

# Lecture 12 Construction of BM and its properties \*

Tiejun Li

## 1 Construction of Wiener Process

Below we will show three approaches to construct the Wiener process. Different forms play different roles in different circumstances.

### A. Construction from invariance principle

The first construction from the invariance principle embodies the idea of taking continuum limit of symmetric random walk.

**Theorem 1.1.** (*Invariance Principle*) Suppose  $\{\xi_i\}$  are i.i.d.  $N(0, 1)$  random variables, define  $S_n = \sum_{i=1}^n \xi_i$  and  $X_t^n$  as follows:

$$X_t^n = \begin{cases} \frac{S_k}{\sqrt{n}}, & t = \frac{k}{n}, \\ (1 - \theta) \frac{S_k}{\sqrt{n}} + \theta \frac{S_{k+1}}{\sqrt{n}}, & t \in \left( \frac{k}{n}, \frac{k+1}{n} \right), \quad \theta = nt - k, \end{cases}$$

then  $X^n \in C[0, \infty)$  and

$$X^n \xrightarrow{d} W,$$

where  $\xrightarrow{d}$  is the weak convergence on the function space  $C[0, \infty)$  to be defined below.

Before stating the sketch of the proof, let us consider a special case by taking

$$\mathbf{P}(\xi_i) = \begin{cases} 1/2, & \xi_i = 1, \\ 1/2, & \xi_i = -1, \end{cases}$$

then  $\mathbb{E}\xi_i = 0$ ,  $\text{var}\xi_i = 1$ . The state of  $X_t^n$  at the time  $t_k = k/n$  is nothing but the random walk considered before. The construction from invariance principle indicates that the standard Brownian motion is just the rescaled limit of the random walk with spatial scale  $l = 1/\sqrt{n}$  and time scale  $\tau = 1/n$ . The relation  $l^2/\tau = 1$  is exactly the regime considered before. This approximation is the most common one in computations.

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\*School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China

**Proposition 1.2.** *The function space  $C[0, \infty)$  equipped with the metric*

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\|x - y\|_{L^\infty([0, k])} \wedge 1), \quad x, y \in C[0, \infty)$$

*is a complete, separable metric space.*

It is also natural to define the  $\sigma$ -algebra  $\mathcal{B}(C[0, \infty))$  in space  $C[0, \infty)$  through finite dimensional cylinder sets

$$C = \{\omega \in C[0, \infty) | (\omega(t_1), \omega(t_2), \dots, \omega(t_n)) \in A\}, \quad n \geq 1, A \in \mathcal{R}^n \quad (1.1)$$

One can show  $\mathcal{B}(C[0, \infty))$  is equivalent to the Borel  $\sigma$ -algebra generated by the open sets in the metric space  $(C[0, \infty), \rho)$ .

**Definition 1.3.** *A family of probability measures  $\{P_n\}_{n=1}^{\infty}$  on the metric space  $S$  with Borel  $\sigma$ -algebra  $\mathcal{B}(S)$  is said to converge weakly to another probability measure  $P$  on the same space if and only if*

$$\lim_{n \rightarrow \infty} \int_S f(s) P_n(ds) = \lim_{n \rightarrow \infty} \int_S f(s) P(ds)$$

*for every bounded, continuous real-valued function  $f$  on  $S$ .*

**Definition 1.4.** *Let  $X_n$  be the random variables defined on the probability space  $(\Omega_n, \mathcal{F}_n, P_n)$  and  $X$  be defined on another probability space  $(\Omega, \mathcal{F}, P)$ . Both  $X_n$  and  $X$  take their values on the metric space  $S$  equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}(S)$ . The random variables  $\{X_n\}$  are said to converge weakly to  $X$  if the corresponding distribution  $\mu_n = P_n \circ X_n^{-1}$  converges weakly to  $\mu = P \circ X^{-1}$ . It is usually denoted as*

$$X_n \xrightarrow{d} X.$$

The proof of the weak convergence in the invariance principle relies on the Prohorov's theorem on the weak compactness of the probability measures and the probabilistic type of Arzela-Ascoli compactness theorem in space  $S = C[0, +\infty)$ . It is quite involved so we will skip the detailed proof. The interested readers may be referred to [2].

The probability measure  $P_*$  as the weak convergence limit of  $P_n \circ X_n^{-1}$  on the space  $S = C[0, \infty)$  is called the *Wiener measure* and the probability space  $(C[0, \infty), \mathcal{B}(C[0, \infty)), P_*)$  is called the *canonical probability space* for Wiener process, under which the coordinate mapping  $W_t(\omega) = \omega(t)$  is a standard Brownian motion.

**Heuristic Check.** Now we give a heuristic check for the validity of invariance principle based on the central limit theorem for some discrete time. From the definition  $S_n = \sum_{i=1}^n \xi_i$ , where  $\{\xi_i\}$  are i.i.d.  $N(0, 1)$  random variables, then by CLT

$$\frac{S_k}{\sqrt{n}} = \frac{\sqrt{k}}{\sqrt{n}} \cdot \frac{S_k}{\sqrt{k}} \xrightarrow{d} N(0, t), \quad \text{as } k, n \rightarrow \infty \text{ and } t = \frac{k}{n}.$$

The limit  $X$  of  $X^n$  is then a Gaussian process formally with  $X_0 = 0$  and

$$\begin{aligned}\mathbb{E}X_t X_s &\sim \mathbb{E}X_t^n X_s^n \\ &= \mathbb{E}X_{t\wedge s}^n (X_{t\vee s}^n - X_{t\wedge s}^n + X_{t\wedge s}^n) \\ &= \mathbb{E}(X_{t\wedge s}^n)^2 + \mathbb{E}X_{t\wedge s}^n (X_{t\vee s}^n - X_{t\wedge s}^n) \\ &\rightarrow t \wedge s. \quad \text{for } t = k/n, s = l/n \text{ and } k, l, n \rightarrow \infty.\end{aligned}$$

The last identity holds because of the independence between  $X_{t\wedge s}^n$  and  $X_{t\vee s}^n - X_{t\wedge s}^n$ , and

$$\mathbb{E}(X_{t\wedge s}^n - X_{t\vee s}^n) = 0.$$

Heuristically the key point in the invariance principle is CLT when  $n, k$  is sufficiently large. This implies the condition  $\xi_n \sim i.i.d. N(0, 1)$  may be relaxed to  $\xi_n$  be *i.i.d.* with mean 0 and variance 1. The distribution of  $\xi_n$  is not important. That is why the theorem is called “invariance” principle.

A realization of Wiener process with finite  $N$  is shown in Fig. 1.

## B. Construction from Karhunen-Loeve Expansion

The construction from Karhunen-Loeve expansion is based on the theory for Gaussian random fields. It can be easily extended to the case of Brownian bridge or high dimensional cases like the Brownian sheet etc [5].

**Theorem 1.5.** (*Karhunen-Loeve expansion*) Let  $X_t$  ( $t \in [0, 1]$ ) be a Gaussian process with mean function  $m(t) = 0$  and continuous covariance function  $K(s, t)$ . Consider the following eigenvalue problem

$$\int_0^1 K(s, t) \phi_k(t) dt = \lambda_k \phi_k(s), \quad k = 1, 2, \dots$$

where  $\int_0^1 \phi_k \phi_j dt = \delta_{kj}$ . We have

$$X_t = \sum_{k=1}^{\infty} \alpha_k \sqrt{\lambda_k} \phi_k(t), \tag{1.2}$$

in the sense that the series

$$X_t^N = \sum_{k=1}^N \alpha_k \sqrt{\lambda_k} \phi_k(t) \rightarrow X_t \quad \text{in } L_t^\infty L_P^2, \tag{1.3}$$

i.e. we have

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, 1]} \mathbb{E}|X_t^N - X_t|^2 = 0.$$

Here  $\alpha_k$  are *i.i.d.*  $N(0, 1)$  random variables.

**Proof.** At first it is easy to find that the operator  $K : L^2[0, 1] \rightarrow L^2[0, 1]$  defined as

$$(K\phi)(s) := \int_0^1 K(s, t)\phi(t)dt$$

through the covariance kernel function is nonnegative, self-adjoint and compact from the non-negativity, symmetry and continuity of  $K(s, t)$  on  $[0, 1]^2$  [3]. From the theory of functional analysis, there are countable real eigenvalues, and 0 is the only possible accumulation point. For each nonzero eigenvalue, the eigensubspace is finite dimensional. This verifies the formal validity of the definition (1.2).

From Mercer's theorem which states that the convergence

$$\sum_{k=1}^N \lambda_k \phi_k(s)\phi_k(t) \rightarrow K(s, t), \quad s, t \in [0, 1], \quad N \rightarrow \infty$$

holds in absolute and uniform sense when  $K$  is continuous [6], we have for  $N > M$

$$\mathbb{E}|X_t^N - X_t^M|^2 = \sum_{k=M+1}^N \lambda_k \phi_k^2(t) \rightarrow 0$$

in the absolute and uniform sense when  $N, M \rightarrow \infty$ . This implies  $X_t^N$  is a Cauchy sequence in the Banach space  $L_t^\infty L_P^2$ , thus the limit  $X_t$  exists and is unique in this space. For each fixed  $t$ , the mean square convergence of the Gaussian random vector  $(X_{t_1}^N, X_{t_2}^N, \dots, X_{t_m}^N)$  to  $(X_{t_1}, X_{t_2}, \dots, X_{t_m})$  implies the convergence in probability for any  $t_1, t_2, \dots, t_m \in [0, 1]$ . Application of the Theorem ?? ensures that the limit  $X_t$  is indeed a Gaussian process. It is not difficult to prove that

$$\begin{aligned} \mathbb{E}X_t &= \lim_{N \rightarrow \infty} \mathbb{E}X_t^N = 0, \\ \mathbb{E}X_s X_t &= \lim_{N \rightarrow \infty} \mathbb{E}X_s^N X_t^N = \sum_{k=1}^{\infty} \lambda_k \phi_k(s)\phi_k(t) = K(s, t) \end{aligned}$$

by the convergence of  $X^N$  to  $X$  in  $L_t^\infty L_P^2$ . The proof is completed.

As an application of Karhunen-Loeve expansion to Brownian motion, one can obtain the eigensystem  $\{\lambda_k, \phi_k(t)\}$  as follows. We have

$$\int_0^1 (s \wedge t)\phi_k(t)dt = \lambda_k \phi_k(s)$$

and thus

$$\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt = \lambda_k \phi_k(s). \quad (1.4)$$

Taking differentiation with respect to  $s$  we obtain

$$\lambda_k \phi_k'(s) = s\phi_k(s) + \int_s^1 s\phi_k(t)dt - s\phi_k(s) = \int_s^1 s\phi_k(t)dt. \quad (1.5)$$

Differentiating once again gives a Sturm-Liouville problem

$$\lambda_k \phi_k''(s) = -\phi_k(s).$$

This naturally suggests  $\lambda_k \neq 0$ . Take  $s = 0$  in (1.4), we obtain  $\phi_k(0) = 0$ ; take  $s = 1$  in (1.5), we have  $\phi_k'(1) = 0$ .

Solving this boundary value problem we obtain

$$\lambda_k = \left( \left( k - \frac{1}{2} \right) \pi \right)^{-2}, \quad \phi_k(s) = \sqrt{2} \sin \left( \left( k - \frac{1}{2} \right) \pi s \right), \quad k = 1, 2, \dots$$

Thus we get another representation of Brownian motion

$$W_t = \sum_{k=1}^{\infty} \alpha_k \frac{\sqrt{2}}{\left( k - \frac{1}{2} \right) \pi} \sin \left( \left( k - \frac{1}{2} \right) \pi t \right). \quad (1.6)$$

It is easy to find that  $W_0 = 0$  with this representation. To understand why it is almost surely continuous, we need the following theorem.

**Theorem 1.6.** *For the Karhunen-Loeve expansion to the Gaussian random field  $X_t$  with the same condition as in Theorem 1.5, if additionally*

$$\int_0^1 (-\ln u)^{1/2} dp(u) < \infty, \quad (1.7)$$

where

$$p(u) := \max\{\sigma(s, t) : |s - t| \leq |u|\}$$

and

$$\sigma(s, t) = \sum_{k=1}^{\infty} \lambda_k (\phi_k(s) - \phi_k(t))^2 = K(s, s) + K(t, t) - 2K(s, t),$$

then  $X_t^N$  converges to  $X_t$  uniformly for  $t \in [0, 1]$  with probability one, and thus  $X$  has continuous trajectory almost surely.

The proof of this theorem may be referred to [1]. For the Wiener process,  $\sigma(s, t) = t \vee s - t \wedge s$  and  $p(u) = |u|$ , so the condition (1.7) is satisfied and we have the continuity of the constructed  $W_t$  almost surely.

A realization with cutoff  $N = 1000$  is shown in Fig. 1.

### C. Construction from Haar basis

The construction below based on the Haar basis is originated from P. Lévy's interpolation method for Brownian motion. At first let us define the mother function

$$\psi(t) = \begin{cases} 1, & t \in [0, 1/2), \\ -1, & t \in [1/2, 1), \\ 0, & \text{otherwise.} \end{cases}$$

The multilevel Haar functions  $\{H_k^{(n)}\}$  are defined as  $H_0^{(0)}(t) = 1$  and

$$H_k^{(n)}(t) = 2^{\frac{n-1}{2}} \psi(2^{n-1}t - k), \quad n \geq 1, \quad k \in I_n := \{0, 1, \dots, 2^{n-1} - 1\}$$

for  $t \in [0, 1]$ , where  $n$  is the level and we take the convention that  $I_0 = \{0\}$ . It is a standard result that the Haar system  $\{H_k^{(n)}\}$  for  $n \in \mathbb{N}$  and  $k \in I_n$  forms an orthonormal basis in  $L^2[0, 1]$  [7]. We have the following theorem.

**Theorem 1.7.** *Let the random variables  $\{\alpha_k^{(n)}\}$  i.i.d.  $N(0, 1)$ . Then*

$$W_t^N = \sum_{n=0}^N \sum_{k \in I_n} \alpha_k^{(n)} \int_0^t H_k^{(n)}(s) ds \longrightarrow W_t, \quad N \rightarrow \infty,$$

uniformly in  $t \in [0, 1]$  in the almost sure sense.

A direct check on the finite terms approximation shows

$$\mathbb{E}W_t^N = \sum_{n=0}^N \sum_{k \in I_n} \mathbb{E}\alpha_k^{(n)} \int_0^t H_k^{(n)}(s) ds = 0,$$

and

$$\begin{aligned} \mathbb{E}W_t^N W_s^N &= \sum_{n,m=0}^N \sum_{k \in I_n, l \in I_m} \mathbb{E}(\alpha_k^{(n)} \alpha_l^{(m)}) \int_0^t H_k^{(n)}(\tau) d\tau \int_0^s H_l^{(m)}(\tau) d\tau \\ &= \sum_{n=0}^N \sum_{k \in I_n} \int_0^t H_k^{(n)}(\tau) d\tau \int_0^s H_k^{(n)}(\tau) d\tau \\ &= \sum_{n=0}^N \sum_{k \in I_n} \int_0^1 H_k^{(n)}(\tau) \chi_{[0,t]}(\tau) d\tau \int_0^1 H_k^{(n)}(\tau) \chi_{[0,s]}(\tau) d\tau \\ &\rightarrow \int_0^1 \chi_{[0,t]} \chi_{[0,s]}(\tau) d\tau = t \wedge s. \end{aligned} \tag{1.8}$$

where  $\chi_{[0,t]}(\tau)$  is the indicator function on  $[0, t]$ . Here the last convergence in the above equations is due to Parseval's identity because  $\{H_k^{(n)}\}$  is an orthonormal basis. Below we give the rigorous proof.

**Proof.** At first, we show  $W_t^N$  uniformly converges to some continuous function  $W_t$  in the almost sure sense. We have the following tail estimate for any Gaussian distributed random variable  $\xi \sim N(0, 1)$ .

$$\mathbb{P}(|\xi| > x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy \leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{y}{x} e^{-\frac{y^2}{2}} dy = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{x^2}{2}}}{x}, \quad x > 0.$$

Define  $a_n = \max_{k \in I_n} \alpha_k^{(n)}$ , then we obtain

$$\mathbb{P}(a_n > n) = \mathbb{P}\left(\bigcup_{k \in I_n} \alpha_k^{(n)} > n\right) \leq 2^{n-1} \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{n^2}{2}}}{n}, \quad n \geq 1.$$

From  $\sum_{n=1}^{\infty} 2^{n-1} \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{n^2}{2}}}{n} < \infty$ , the Borel-Cantelli lemma implies that there exists a set  $\tilde{\Omega}$  with  $\mathbb{P}(\tilde{\Omega}) = 1$  such that for any  $\omega \in \tilde{\Omega}$  there is a  $N(\omega)$  satisfying  $a_m(\omega) \leq m$  for any  $m \geq N(\omega)$ . In this case

$$\left| \sum_{m=N(\omega)}^{\infty} \sum_{k \in I_m} \alpha_k^{(m)} \int_0^t H_k^{(m)}(s) ds \right| \leq \sum_{m=N(\omega)}^{\infty} m \sum_{k \in I_m} \int_0^t H_k^{(m)}(s) ds \leq \sum_{m=N(\omega)}^{\infty} m 2^{-\frac{m+1}{2}} < \infty,$$

which shows the uniform convergence of  $W_t^N$  to a continuous function  $W_t$  in the almost sure sense.

Now we prove  $W_t$  is indeed the standard Brownian motion. From the uniform convergence of  $W_t^N$  with respect to  $t$  in a almost sure sense, the limit  $W_t$  is indeed a Gaussian process from Theorem ???. From the initial condition  $W_0 = 0$  and the covariance function relation (1.8), we obtain a new representation of the Wiener process  $W_t$ .

A realization with finite cutoff is shown in Fig. 1.

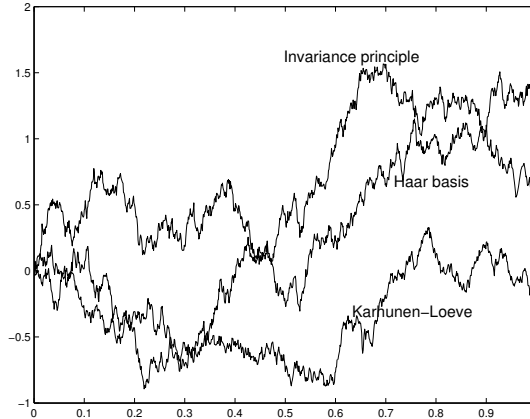


Figure 1: Numerical constructions of Brownian motion

**Example 1.8.** *Compute the expectation*

$$\mathbb{E} \exp\left(-\frac{1}{2} \int_0^1 W_t^2 dt\right).$$

**Solution.** Note that it is not straightforward to compute this expectation since the integrand involves the whole Wiener path, i.e. a Wiener functional. From the Karhunen-Loeve

expansion,

$$\begin{aligned}\int_0^1 W_t^2 dt &= \int_0^1 \sum_{k,l} \sqrt{\lambda_k \lambda_l} \alpha_k \alpha_l \phi_k(t) \phi_l(t) dt \\ &= \sum_k \int_0^1 \lambda_k \alpha_k^2 \phi_k^2(t) dt = \sum_k \lambda_k \alpha_k^2.\end{aligned}$$

Then

$$\mathbb{E} \exp\left(-\frac{1}{2} \int_0^1 W_t^2 dt\right) = \mathbb{E}\left(\prod_k \exp\left(-\frac{1}{2} \lambda_k \alpha_k^2\right)\right) = \prod_k \mathbb{E} \exp\left(-\frac{1}{2} \lambda_k \alpha_k^2\right).$$

From the identity

$$\mathbb{E} \exp\left(-\frac{1}{2} \lambda_k \alpha_k^2\right) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{1}{2} \lambda_k x^2} dx = \sqrt{\frac{1}{1 + \lambda_k}}$$

we obtain

$$\mathbb{E} \exp\left(-\frac{1}{2} \int_0^1 W_t^2 dt\right) = \prod_k \sqrt{\frac{1}{1 + \lambda_k}} := M,$$

where

$$M^{-2} = \prod_{k=1}^{\infty} \left(1 + \frac{4}{(2k-1)^2 \pi^2}\right).$$

From the identities for infinite product series we have

$$\cosh(x) = \prod_{n=1}^{\infty} \left(1 + \frac{4x^2}{(2n-1)^2 \pi^2}\right),$$

where  $\cosh(x) = (e^x + e^{-x})/2$ . Thus

$$M = (\cosh(1))^{-\frac{1}{2}} = \sqrt{\frac{2e}{1 + e^2}}.$$

## 2 Properties of Wiener path

In this section, we investigate some basic properties and the regularity of the Wiener path.

**Theorem 2.1** (Basic properties). *Suppose  $W_t$  is a standard Brownian motion, then*

1. *Time-homogeneity: For any  $s > 0$ ,  $W_{t+s} - W_s$ ,  $t \geq 0$ , is a Brownian motion;*
2. *Symmetry: The process  $-W_t$ ,  $t \geq 0$ , is a Brownian motion;*



3. *Scaling:* For every  $c > 0$ , the process  $cW_{t/c^2}$ ,  $t \geq 0$ , is a Brownian motion;
4. *Time-inversion:* The process  $X$  defined by  $X_0 = 0$ ,  $X_t = tW_{1/t}$  for  $t > 0$ , is a Brownian motion.

The proof of these properties are left as exercise. Specially, the scaling property 3 has important implication for the dimensional analysis involving Brownian motion, which states

$$W_{kt} \sim \sqrt{k}W_t, \quad \dot{W}_{kt} \sim \frac{1}{\sqrt{k}}\dot{W}_t, \quad (2.1)$$

where  $\dot{W}_t$  means the formal derivative of  $W_t$  with respect to  $t$  as discussed later. Note that for a standard smooth function  $f(t)$  with the change of variable  $t = k\tau$ , we have the relation

$$\frac{df}{dt}(k\tau) = \frac{1}{k} \frac{df}{d\tau}(k\tau), \quad (2.2)$$

instead of (2.1).

Now let us investigate the regularity of the Brownian motion. The total variation of a specific path of the process  $X$  on  $[a, b]$  is defined as

$$V(X(\omega); [a, b]) = \sup_{\Delta} \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|,$$

where  $\Delta = \cup_k [t_k, t_{k+1}]$  is any fixed subdivision of  $[a, b]$ . The discrete quadratic variation of  $X$  on  $[0, t]$  with subdivision  $\Delta$  is defined as

$$Q_t^\Delta = \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|^2.$$

If for any  $t$  and any sequence  $\Delta_n$  of subdivisions of  $[0, t]$  such that  $|\Delta_n|$  goes to zero, there exists a finite process  $\langle X, X \rangle$  such that

$$Q_t^{\Delta_n} \rightarrow \langle X, X \rangle_t \quad \text{in Probability as } n \rightarrow \infty,$$

then  $\langle X, X \rangle$  is called the *quadratic variation process* of  $X$ . It is obvious that  $\langle X, X \rangle$  is increasing. The definition can be straightforwardly extended to the case on the interval  $[a, b]$  as

$$Q_{[a,b]}^{\Delta_n} \rightarrow \langle X, X \rangle_b - \langle X, X \rangle_a \quad \text{as } n \rightarrow \infty.$$

**Proposition 2.2.** For any  $t$  and subdivision  $\Delta$  of  $[0, t]$ , we have for Wiener process  $W$

$$\mathbb{E}(Q_t^\Delta - t)^2 = 2 \sum_k (t_{k+1} - t_k)^2, \quad (2.3)$$

thus we get

$$Q_t^\Delta \longrightarrow t \text{ in } L^2(\mathbb{P}) \text{ as } |\Delta| \rightarrow 0$$

and  $\langle W, W \rangle_t = t$  a.s.

The proof of Proposition 2.2 is straightforward and left as an exercise. This result is sometimes formally stated as  $dW_t^2 = dt$ .

**Theorem 2.3** (Unbounded variation of the Wiener path). *The Wiener paths are a.s. of infinite variations on any interval.*

**Proof.** Suppose the probability space is  $(\Omega, \mathcal{F}, \mathbb{P})$ . Based on (2.3) and the subsequence argument, there is a set  $\Omega_0 \subset \Omega$  such that  $\mathbb{P}(\Omega_0) = 1$ , and there exists a subsequence of the subdivisions, still denoted as  $\Delta_n$ , such that for any rational pair  $p < q$ ,

$$Q_{[p,q]}^{\Delta_n} \rightarrow q - p, \quad \text{on } \Omega_0.$$

Now for any rational interval  $[p, q]$ , we have

$$q - p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 \leq \sup_k |W_{t_{k+1}} - W_{t_k}| \cdot V(W(\omega), [p, q]).$$

From the uniform continuity of  $W$  on  $[p, q]$ ,  $\sup_k |W_{t_{k+1}} - W_{t_k}| \rightarrow 0$ , thus we complete the proof.

The following result shows the Brownian motion has very curious smoothness.

**Theorem 2.4** (Smoothness of the Wiener path). *Consider the Wiener process on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Define  $\Omega_\alpha$  the set of functions that are Hölder continuous with exponent  $\alpha$  ( $0 < \alpha < 1$ )*

$$\Omega_\alpha = \left\{ f \in C[0, 1], \sup_{0 \leq s, t \leq 1} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty \right\}.$$

*Then if  $0 \leq \alpha < \frac{1}{2}$ ,  $\mathbb{P}(W_t \in \Omega_\alpha) = 1$ ; if  $\alpha \geq \frac{1}{2}$ ,  $\mathbb{P}(W_t \in \Omega_\alpha) = 0$ .*

The proof of Theorem 2.4 relies on the following generalized Kolmogorov's continuity theorem, which can be referred to [5].

**Theorem 2.5.** *Let  $X_t$  ( $t \in [0, 1]^d$ ) be a Banach-valued process for which there exist three strictly positive constants  $\gamma, c, \varepsilon$  such that*

$$\mathbb{E}(|X_t - X_s|^\gamma) \leq c|t - s|^{d+\varepsilon},$$

*then there is a modification  $\tilde{X}$  of  $X$  such that*

$$\mathbb{E} \left( \sup_{s \neq t} (|\tilde{X}_t - \tilde{X}_s| / |t - s|^\alpha)^\gamma \right) < \infty$$

*for every  $\alpha \in [0, \varepsilon/\gamma)$ . In particular, the paths of  $\tilde{X}$  are Hölder continuous of order  $\alpha$ .*

**Proof of Theorem 2.4.** When  $\alpha < 1/2$ , according to the generalized Kolmogorov continuity theorem and the following identity for 1D Gaussian R.V.

$$\mathbb{E}|B_t|^{2p} = Ct^p$$

for any  $p \in \mathbb{N}$ , we have  $\epsilon/\gamma = (p-1)/2p = 1/2 - 1/2p$ . Thus for  $\alpha < 1/2$ ,  $\mathbb{P}(W_t \in \Omega_\alpha) = 1$ .

When  $\alpha > 1/2$ , if there exists rational interval  $[p, q]$  such that  $|W_t - W_s| \leq c|t - s|^\alpha$  for any  $p \leq s, t \leq q$  then by Proposition 2.2

$$q-p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 \leq c^2 \sum_k |t_{k+1} - t_k|^{2\alpha-1} |t_{k+1} - t_k| \leq c^2(q-p) \sup_k |t_{k+1} - t_k|^{2\alpha-1} \rightarrow 0,$$

which is a contradiction.

For the critical case  $\alpha = 1/2$ , one should apply the deep theorem on Lévy's modulus of continuity. The readers may be referred to [5].

From Fig. 1 we may observe the Brownian path is always fluctuating and it is a very noisy curve. Theorem 2.3 and 2.4 tell us that each trajectory is continuous and nowhere differentiable and it has unbounded variation in any finite interval! All of these results show the Brownian motion is a very subtle and strange mathematical object.

The following theorem due to A. Khinchin, characterizes the local behavior of  $W_t$  when  $t$  goes to zero.

**Theorem 2.6** (Local law of the iterated logarithm). *For the standard Brownian motion, we have*

$$\mathbb{P}\left(\limsup_{t \rightarrow 0} \frac{W_t}{\sqrt{-2t \ln \ln t}} = 1\right) = 1.$$

Correspondingly

$$\mathbb{P}\left(\liminf_{t \rightarrow 0} \frac{W_t}{\sqrt{-2t \ln \ln t}} = -1\right) = 1.$$

For the long time behavior of the Brownian motion, we have the following type of strong law of large numbers.

**Theorem 2.7** (Strong Law of Large Numbers). *For the standard Brownian motion, we have*

$$\lim_{t \rightarrow \infty} \frac{W_t}{t} = 0, \quad a.s.$$

The readers may be referred to [2, 4, 5] for more properties of Brownian motion.

### 3 Homeworks

- HW1. (Scaling invariance of Wiener Process) Let  $W_t$  be a Wiener process. Show that

$$X_t = \begin{cases} 0 & \text{if } t = 0 \\ tW_{1/t} & \text{if } t \in (0, 1] \end{cases}$$

$$Y_t = \frac{1}{\sqrt{c}} W_{ct}, \quad t > 0, c > 0$$

$$Z_t = W(T) - W(T - t), \quad 0 < t \leq T.$$

are all Wiener processes in the sense that they have the same finite dimensional distributions.

- HW2. Prove Proposition 2.2 and if we set the points  $t_k = k2^{-n}t, k = 0, 1, \dots, 2^n$  and consider the discrete quadratic variation of Brownian motion in  $[0, t]$ , prove the following sharpening of the Proposition 2.2.

$$\lim_{n \rightarrow \infty} Y_N(t, \omega) \rightarrow t, \quad \text{a.s.}$$

- HW3. Prove that  $C[0, \infty)$  is a complete, separable metric space with the metric defined as

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left( \|x - y\|_{L^\infty[0, n]} \wedge 1 \right)$$

- HW4. Prove that that given  $0 \leq s < t, W_s = x, W_t = y$ , then the conditional distribution

$$W_{\frac{s+t}{2}} | W_s = x, W_t = y \sim N\left(\frac{x+y}{2}, \frac{t-s}{4}\right).$$

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