm(x)$.

Let $w = u_+ + u_-$, we have

$$\epsilon^2 \frac{\partial^2 w}{\partial t^2} = \alpha^2 \frac{\partial^2 w}{\partial x^2} - 2\beta \frac{\partial w}{\partial t}, \quad w|_{t=0} = f_+ + f_-, \; \partial_t w|_{t=0} = \frac{\alpha}{\epsilon}\partial_x(f_+ - f_-).$$

Consider the case when $f_+ = f_- = f$. In this case the time derivative of w vanishes at $t = 0$, hence we avoid the extra complication coming from the initial layer. Following the standard approach in asymptotic analysis, we make the ansatz:

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots .$$

To leading order, this gives:

$$\frac{\partial w_0}{\partial t} = \frac{\alpha^2}{2\beta}\frac{\partial^2 w_0}{\partial x^2}, \quad w_0|_{t=0} = 2f. \tag{2.3}$$

This means that to leading order, x^ϵ behaves like Brownian motion with diffusion constant $D = \alpha/\sqrt{\beta}$. This is not surprising since it is what the central limit theorem tells us for the process

$$\tilde{x}^\epsilon(t) = \frac{1}{\epsilon}\int_0^t y^\epsilon(s)ds \quad \text{as } \epsilon \to 0.$$

We turn now to the general case. Suppose the stochastic process $\boldsymbol{X}_t^\epsilon$ possess the backward equation for $u^\epsilon(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{x}} f(\boldsymbol{X}_t^\epsilon)$ as

$$\frac{\partial u^\epsilon}{\partial t} = \frac{1}{\epsilon^2}\mathcal{L}_1 u^\epsilon + \frac{1}{\epsilon}\mathcal{L}_2 u^\epsilon + \mathcal{L}_3 u^\epsilon, \quad u^\epsilon(0) = f, \tag{2.4}$$

where $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are differential operators defined on some Banach space B, whose properties will be specified below. We would like to study the asymptotic behavior of u^ϵ when $\epsilon \to 0$ for $0 \le t \le T$, $T < \infty$. This discussion follows the work of Khasminski, Kurtz, Papanicolaou, etc [3].

We do not have to limit ourselves to SDEs. In fact, as long as the backward equation takes the form of (2.4), the analysis that we present below applies, e.g. the homogenization problem [4]. In particular, we can apply this procedure to the model problem studied above. We will return to this specific example after we present the general procedure.

As a general framework we assume that the following conditions hold.

(a) $\mathcal{L}_1$ is an infinitesimal generator of a stationary Markov process, and the semi-group $\exp(\mathcal{L}_1 t)$ generated by $\mathcal{L}_1$ converges to a projection operator to the null space of $\mathcal{L}_1$, which we will denote as P.

$$\exp(\mathcal{L}_1 t) \to P, \quad t \to \infty.$$

(b) Solvability condition: $P\mathcal{L}_2 P = 0$.

(c) Consistency condition for the initial value: $Pf = f$.

Note that (a) implies that $P^2 = P$, which is a projection operator and

$$\text{Range}(P) = \text{Null}(\mathcal{L}_1), \quad \text{Null}(P) = \overline{\text{Range}(\mathcal{L}_1)}. \tag{2.5}$$

But P does not need to be an orthogonal projection since generally $P^* \neq P$ (see Exercise 1 for more details).

Assume that u^ϵ can be expressed in the following form:

$$u^\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots.$$

Substituting it into (2.4) and collecting terms of the same order in ϵ, we get

$$\begin{aligned} O(\epsilon^{-2}): &\quad \mathcal{L}_1 u_0 = 0, && (2.6)\\ O(\epsilon^{-1}): &\quad \mathcal{L}_1 u_1 = -\mathcal{L}_2 u_0, && (2.7)\\ O(\epsilon^{0}): &\quad \mathcal{L}_1 u_2 = -\mathcal{L}_2 u_1 - \mathcal{L}_3 u_0 + \frac{\partial u_0}{\partial t}, && (2.8)\\ &\quad \dots\dots \end{aligned}$$

and $u_0(0) = f$ from the initial condition.

From (2.6) and (2.5), we obtain that u_0 is in the null space of $\mathcal{L}_1$, which is the same as the range of P, i.e.

$$P u_0 = u_0. \tag{2.9}$$

The consistency condition $Pf = f$, i.e. $Pu_0(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}, 0)$ allows us to avoid the initial layer problem.

To solve u_1 from (2.7), we assume the Fredholm alternative holds for the operator $\mathcal{L}_1$, which should be rigorously proved for each concrete problem [1]. The Fredholm alternative states that Eq. (2.7) has a solution if we have $\langle g, \mathcal{L}_2 u_0\rangle = 0$ for any $g \in \mathrm{Null}(\mathcal{L}_1^*) \subset B^*$. But this is true since $g \in \mathrm{Null}(\mathcal{L}_1^*)$ implies that

$$g \in {}^\perp\overline{\mathrm{Range}(\mathcal{L}_1)} = {}^\perp\mathrm{Null}(P),$$

thus $\langle g, \mathcal{L}_2 u_0\rangle = 0$ by (2.9) and solvability condition (b). Here $g \in {}^\perp\overline{\mathrm{Range}(\mathcal{L}_1)}$ or ${}^\perp\mathrm{Null}(P)$ means that for any $h \in \overline{\mathrm{Range}(\mathcal{L}_1)}$ or $\mathrm{Null}(P)$, we have $\langle g, h\rangle = 0$.

With the solvability condition and the Fredholm alternative, we denote a solution of (2.7) as

$$u_1 = -\mathcal{L}_1^{-1}\mathcal{L}_2 P u_0. \tag{2.10}$$

Substituting this into (2.8) and applying P on both sides, we obtain the effective equation for the leading order u_0

$$\frac{\partial u_0}{\partial t} = (P\mathcal{L}_3 P - P\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2 P)u_0 := \bar{\mathcal{L}} u_0, \quad u_0(0) = f \tag{2.11}$$

in the range of P. It is important to note that although (2.7) might have multiple solutions, the solvability condition ensures that the choice of the solution of (2.10) does not affect the final reduced system (2.11). One can also derive effective equations for the higher order terms u_1, u_2, etc. But it is more complicated and usually not very useful.

To see this abstract framework actually works, we use it for the simple model introduced at the beginning of this section. We have

$$\mathcal{L}_1 = A,\ \mathcal{L}_2 = \begin{pmatrix} +\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \frac{\partial}{\partial x}, \quad \mathcal{L}_3 = 0.$$

Thus the projection operator P is given by

$$P = \lim_{t\to\infty} \exp(\mathcal{L}_1 t) = \lim_{t\to\infty} \frac{1}{2} \begin{pmatrix} 1+e^{-2\beta t} & 1-e^{-2\beta t} \\ 1-e^{-2\beta t} & 1+e^{-2\beta t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

In the current example, we can simply pick a version of $\mathcal{L}_1^{-1}$ as

$$\mathcal{L}_1^{-1} = -\int_0^\infty (\exp(\mathcal{L}_1 t) - P) dt = -\frac{1}{4\beta} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is easy to verify that the solvability condition $P\mathcal{L}_2 P = 0$ is satisfied. The consistency condition $Pf = f$ gives $f_+ = f_-$, which we still denote as f. Finally the effective operator

$$-P\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2 P = \frac{\alpha^2}{4\beta} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2}.$$

Combining these we obtain the effective equation

$$\frac{\partial}{\partial t}(u_0^+ + u_0^-) = \frac{\alpha^2}{2\beta}\frac{\partial^2}{\partial x^2}(u_0^+ + u_0^-), \quad (u_0^+ + u_0^-)|_{t=0} = f_+ + f_- = 2f,$$

where $u_0 = (u_0^+, u_0^-)$. Set $w_0 = u_0^+ + u_0^-$, we recover (2.3).

Other interesting applications can be found in [2, 3, 5, 6] and the references therein.

Homeworks

1. Consider a strongly continuous contraction semigroup $S(t)$ generated by $\mathcal{L}$ on a Banach space B, which means $\|S(t)\| \le 1$ and

$$\|S(t)f - f\|_B \to 0 \text{ as } t \to 0, \text{ for all } f \in B.$$

   Assume $S(t) \to P$ as $t \to \infty$, then we have

   (a) P is a linear contraction on B and $P^2 = P$, a projection operator.

   (b) $S(t)P = PS(t) = P$ for any $t \ge 0$.

   (c) $\mathrm{Range}(P) = \mathrm{Null}(\mathcal{L})$.

   (d) $\mathrm{Null}(P) = \overline{\mathrm{Range}(\mathcal{L})}$.

References

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