

Lecture 10 Random Walk and Brownian motion *

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1 1D Symmetric Random Walk

Example 1. (1D Random Walk) Suppose a particle suffers displacements along a straight line from the origin, denote its position $X_n \in \mathbb{Z}$. Let ξ_i are i.i.d. random moves such that $\xi_i = \pm 1$ with probability $\frac{1}{2}$, and let

$$X_n = \xi_1 + \xi_2 + \dots + \xi_n \quad (\text{i.e. } X_0 = 0)$$

$\{X_n\}$ is called a unconstrained symmetric random walk on \mathbb{Z} . Given $X_n = i$, we have

$$\begin{aligned} P\{X_{n+1} = i \pm 1 | X_n = i\} &= \frac{1}{2}, \\ P\{X_{n+1} = \text{anything else} | X_n = i\} &= 0. \end{aligned}$$

It is a typical example of the simplest Markov chains.

After taking N steps, the particle could be at any of the points

$$-N, -N + 2, \dots, \dots, N - 2, N.$$

1.1 Distribution of X_N

One basic question is the probability $W(m, N) = \text{Prob}\{X_N = m\}$ that the particle arrives at the point m after suffering N displacements.

It is not difficult to find that $W(m, N)$ obeys binomial distribution

$$W(m, N) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \left(\frac{1}{2}\right)^N,$$

and it is easy to note that m can be odd or even only according as N is odd or even.

The expectation position and mean square deviation are

$$\mathbb{E}X_N = 0, \quad \mathbb{E}X_N^2 = N,$$

then the root mean square displacement is \sqrt{N} .

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Definition 1. (*Diffusion coefficient*) The 1D diffusion coefficient D is defined as

$$D = \frac{\langle (X_N - X_0)^2 \rangle}{2N}.$$

It is assumed $\mathbb{E}X_N = X_0$ here. In general continuous case, it is defined as

$$D = \lim_{t \rightarrow \infty} \frac{\langle (X_t - X_0)^2 \rangle}{2dt},$$

where d is the space dimension.

For this simplest random walk, $D = \frac{1}{2}$.

Next we consider the case $N, m \gg 1$, and $m \ll N$ since we will rescale the process with the relation $x = ml, t = N\tau$ and $l \sim \mathcal{O}(\sqrt{\tau})$, $\tau \rightarrow 0$. Thus $m/N = x/t \cdot \tau/l \rightarrow 0$. So only the range $m \ll N$ matters. By Stirling's formula

$$\log n! = (n + \frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + O(n^{-1}) \quad (n \rightarrow +\infty),$$

we have

$$\begin{aligned} \log W(m, N) &\approx (N + \frac{1}{2}) \log N - \frac{1}{2}(N + m + 1) \log \left[\frac{N}{2} \left(1 + \frac{m}{N} \right) \right] \\ &\quad - \frac{1}{2}(N - m + 1) \log \left[\frac{N}{2} \left(1 - \frac{m}{N} \right) \right] - \frac{1}{2} \log 2\pi - N \log 2. \end{aligned}$$

Since $m \ll N$ we have Taylor series expansion for $x \ll 1$

$$\log(1 + x) = x - \frac{1}{2}x^2 + O(x^3),$$

thus

$$\log W(m, N) \approx -\frac{1}{2} \log N + \log 2 - \frac{1}{2} \log 2\pi - \frac{m^2}{2N} + O\left(\left(\frac{m}{N}\right)^2\right).$$

In other words, one obtains the asymptotic formula

$$W(m, N) \approx \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} \exp\left(-\frac{m^2}{2N}\right).$$

An interesting thing is to take the continuum limit of random walk. Now suppose we rescale the random walk with the spatial steplength l and the time spacing τ for each movement, we take the limit in the following sense when considering the point (x, t)

$$N, m \rightarrow \infty, \quad l, \tau \rightarrow 0, \quad \text{and} \quad N\tau = t, \quad ml = x. \quad (1)$$

To make the continuum limit physically reasonable, we also ask to fix the diffusion coefficient

$$D = \frac{\langle (X_{N\tau} - X_0)^2 \rangle}{2N\tau} = \frac{l^2}{2\tau}$$

in the limit. Consider the intervals Δx which are large compared with the length l , we have the probability that $y \in (x - \Delta x/2, x + \Delta x/2)$ for the continuous probability density $W(x, t)$ satisfies

$$W(x, t)\Delta x \approx \int_{x-\Delta x/2}^{x+\Delta x/2} W(y, t)dy \approx \sum_{\substack{m' \in \{m, m \pm 2, m \pm 4, \dots\} \\ m'l \in (x-\Delta x/2, x+\Delta x/2)}} W(m', N) \approx W(m, N) \frac{\Delta x}{2l} \quad (x = ml)$$

figure. From $1 = 1/2 + 1/2$, this decomposition actually decompose the paths into those go leftwards and rightwards with equal probability, which corresponds to all the paths just stated.

- The number of the paths hitting m_1 and hitting m finally is equal to that of the paths hitting $2m_1 - m$ finally. This can be understood because these paths have to cross m_1 and we can denote the final hitting time as N_2 as shown in the figure. So after N_2 , the paths go leftwards or rightwards with mirror symmetry to hit m or $2m_1 - m$. Before N_1 , the paths can be either branch.

These assertions are called the **reflection principle**, which is the basis of the following calculations for reflection and absorbing barrier problem.

So we have the following identity

$$W_r(m, N; m_1) = W(m, N) + W(2m_1 - m, N).$$

If we take large N limit we have

$$W_r(m, N; m_1) \approx \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} \left[\exp\left(-\frac{m^2}{2N}\right) + \exp\left(-\frac{(2m_1 - m)^2}{2N}\right) \right],$$

then passing to the continuum limit we have

$$W_r(x, t; x_1) = \frac{1}{\sqrt{4\pi Dt}} \left[\exp\left(-\frac{x^2}{4Dt}\right) + \exp\left(-\frac{(2x_1 - x)^2}{4Dt}\right) \right], \quad (2)$$

and we may note in this case

$$\left. \frac{\partial W_r}{\partial x} \right|_{x=x_1} = 0.$$

with W is defined in (2).

Case 2: Absorbing wall at $m = m_1$;

Similarly as before we easily deduce that

$$W_a(m, N; m_1) = W(m, N) - W(2m_1 - m, N).$$

by reflection principle.

In the large N limit we have

$$W_a(m, N; m_1) \approx \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} \left[\exp\left(-\frac{m^2}{2N}\right) - \exp\left(-\frac{(2m_1 - m)^2}{2N}\right) \right],$$

and the continuum limit is

$$W_a(x, t; x_1) = \frac{1}{\sqrt{4\pi Dt}} \left[\exp\left(-\frac{x^2}{4Dt}\right) - \exp\left(-\frac{(2x_1 - x)^2}{4Dt}\right) \right], \quad (3)$$

and we may note in this case

$$W_a(x, t; x_1) = 0.$$

with W_a is defined in (3).

Define the first hitting probability $a(m_1, N) = \text{Prob}\{X_N = m_1, \text{ and } X_n < m_1, \forall n < N\}$ that taking N steps the particle will arrive at m_1 without ever hitting $m = m_1$ at any earlier step. Then we have

$$a(m_1, N) = \frac{1}{2} W_a(m_1 - 1, N - 1; m_1) = \frac{m_1}{N} W(m_1, N)$$

by the above formula and the relation $W(m, N) = \frac{N}{N+m}W(m-1, N-1)$. In the large N limit we have

$$a(m_1, N) \approx \frac{m_1}{N} \left(\frac{2}{\pi N} \right)^{\frac{1}{2}} \exp\left(-\frac{m_1^2}{2N}\right).$$

The continuous probability density $a(m_1, t)$ becomes

$$a(m_1, t)\Delta t \approx a(m_1, N) \frac{\Delta t}{2\tau} \quad (t = N\tau)$$

In the continuum limit one obtains

$$a(x_1, t) = \frac{x_1}{t} \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x_1^2}{4Dt}\right).$$

We may note in this case

$$a(x_1, t) = -D \frac{\partial W}{\partial x} \Big|_{x=x_1}.$$

with W is defined in (3).

2 Arcsine law and the law of iterated logarithm

To simplify the notations, we define the first hitting time

$$\sigma_{2n} = \min\{1 \leq k \leq 2n : S_k = 0\}$$

and we define $\sigma_{2n} = +\infty$ if $S_k \neq 0$ for $1 \leq k \leq 2n$. For $0 \leq k \leq n$ we define

$$u_{2k} = \mathbb{P}(S_{2k} = 0), \quad f_{2k} = \mathbb{P}(\sigma_{2n} = 2k). \quad (4)$$

It is clear that $u_{2k} = C_{2k}^k \cdot 2^{-2k}$. From the reflection principle, we have

$$f_{2k} = 2 \frac{1}{2} \cdot \frac{1}{2k-1} W(1, 2k-1) = \frac{1}{2k} u_{2(k-1)} = u_{2(k-1)} - u_{2k}. \quad (5)$$

Now define $P_{2k, 2n}$ be the probability that during the interval $[0, 2n]$ the particle spends $2k$ units of time on the positive side (We say that the particle is on the positive side in the interval $[m-1, m]$ if one, at least, of the value S_{m-1} and S_m is positive).

Lemma 1. *Let $u_0 = 1$ and $0 \leq k \leq n$. Then*

$$P_{2k, 2n} = u_{2k} \cdot u_{2n-2k}. \quad (6)$$

Proof. At first let us show that (6) holds for $k = 0$. Suppose we have a path with $S_{2n} = 0$ and $\min_{0 \leq k \leq 2n} X_k = -m$, where $m > 0$. Denote $l = \min\{k | X_k = -m\}$. We can map this path into a path only in the positive side. Take a reflection of the path $\{X_k\}_{0 \leq k \leq l}$ with respect to the axis $t = l$ and denote the new path by $\{\tilde{X}_k\}_{0 \leq k \leq l}$ such that $\tilde{X}_k = X_{l-k}$. Concatenate \tilde{X}_0 to the point $(2n, 0)$ and translate the left endpoint of the new path into the origin. With such manipulation, we get a path on the positive side and the right endpoint is $(2n, 2m)$. Conversely, for each path on the positive side with the right endpoint is $(2n, 2m)$, we take $l = \max\{k | X_k = m\}$. We can cut the part beyond $t = k$, make a reflection with respect

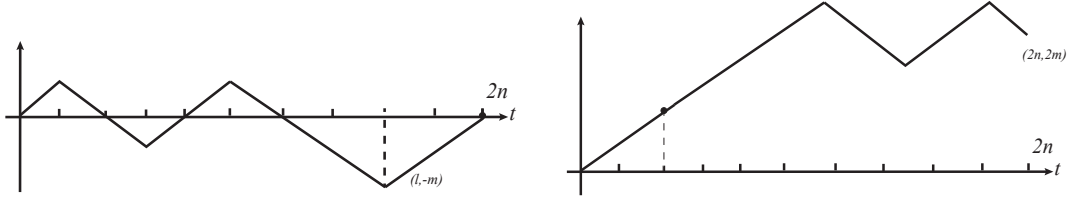


Figure 2: Schematics of construction from a path with $S_{2n} = 0$ to a new path on the positive side.

to $t = l$, concatenate it to the left endpoint of the rest part and translate the whole path into the origin, we then get a new path with $S_{2n} = 0$. A special case is illustrate in Figure 2. The case for $k = n$ is trivially true by symmetry and the case $k = 0$.

Then let us prove the following relation

$$u_{2k} = \sum_{r=1}^k f_{2r} \cdot u_{2(k-r)}. \quad (7)$$

Since $\{S_{2k} = 0\} \subset \{\sigma_{2n} \leq 2k\}$, we have

$$\{S_{2k} = 0\} = \{S_{2k} = 0\} \cap \{\sigma_{2n} \leq 2k\} = \sum_{r=1}^k \{S_{2k} = 0\} \cap \{\sigma_{2n} = 2r\} \quad (8)$$

Consequently

$$\begin{aligned} u_{2k} &= \mathbb{P}(S_{2k} = 0) = \sum_{r=1}^k \mathbb{P}(S_{2k} = 0, \sigma_{2n} = 2r) \\ &= \sum_{r=1}^k \mathbb{P}(S_{2k} = 0 | \sigma_{2n} = 2r) \mathbb{P}(\sigma_{2n} = 2r). \end{aligned} \quad (9)$$

But

$$\begin{aligned} \mathbb{P}(S_{2k} = 0 | \sigma_{2n} = 2r) &= \mathbb{P}(S_{2k} = 0 | S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0) \\ &= \mathbb{P}(S_{2r} + (\xi_{2r+1} + \dots + \xi_{2k}) = 0 | S_1 \neq 0, \dots, S_{2r-1} \neq 0, S_{2r} = 0) \\ &= \mathbb{P}(S_{2r} + (\xi_{2r+1} + \dots + \xi_{2k}) = 0 | S_{2r} = 0) \\ &= \mathbb{P}(\xi_{2r+1} + \dots + \xi_{2k} = 0) = \mathbb{P}(S_{2(k-r)} = 0). \end{aligned} \quad (10)$$

Combing (9) and (10) we obtain (7). To prove (6), we apply the induction method. Now let $1 \leq k \leq n - 1$. If the particle is on the positive side for exactly $2k$ instants, it must pass through zero. Let $2r$ be the time of first passage through zero. There are two possibilities: either $S_k \geq 0$, $k \leq 2r$, or $S_k \leq 0$, $k \leq 2r$.

The number of paths of the first kind is

$$(2^{2r} \cdot \frac{1}{2} f_{2r}) \cdot (2^{2(n-r)} \cdot P_{2(k-r), 2(n-r)}) = \frac{1}{2} 2^{2n} f_{2r} P_{2(k-r), 2(n-r)}.$$

The number of paths of the second kind is

$$\frac{1}{2} 2^{2n} f_{2r} P_{2k, 2(n-r)}.$$

Consequently, for $1 \leq k \leq n-1$,

$$P_{2k,2n} = \frac{1}{2} \sum_{r=1}^k f_{2r} P_{2(k-r),2(n-r)} + \frac{1}{2} \sum_{r=1}^k f_{2r} P_{2k,2(n-r)}. \quad (11)$$

Suppose that $P_{2k,2m} = u_{2k} \cdot u_{2m-2k}$ holds for $m = k, k+1, \dots, n-1$. Then by (7) and (11) we have (How is the induction applied here?)

$$\begin{aligned} P_{2k,2n} &= \frac{1}{2} u_{2n-2k} \sum_{r=1}^k f_{2r} u_{2k-2r} + \frac{1}{2} u_{2k} \sum_{r=1}^k f_{2r} u_{2n-2k-2r} \\ &= \frac{1}{2} u_{2n-2k} u_{2k} + \frac{1}{2} u_{2k} u_{2n-2k} = u_{2k} u_{2n-2k}. \end{aligned}$$

This completes the proof. \square

Now let $\gamma(2n)$ be the number of time units that the particle spends on the positive axis in the interval $[0, 2n]$. Then when $x < 1$,

$$\mathbb{P}\left\{\frac{1}{2} < \frac{\gamma(2n)}{2n} \leq x\right\} = \sum_{k, 1/2 < 2k/2n \leq x} P_{2k,2n}.$$

Since

$$u_{2k} \sim \frac{1}{\sqrt{\pi k}}$$

by Stirling's formula as $k \rightarrow \infty$, we have

$$P_{2k,2n} \sim \frac{1}{\pi \sqrt{k(n-k)}}$$

as $k, n-k \rightarrow \infty$.

Therefore

$$\sum_{\{k, 1/2 < 2k/2n \leq x\}} P_{2k,2n} - \sum_{k, 1/2 < 2k/2n \leq x} \frac{1}{\pi n} \cdot \left[\frac{k}{n} \left(1 - \frac{k}{n}\right)\right]^{-\frac{1}{2}} \rightarrow 0, \quad n \rightarrow \infty,$$

Whence

$$\sum_{\{k, 1/2 < 2k/2n \leq x\}} P_{2k,2n} \rightarrow \frac{1}{\pi} \int_{\frac{1}{2}}^x \frac{dt}{\sqrt{t(1-t)}}, \quad n \rightarrow \infty.$$

From the symmetry,

$$\sum_{\{k, 2k/2n \leq 1/2\}} P_{2k,2n} \rightarrow \frac{1}{2}$$

and

$$\frac{1}{\pi} \int_{\frac{1}{2}}^x \frac{dt}{\sqrt{t(1-t)}} = \frac{2}{\pi} \arcsin \sqrt{x} - \frac{1}{2}.$$

Thus we have the following theorem:

Theorem 1 (Arcsine Law). *The probability that the fraction of the time spent by the particle on the positive side is at most x tends to $\frac{2}{\pi} \arcsin \sqrt{x}$:*

$$\sum_{\{k, k/n \leq x\}} P_{2k,2n} \rightarrow \frac{2}{\pi} \arcsin \sqrt{x}.$$

The following deep theorem is due to Hartman and Wintler (1941).

Theorem 2 (Law of Iterated Logarithm). *Let ξ_1, \dots, ξ_n are i.i.d. R.V. with $\mathbb{E}\xi_i = 0, \text{Var}\xi_i = \sigma^2 > 0$, Then*

$$\mathbb{P}\left\{\limsup \frac{S_n}{\sqrt{2\sigma^2 n \ln \ln n}} = 1\right\} = 1.$$

Remark 1. *Application of the above result to $-\xi_i$, one also obtains*

$$\mathbb{P}\left\{\liminf \frac{S_n}{\sqrt{2\sigma^2 n \ln \ln n}} = -1\right\} = 1.$$

3 Random Flights With Gaussian Displacements

In the general problem of random flights, the position \mathbf{R} of the particle after N displacements is given by

$$\mathbf{R} = \sum_{i=1}^N \mathbf{r}_i,$$

where the $\mathbf{r}_i = (r_i^1, r_i^2, r_i^3)$'s denote the different displacements. Assume the probability that the i th displacement between \mathbf{r}_i and $\mathbf{r}_i + d\mathbf{r}_i$ is given by

$$\tau_i(r_i^1, r_i^2, r_i^3) dr_i^1 dr_i^2 dr_i^3 = \tau_i d\mathbf{r}_i \quad (i = 1, \dots, N).$$

Now we ask the probability $W_N(\mathbf{R})d\mathbf{R}$ that the position of the particle after N displacements lies in the interval $\mathbf{R}, \mathbf{R} + d\mathbf{R}$. The method presented in the following is originally devised by A.A. Markov.

It is a standard exercise to have

$$\widehat{W}_N(\mathbf{R}) = \int W_N(\mathbf{R}) \exp(i\boldsymbol{\rho} \cdot \mathbf{R}) d\mathbf{R} = \prod_{i=1}^N \int \tau_i(\mathbf{r}_i) \exp(i\boldsymbol{\rho} \cdot \mathbf{r}_i) d\mathbf{r}_i = \prod_{i=1}^N \widehat{\tau}_i(\mathbf{r}_i).$$

In the case of Gaussian distribution of random displacement \mathbf{r}_i , we have the pdf

$$\tau_i(\mathbf{r}_i) = \frac{1}{(2\pi l^2)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{r}_i|^2}{2l^2}\right),$$

From the property of Fourier transform for Gaussian distribution, we have

$$W_N(\mathbf{R}) = \frac{1}{(2\pi Nl^2)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{R}|^2}{2Nl^2}\right).$$

Suppose the time spacing is τ each time and define the diffusion coefficient as before

$$D = \lim_{t \rightarrow 0} \frac{\langle (X_t - X_0)^2 \rangle}{2dt} = \frac{3Nl^2}{6N\tau} = \frac{l^2}{2\tau},$$

Then we have the continuum limit pdf for free Gaussian random flight

$$W(\mathbf{R}, t) = \frac{1}{(4\pi Dt)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{R}|^2}{4Dt}\right) \quad (t = N\tau).$$

Remark 2. *Similar results for $W_N(\mathbf{R})$ holds for other distributions which can be referred in [1]. These results will be further clarified in next lecture on Brownian motion.*

Question 1. *How about more general reflecting and absorbing barrier problem in high dimensions?*

4 Einstein's work on the theory of Brownian motion

In 1905, A. Einstein published a seminal paper on the theory of Brownian motion (he also publishes two other seminal papers on Special Relativity and photoemission in this year). Two major points in Einstein's solution to Brownian motion are

(i) The motion is caused by the exceedingly frequent impacts on the pollen grain of the incessantly moving molecules of liquid in which it is suspended;

(ii) The motion of these molecules is so complicated that its effect on the pollen grain can only be described probabilistically in terms of exceedingly frequent statistically independent impacts.

His mathematical interpretation is as follows (1D version).

In a small time interval τ , the X -coordinates of an individual particle will increase by an amount Δ . There will be a certain "frequency law" for Δ

$$dn = n\phi(\Delta)d\Delta$$

where

$$\int_{-\infty}^{+\infty} \phi(\Delta)d\Delta = 1, \quad \phi(-\Delta) = \phi(\Delta),$$

and ϕ is only different from 0 for very small values of Δ .

Let $f(x, t)$ be the number of particles per unit volume, then

$$f(x, t + \tau)dx = \int_{-\infty}^{+\infty} f(x - \Delta, t)dx\phi(\Delta)d\Delta.$$

Since τ is small

$$f(x, t + \tau) = f(x, t) + \frac{\partial f}{\partial t}\tau,$$

furthermore

$$f(x - \Delta, t) = f(x, t) - \Delta \frac{\partial f}{\partial x} + \frac{\Delta^2}{2} \frac{\partial^2 f}{\partial x^2} + \dots$$

Thus

$$f(x, t) + \frac{\partial f}{\partial t}\tau = f \int_{-\infty}^{+\infty} \phi(\Delta)d\Delta + \frac{\partial f}{\partial x} \int_{-\infty}^{+\infty} \Delta\phi(\Delta)d\Delta + \frac{\partial^2 f}{\partial x^2} \int_{-\infty}^{+\infty} \frac{\Delta^2}{2}\phi(\Delta)d\Delta + \dots$$

Set

$$\frac{1}{\tau} \int_{-\infty}^{+\infty} \frac{\Delta^2}{2}\phi(\Delta)d\Delta = D$$

throwing h.o.t., we have

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}.$$

His description contains very many of the major concepts which have been developed more and more generally and rigorously since then, such as

(i) Chapman-Kolmogorov equation;

(ii) Fokker-Planck equation;

(iii) Kramers-Moyal expansion;

etc.

5 Homeworks

- HW1. Prove that the continuum limit pdf $W(x, t)$ for free random walk satisfies the PDE

$$\begin{cases} \frac{\partial W}{\partial t} = D \frac{\partial^2 W}{\partial x^2}, & x \in R, t \geq 0 \\ W(x, t) \Big|_{t=0} = \delta(x). \end{cases}$$

- HW2. Prove that the continuum limit pdf $W(x, t)$ with reflecting barrier satisfies the PDE

$$\begin{cases} \frac{\partial W}{\partial t} = D \frac{\partial^2 W}{\partial x^2}, & x \leq x_1, t \geq 0 \\ W(x, t) \Big|_{t=0} = \delta(x), \\ \frac{\partial W}{\partial x}(x, t) \Big|_{x=x_1} = 0. \end{cases}$$

- HW3. Prove that the continuum limit pdf $W(x, t)$ with absorbing barrier satisfies the PDE

$$\begin{cases} \frac{\partial W}{\partial t} = D \frac{\partial^2 W}{\partial x^2}, & x \leq x_1, t \geq 0 \\ W(x, t) \Big|_{t=0} = \delta(x), \\ W(x, t) \Big|_{x=x_1} = 0. \end{cases}$$

- HW4 (optional). Give the detailed procedure for the induction in proving the Arcsine Law.

References

- [1] S. Chandrasekhar, Stochastic problems and in physics and astronomy, Rev. Mod. Phys. 15(1943), 1-89.
- [2] A.N. Shiryaev, Probability, Springer-Verlag, Berlin and Heidelberg, 1996.