Lecture 13. Construction of BM and its Properties

Tiejun Li[1,2]

[1]School of Mathematical Sciences (SMS),
&
[2]Center for Machine Learning Research (CMLR),
Peking University,
Beijing 100871,
P.R. China
tieli@pku.edu.cn

Office: No. 1 Science Building, Room 1376E

Table of Contents

Construction of Wiener Process

We will show three approaches to construct the Wiener process. Different forms play different roles in different circumstances.

- Construction from invariance principle
- Construction from Karhunen-Loeve Expansion
- Construction from Haar basis

Construction from invariance principle

The construction from the invariance principle embodies the idea of taking continuum limit of symmetric random walk.

Theorem (Invariance Principle)

Suppose $\{\xi_i\}$ *are* $i.i.d.$ $N(0,1)$ *random variables, define* $S_n = \sum_{i=1}^{n} \xi_i$ *and* X_t^n *as follows:*

$$X_t^n = \begin{cases} \frac{s_k}{\sqrt{n}}, & t = \frac{k}{n}, \\ (1-\theta)\frac{s_k}{\sqrt{n}} + \theta\frac{s_{k+1}}{\sqrt{n}}, & t \in \left(\frac{k}{n}, \frac{k+1}{n}\right), \quad \theta = nt - k, \end{cases}$$

then $X^n \in C[0,\infty)$ *and*

$$X^n \xrightarrow{d} W,$$

where $\stackrel{d}{\rightarrow}$ *is the weak convergence on the function space* $C[0,\infty)$ *to be defined below.*

Construction from invariance principle

- Let us consider a special case by taking

$$\boldsymbol{P}(\xi_i) = \begin{cases} 1/2, & \xi_i = 1, \\ 1/2, & \xi_i = -1, \end{cases}$$

then $\mathbb{E}\xi_i = 0$, $\mathrm{var}\xi_i = 1$. The state of X_t^n at the time $t_k = k/n$ is nothing but the random walk considered before.

Construction from invariance principle

- Let us consider a special case by taking

$$\boldsymbol{P}(\xi_i) = \begin{cases} 1/2, & \xi_i = 1, \\ 1/2, & \xi_i = -1, \end{cases}$$

  then $\mathbb{E}\xi_i = 0$, $\mathrm{var}\xi_i = 1$. The state of X_t^n at the time $t_k = k/n$ is nothing but the random walk considered before.
- The construction from invariance principle indicates that the standard Brownian motion is just the rescaled limit of the random walk with spatial scale $l = 1/\sqrt{n}$ and time scale $\tau = 1/n$. The relation $l^2/\tau = 1$ is exactly the regime considered before.

Construction from invariance principle

- Let us consider a special case by taking

$$\boldsymbol{P}(\xi_i) = \begin{cases} 1/2, & \xi_i = 1, \\ 1/2, & \xi_i = -1, \end{cases}$$

  then $\mathbb{E}\xi_i = 0$, $\mathrm{var}\xi_i = 1$. The state of X_t^n at the time $t_k = k/n$ is nothing but the random walk considered before.
- The construction from invariance principle indicates that the standard Brownian motion is just the rescaled limit of the random walk with spatial scale $l = 1/\sqrt{n}$ and time scale $\tau = 1/n$. The relation $l^2/\tau = 1$ is exactly the regime considered before.
- This approximation is the most common one in computations.

Construction from invariance principle

Heuristic check for invariance principle:

- From the definition $S_n = \sum_{i=1}^{n} \xi_i$, where $\{\xi_i\}$ are i.i.d. $N(0,1)$ random variables, then by the central limit theorem

$$\frac{S_k}{\sqrt{n}} = \frac{\sqrt{k}}{\sqrt{n}} \cdot \frac{S_k}{\sqrt{k}} \xrightarrow{d} N(0,t), \text{ as } k,n \to \infty \text{ and } t = \frac{k}{n}.$$

Construction from invariance principle

Heuristic check for invariance principle:

- From the definition $S_n = \sum_{i=1}^n \xi_i$, where $\{\xi_i\}$ are i.i.d. $N(0,1)$ random variables, then by the central limit theorem

$$\frac{S_k}{\sqrt{n}} = \frac{\sqrt{k}}{\sqrt{n}} \cdot \frac{S_k}{\sqrt{k}} \xrightarrow{d} N(0,t), \text{ as } k,n \to \infty \text{ and } t = \frac{k}{n}.$$

- The limit X of X^n is then a Gaussian process formally with $X_0 = 0$ and

$$\begin{aligned}\mathbb{E}X_t X_s &\sim \mathbb{E}X_t^n X_s^n \\ &= \mathbb{E}X_{t\wedge s}^n (X_{t\vee s}^n - X_{t\wedge s}^n + X_{t\wedge s}^n) \\ &= \mathbb{E}(X_{t\wedge s}^n)^2 + \mathbb{E}X_{t\wedge s}^n (X_{t\vee s}^n - X_{t\wedge s}^n) \\ &\to t \wedge s. \quad \text{for } t = k/n, s = l/n \text{ and } k,l,n \to \infty.\end{aligned}$$

 The last identity holds because of the independence between $X_{t\wedge s}^n$ and $X_{t\vee s}^n - X_{t\wedge s}^n$, and $\mathbb{E}(X_{t\wedge s}^n - X_{t\vee s}^n) = 0$.

Construction from invariance principle

Remark.

- Heuristically the key point in the invariance principle is CLT when n, k is sufficiently large.
- This implies the condition $\xi_n \sim i.i.d.\ N(0,1)$ may be relaxed to ξ_n be $i.i.d.$ with mean 0 and variance 1. The distribution of ξ_n is not important.
- That is why the theorem is called "invanriance" principle.

Construction from Karhunen-Loeve Expansion

Theorem (Karhunen-Loeve expansion)

Let X_t $(t \in [0,1])$ *be a Gaussian process with mean function* $m(t) = 0$ *and continuous covariance function* $K(s,t)$. *Consider the following eigenvalue problem*

$$\int_0^1 K(s,t)\phi_k(t)dt = \lambda_k \phi_k(s), \qquad k = 1, 2, \cdots$$

where $\int_0^1 \phi_k \phi_j dt = \delta_{kj}$. *We have*

$$X_t = \sum_{k=1}^{\infty} \alpha_k \sqrt{\lambda_k} \phi_k(t),$$

in the sense that the series $X_t^N = \sum_{k=1}^N \alpha_k \sqrt{\lambda_k} \phi_k(t) \to X_t$, *in* $L_t^\infty L_P^2$, *i.e.* $\lim_{N\to\infty} \sup_{t\in[0,1]} \mathbb{E}|X_t^N - X_t|^2 = 0$. *Here* α_k *are* $i.i.d.$ $N(0,1)$ *random variables.*

Construction from Karhunen-Loeve Expansion

Proof.

- The operator $K : L^2[0,1] \to L^2[0,1]$ defined as

$$(K\phi)(s) := \int_0^1 K(s,t)\phi(t)dt$$

  is nonnegative, self-adjoint and compact. due to the non-negativity, symmetry and continuity of $K(s,t)$ on $[0,1]^2$.

Construction from Karhunen-Loeve Expansion

Proof.

- The operator $K : L^2[0,1] \to L^2[0,1]$ defined as

$$(K\phi)(s) := \int_0^1 K(s,t)\phi(t)dt$$

  is nonnegative, self-adjoint and compact. due to the non-negativity, symmetry and continuity of $K(s,t)$ on $[0,1]^2$.
- From the theory of functional analysis, there are countable real eigenvalues, and 0 is the only possible accumulation point. For each nonzero eigenvalue, the eigensubspace is finite dimensional.

Construction from Karhunen-Loeve Expansion

- From Mercer's theorem (stronger version of Hilbert-Schmidt Theorem) which states that the convergence

$$\sum_{k=1}^{N} \lambda_k \phi_k(s)\phi_k(t) \to K(s,t), \quad s,t \in [0,1],\ N \to \infty$$

holds in the uniform sense (i.e., $L^\infty[0,1]$) when K is continuous, we have for $N > M$

$$\mathbb{E}|X_t^N - X_t^M|^2 = \sum_{k=M+1}^{N} \lambda_k \phi_k^2(t) \to 0$$

in the uniform sense when $N, M \to \infty$.

Construction from Karhunen-Loeve Expansion

- From Mercer's theorem (stronger version of Hilbert-Schmidt Theorem) which states that the convergence

$$\sum_{k=1}^{N} \lambda_k \phi_k(s)\phi_k(t) \to K(s,t), \quad s,t \in [0,1], \; N \to \infty$$

  holds in the uniform sense (i.e., $L^\infty[0,1]$) when K is continuous, we have for $N > M$

$$\mathbb{E}|X_t^N - X_t^M|^2 = \sum_{k=M+1}^{N} \lambda_k \phi_k^2(t) \to 0$$

  in the uniform sense when $N, M \to \infty$.
- This implies X_t^N is a Cauchy sequence in the Banach space $L_t^\infty L_P^2$, thus the limit X_t exists and is unique in this space.

Construction from Karhunen-Loeve Expansion

- For each fixed t, the mean square convergence of the Gaussian random vector $(X_{t_1}^N, X_{t_2}^N, \ldots, X_{t_m}^N)$ to $(X_{t_1}, X_{t_2}, \ldots, X_{t_m})$ implies the convergence in probability for any $t_1, t_2, \ldots, t_m \in [0, 1]$.

Construction from Karhunen-Loeve Expansion

- For each fixed t, the mean square convergence of the Gaussian random vector $(X_{t_1}^N, X_{t_2}^N, \dots, X_{t_m}^N)$ to $(X_{t_1}, X_{t_2}, \dots, X_{t_m})$ implies the convergence in probability for any $t_1, t_2, \dots, t_m \in [0, 1]$.
- The closure property ensures that the limit X_t is indeed a Gaussian process. It is not difficult to prove that

$$\mathbb{E}X_t = \lim_{N\to\infty} \mathbb{E}X_t^N = 0,$$

$$\mathbb{E}X_s X_t = \lim_{N\to\infty} \mathbb{E}X_s^N X_t^N = \sum_{k=1}^{\infty} \lambda_k \phi_k(s)\phi_k(t) = K(s,t)$$

by the convergence of X^N to X in $L_t^\infty L_P^2$. $\square$

Construction from Karhunen-Loeve Expansion

Application to Brownian motion. Obtain the eigensystem $\{\lambda_k, \phi_k(t)\}$

- We have

$$\int_0^1 (s \wedge t)\phi_k(t)dt = \lambda_k \phi_k(s)$$

  and thus

$$\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt = \lambda_k \phi_k(s).$$

Construction from Karhunen-Loeve Expansion

Application to Brownian motion. Obtain the eigensystem $\{\lambda_k, \phi_k(t)\}$

- We have

$$\int_0^1 (s \wedge t)\phi_k(t)dt = \lambda_k \phi_k(s)$$

  and thus

$$\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt = \lambda_k \phi_k(s).$$

- Taking differentiation with respect to s we obtain

$$\lambda_k \phi_k'(s) = s\phi_k(s) + \int_s^1 s\phi_k(t)dt - s\phi_k(s) = \int_s^1 s\phi_k(t)dt.$$

Construction from Karhunen-Loeve Expansion

Application to Brownian motion. Obtain the eigensystem $\{\lambda_k, \phi_k(t)\}$

- We have

$$\int_0^1 (s \wedge t)\phi_k(t)dt = \lambda_k \phi_k(s)$$

  and thus

$$\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt = \lambda_k \phi_k(s).$$

- Taking differentiation with respect to s we obtain

$$\lambda_k \phi_k'(s) = s\phi_k(s) + \int_s^1 s\phi_k(t)dt - s\phi_k(s) = \int_s^1 s\phi_k(t)dt.$$

- Differentiating once again gives a Sturm-Liouville problem

$$\lambda_k \phi_k''(s) = -\phi_k(s).$$

Construction from Karhunen-Loeve Expansion

Application to Brownian motion. Obtain the eigensystem $\{\lambda_k, \phi_k(t)\}$

- We have
$$\int_0^1 (s \wedge t)\phi_k(t)dt = \lambda_k \phi_k(s)$$
and thus
$$\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt = \lambda_k \phi_k(s).$$
- Taking differentiation with respect to s we obtain
$$\lambda_k \phi_k'(s) = s\phi_k(s) + \int_s^1 s\phi_k(t)dt - s\phi_k(s) = \int_s^1 s\phi_k(t)dt.$$
- Differentiating once again gives a Sturm-Liouville problem
$$\lambda_k \phi_k''(s) = -\phi_k(s).$$
- It's easy to check that $\lambda_k \neq 0, \phi_k(0) = 0, \phi_k'(1) = 0.$

Construction from Karhunen-Loeve Expansion

- Solving this boundary value problem, we obtain

$$\lambda_k = \left((k - \frac{1}{2})\pi \right)^{-2}, \quad \phi_k(s) = \sqrt{2}\sin\left((k - \frac{1}{2})\pi s \right).$$

Construction from Karhunen-Loeve Expansion

- Solving this boundary value problem, we obtain

$$\lambda_k = \left((k - \frac{1}{2})\pi \right)^{-2}, \quad \phi_k(s) = \sqrt{2} \sin\left((k - \frac{1}{2})\pi s \right).$$

- Thus we get another representation of Brownian motion

$$W_t = \sum_{k=1}^{\infty} \alpha_k \frac{\sqrt{2}}{(k - \frac{1}{2})\pi} \sin\left((k - \frac{1}{2})\pi t \right).$$

It is easy to find that $W_0 = 0$ with this representation.

Construction from Karhunen-Loeve Expansion

- Solving this boundary value problem, we obtain

$$\lambda_k = \left((k - \frac{1}{2})\pi \right)^{-2}, \quad \phi_k(s) = \sqrt{2}\sin\left((k - \frac{1}{2})\pi s \right).$$

- Thus we get another representation of Brownian motion

$$W_t = \sum_{k=1}^{\infty} \alpha_k \frac{\sqrt{2}}{(k - \frac{1}{2})\pi} \sin\left((k - \frac{1}{2})\pi t \right).$$

  It is easy to find that $W_0 = 0$ with this representation.
- To understand why it is almost surely continuous, we need the following theorem.

Construction from Karhunen-Loeve Expansion

Theorem

For the Karhunen-Loeve expansion to the Gaussian random field X_t with the same condition as in Theorem (KLE), if additionally

$$\int_0^1 (-\ln u)^{1/2} dp(u) < \infty,$$

where $p(u) := \max\{\sigma(s,t) : |s-t| \le |u|\}$ *and*

$$\sigma(s,t) = \sum_{k=1}^{\infty} \lambda_k(\phi_k(s) - \phi_k(t))^2 = K(s,s) + K(t,t) - 2K(s,t),$$

then X_t^N *converges to* X_t *uniformly for* $t \in [0,1]$ *with probability one, and thus* X *has continuous trajectory almost surely.*

Construction from Karhunen-Loeve Expansion

Theorem

For the Karhunen-Loeve expansion to the Gaussian random field X_t with the same condition as in Theorem (KLE), if additionally

$$\int_0^1 (-\ln u)^{1/2} dp(u) < \infty,$$

where $p(u) := \max\{\sigma(s,t) : |s-t| \leq |u|\}$ *and*

$$\sigma(s,t) = \sum_{k=1}^{\infty} \lambda_k(\phi_k(s) - \phi_k(t))^2 = K(s,s) + K(t,t) - 2K(s,t),$$

then X_t^N *converges to* X_t *uniformly for* $t \in [0,1]$ *with probability one, and thus* X *has continuous trajectory almost surely.*

- For the Wiener process, $\sigma(s,t) = t \vee s - t \wedge s$ and $p(u) = |u|$, so the condition is satisfied and we have the continuity of the constructed W_t almost surely.

Construction from Karhunen-Loeve Expansion

Example

Compute the expectation

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big).$$

Construction from Karhunen-Loeve Expansion

Example

Compute the expectation

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big).$$

Solution. From the Karhunen-Loeve expansion,

$$\begin{aligned}\int_0^1 W_t^2 dt &= \int_0^1 \sum_{k,l} \sqrt{\lambda_k\lambda_l}\alpha_k\alpha_l\phi_k(t)\phi_l(t)dt \\ &= \sum_k \int_0^1 \lambda_k\alpha_k^2\phi_k^2(t)dt = \sum_k \lambda_k\alpha_k^2.\end{aligned}$$

Construction from Karhunen-Loeve Expansion

Example

Compute the expectation

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big).$$

Solution. From the Karhunen-Loeve expansion,

$$\begin{aligned}\int_0^1 W_t^2 dt &= \int_0^1 \sum_{k,l} \sqrt{\lambda_k\lambda_l}\alpha_k\alpha_l\phi_k(t)\phi_l(t)dt \\ &= \sum_k \int_0^1 \lambda_k\alpha_k^2\phi_k^2(t)dt = \sum_k \lambda_k\alpha_k^2.\end{aligned}$$

Then

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big) = \mathbb{E}\Big(\prod_k \exp(-\frac{1}{2}\lambda_k\alpha_k^2)\Big) = \prod_k \mathbb{E}\exp(-\frac{1}{2}\lambda_k\alpha_k^2).$$

Construction from Karhunen-Loeve Expansion

$$\mathbb{E}\exp(-\frac{1}{2}\lambda_k\alpha_k^2) = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot e^{-\frac{1}{2}\lambda_k x^2}dx = \sqrt{\frac{1}{1+\lambda_k}}$$

we obtain

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big) = \prod_k \sqrt{\frac{1}{1+\lambda_k}} := M,$$

where

$$M^{-2} = \prod_{k=1}^{\infty}\Big(1+\frac{4}{(2k-1)^2\pi^2}\Big).$$

Construction from Karhunen-Loeve Expansion

$$\mathbb{E}\exp(-\frac{1}{2}\lambda_k\alpha_k^2) = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot e^{-\frac{1}{2}\lambda_k x^2}dx = \sqrt{\frac{1}{1+\lambda_k}}$$

we obtain

$$\mathbb{E}\exp\Big(-\frac{1}{2}\int_0^1 W_t^2 dt\Big) \quad = \quad \prod_k \sqrt{\frac{1}{1+\lambda_k}} := M,$$

where

$$M^{-2} = \prod_{k=1}^{\infty}\Big(1+\frac{4}{(2k-1)^2\pi^2}\Big).$$

From the identities for infinite product series we have

$$\cosh(x) = \prod_{n=1}^{\infty}\Big(1+\frac{4x^2}{(2n-1)^2\pi^2}\Big),$$

where $\cosh(x) = (e^x + e^{-x})/2$. Thus

$$M = (\cosh(1))^{-\frac{1}{2}} = \sqrt{\frac{2e}{1+e^2}}.$$

Construction from Haar basis

- At first we define the mother function

$$\psi(t) = \begin{cases} 1, & t \in [0, 1/2), \\ -1, & t \in [1/2, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Construction from Haar basis

- At first we define the mother function

$$\psi(t) = \begin{cases} 1, & t \in [0, 1/2), \\ -1, & t \in [1/2, 1), \\ 0, & \text{otherwise.} \end{cases}$$

- Defined the multilevel Haar functions $\{H_k^{(n)}\}$ as $H_0^{(0)}(t) = 1$

$$H_k^{(n)}(t) = 2^{\frac{n-1}{2}} \psi(2^{n-1}t - k), n \geq 1,\ k \in I_n := \{0, \ldots, 2^{n-1} - 1\}$$

for $t \in [0, 1]$, where n is the level and we take the convention that $I_0 = \{0\}$.

Construction from Haar basis

- At first we define the mother function

$$\psi(t) = \begin{cases} 1, & t \in [0, 1/2), \\ -1, & t \in [1/2, 1), \\ 0, & \text{otherwise}. \end{cases}$$

- Defined the multilevel Haar functions $\{H_k^{(n)}\}$ as $H_0^{(0)}(t) = 1$

  $$H_k^{(n)}(t) = 2^{\frac{n-1}{2}}\psi(2^{n-1}t-k), n \geq 1,\ k \in I_n := \{0, \dots, 2^{n-1}-1\}$$

  for $t \in [0, 1]$, where n is the level and we take the convention that $I_0 = \{0\}$.

- It is a standard result that the Haar system $\{H_k^{(n)}\}$ for $n \in \mathbb{N}$ and $k \in I_n$ forms an orthonormal basis in $L^2[0, 1]$.

Construction from Haar basis

Theorem

Let the random variables $\{\alpha_k^{(n)}\}$ $i.i.d.$ $N(0,1)$. *Then*

$$W_t^N = \sum_{n=0}^{N} \sum_{k \in I_n} \alpha_k^{(n)} \int_0^t H_k^{(n)}(s) ds \longrightarrow W_t, \quad N \to \infty,$$

uniformly in $t \in [0,1]$ *in the almost sure sense.*

Construction from Haar basis

Theorem

Let the random variables $\{\alpha_k^{(n)}\}$ *i.i.d.* $N(0,1)$. *Then*

$$W_t^N = \sum_{n=0}^{N} \sum_{k \in I_n} \alpha_k^{(n)} \int_0^t H_k^{(n)}(s)ds \longrightarrow W_t, \quad N \to \infty,$$

uniformly in $t \in [0,1]$ *in the almost sure sense.*

A direct check on the finite terms approximation:

- $$\mathbb{E}W_t^N = \sum_{n=0}^{N} \sum_{k \in I_n} \mathbb{E}\alpha_k^{(n)} \int_0^t H_k^{(n)}(s)ds = 0,$$

Construction from Haar basis

- $$\begin{aligned}
&\mathbb{E}W_t^N W_s^N \\
&= \sum_{n,m=0}^{N} \sum_{k\in I_n, l\in I_m} \mathbb{E}(\alpha_k^{(n)}\alpha_l^{(m)}) \int_0^t H_k^{(n)}(\tau)d\tau \int_0^s H_l^{(m)}(\tau)d\tau \\
&= \sum_{n=0}^{N} \sum_{k\in I_n} \int_0^t H_k^{(n)}(\tau)d\tau \int_0^s H_k^{(n)}(\tau)d\tau \\
&= \sum_{n=0}^{N} \sum_{k\in I_n} \int_0^1 H_k^{(n)}(\tau)\chi_{[0,t]}(\tau)d\tau \int_0^1 H_k^{(n)}(\tau)\chi_{[0,s]}(\tau)d\tau \\
&\to \int_0^1 \chi_{[0,t]}\chi_{[0,s]}(\tau)d\tau = t \wedge s.
\end{aligned}$$

  Here $\chi_{[0,t]}(\tau)$ is the indicator function on $[0,t]$. The last convergence in the above equations is due to Parseval's identity because $\{H_k^{(n)}\}$ is an orthonormal basis.

Construction from Haar basis

Proof. At first, we show W_t^N uniformly converges to some continuous function W_t in the almost sure sense.

- We have the following tail estimate for any Gaussian distributed random variable $\xi \sim N(0,1)$. For $x > 0$,

$$\mathbb{P}(|\xi| > x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$$
$$\leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{y}{x} e^{-\frac{y^2}{2}} dy = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{x^2}{2}}}{x}.$$

Construction from Haar basis

Proof. At first, we show W_t^N uniformly converges to some continuous function W_t in the almost sure sense.

- We have the following tail estimate for any Gaussian distributed random variable $\xi \sim N(0,1)$. For $x > 0$,

$$\mathbb{P}(|\xi| > x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy \\ \leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{y}{x} e^{-\frac{y^2}{2}} dy = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{x^2}{2}}}{x}.$$

- Define $a_n = \max_{k \in I_n} |\alpha_k^{(n)}|$, then we obtain

$$\mathbb{P}(a_n > n) = \mathbb{P}\left(\bigcup_{k \in I_n} |\alpha_k^{(n)}| > n \right) \leq 2^{n-1} \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{n^2}{2}}}{n}, \quad n \geq 1.$$

Construction from Haar basis

- From $\sum_{n=1}^{\infty} 2^{n-1}\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{n^2}{2}}}{n} < \infty$, the Borel-Cantelli lemma implies that there exists a set $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for any $\omega \in \tilde{\Omega}$ there is a $N(\omega)$ satisfying $a_m(\omega) \leq m$ for any $m \geq N(\omega)$.

Construction from Haar basis

- From $\sum_{n=1}^{\infty} 2^{n-1}\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{n^2}{2}}}{n} < \infty$, the Borel-Cantelli lemma implies that there exists a set $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for any $\omega \in \tilde{\Omega}$ there is a $N(\omega)$ satisfying $a_m(\omega) \leq m$ for any $m \geq N(\omega)$.
- In this case,

$$\left| \sum_{m=N(\omega)}^{\infty} \sum_{k \in I_m} \alpha_k^{(m)} \int_0^t H_k^{(m)}(s)ds \right|$$
$$\leq \sum_{m=N(\omega)}^{\infty} m \sum_{k \in I_m} \int_0^t H_k^{(m)}(s)ds$$
$$\leq \sum_{m=N(\omega)}^{\infty} m 2^{-\frac{m+1}{2}} < \infty,$$

which shows the uniform convergence of W_t^N to a continuous function W_t in the almost sure sense.

Construction from Haar basis

Now we show that W_t is indeed a standard Brownian motion.

- From the uniform convergence of W_t^N with respect to t in the almost sure sense, the limit W_t is indeed a Gaussian process.

Construction from Haar basis

Now we show that W_t is indeed a standard Brownian motion.

- From the uniform convergence of W_t^N with respect to t in the almost sure sense, the limit W_t is indeed a Gaussian process.
- From the initial condition $W_0 = 0$ and the covariance function relation, we obtain a new representation of the Wiener process W_t.

□

Numerical constructions of Brownian motion

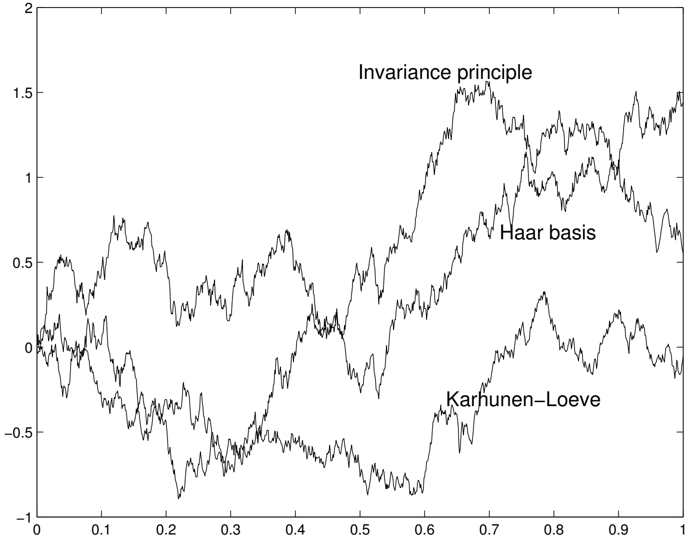

Table of Contents

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t *is a standard Brownian motion, then*

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t *is a standard Brownian motion, then*

1. *Time-homogeneity: For any* $s > 0$, $W_{t+s} - W_s,\ t \geq 0$, *is a Brownian motion;*

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t *is a standard Brownian motion, then*

1. *Time-homogeneity: For any* $s > 0$, $W_{t+s} - W_s,\ t \geq 0$, *is a Brownian motion;*
2. *Symmetry: The process* $-W_t$, $t \geq 0$, *is a Brownian motion;*

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t is a standard Brownian motion, then

1. *Time-homogeneity: For any $s > 0$, $W_{t+s} - W_s,\ t \geq 0$, is a Brownian motion;*
2. *Symmetry: The process $-W_t$, $t \geq 0$, is a Brownian motion;*
3. *Scaling: For every $c > 0$, the process cW_{t/c^2}, $t \geq 0$, is a Brownian motion;*

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t is a standard Brownian motion, then

1. *Time-homogeneity: For any $s > 0$, $W_{t+s} - W_s,\ t \geq 0$, is a Brownian motion;*
2. *Symmetry: The process $-W_t$, $t \geq 0$, is a Brownian motion;*
3. *Scaling: For every $c > 0$, the process cW_{t/c^2}, $t \geq 0$, is a Brownian motion;*
4. *Time-inversion: The process X defined by $X_0 = 0$, $X_t = tW_{1/t}$ for $t > 0$, is a Brownian motion.*

Properties of Wiener path

Theorem (Basic properties)

Suppose W_t is a standard Brownian motion, then

1. *Time-homogeneity: For any $s > 0$, $W_{t+s} - W_s,\ t \geq 0$, is a Brownian motion;*
2. *Symmetry: The process $-W_t$, $t \geq 0$, is a Brownian motion;*
3. *Scaling: For every $c > 0$, the process cW_{t/c^2}, $t \geq 0$, is a Brownian motion;*
4. *Time-inversion: The process X defined by $X_0 = 0$, $X_t = tW_{1/t}$ for $t > 0$, is a Brownian motion.*

- **Remark.** The scaling property 3 states $W_{kt} \sim \sqrt{k} W_t$, $\dot{W}_{kt} \sim \frac{1}{\sqrt{k}} \dot{W}_t$, where $\dot{W}_t$ means the formal derivative. For a standard smooth function $f(t)$ with the change of variable $t = k\tau$, we have $\frac{df}{dt}(k\tau) = \frac{1}{k}\frac{df}{d\tau}(k\tau)$.

The regularity of the Brownian motion

- The total variation of a specific path of the process X on $[a, b]$ is defined as

$$V(X(\omega); [a, b]) = \sup_{\Delta} \sum_{k} |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|,$$

where $\Delta = \cup_k [t_k, t_{k+1}]$ is any fixed subdivision of $[a, b]$.

The regularity of the Brownian motion

- The total variation of a specific path of the process X on $[a, b]$ is defined as

$$V(X(\omega); [a,b]) = \sup_{\Delta} \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|,$$

where $\Delta = \cup_k [t_k, t_{k+1}]$ is any fixed subdivision of $[a, b]$.

- The discrete quadratic variation of X on $[0, t]$ with subdivision Δ is defined as $Q_t^{\Delta} = \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|^2$.

The regularity of the Brownian motion

- The total variation of a specific path of the process X on $[a, b]$ is defined as

$$V(X(\omega); [a, b]) = \sup_{\Delta} \sum_{k} |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|,$$

where $\Delta = \cup_k [t_k, t_{k+1}]$ is any fixed subdivision of $[a, b]$.
- The discrete quadratic variation of X on $[0, t]$ with subdivision Δ is defined as $Q_t^{\Delta} = \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|^2$.
- If for any t and any sequence Δ_n of subdivisions of $[0, t]$ such that $|\Delta_n|$ goes to zero, there exists a finite process $\langle X, X \rangle$ such that

$$Q_t^{\Delta_n} \to \langle X, X \rangle_t \quad \text{in Probability as } n \to \infty,$$

then $\langle X, X \rangle$ is called the *quadratic variation process* of X.

The regularity of the Brownian motion

- The total variation of a specific path of the process X on $[a, b]$ is defined as

$$V(X(\omega); [a, b]) = \sup_{\Delta} \sum_{k} |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|,$$

where $\Delta = \cup_k [t_k, t_{k+1}]$ is any fixed subdivision of $[a, b]$.

- The discrete quadratic variation of X on $[0, t]$ with subdivision Δ is defined as $Q_t^{\Delta} = \sum_k |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|^2$.
- If for any t and any sequence Δ_n of subdivisions of $[0, t]$ such that $|\Delta_n|$ goes to zero, there exists a finite process $\langle X, X \rangle$ such that

$$Q_t^{\Delta_n} \to \langle X, X \rangle_t \quad \text{in Probability as } n \to \infty,$$

then $\langle X, X \rangle$ is called the *quadratic variation process* of X.

- Obviously, $\langle X, X \rangle$ is increasing. The definition can be straightforwardly extended to the case on the interval $[a, b]$ as

$$Q_{[a,b]}^{\Delta_n} \to \langle X, X \rangle_b - \langle X, X \rangle_a \quad \text{as } n \to \infty.$$

The regularity of the Brownian motion

Proposition

For any t and subdivision Δ of $[0,t]$, we have for Wiener process W

$$\mathbb{E}(Q_t^\Delta - t)^2 = 2\sum_k (t_{k+1} - t_k)^2,$$

thus we get

$$Q_t^\Delta \longrightarrow t \text{ in } L^2(\mathbb{P}) \text{ as } |\Delta| \to 0$$

and $\langle W, W\rangle_t = t$ a.s.

This result is sometimes formally stated as $(dW_t)^2 = dt$.

The regularity of the Brownian motion

Theorem (Unbounded variation of the Wiener path)

The Wiener paths are a.s. of infinite variations on any interval.

The regularity of the Brownian motion

Theorem (Unbounded variation of the Wiener path)

The Wiener paths are a.s. of infinite variations on any interval.

Proof. Suppose the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. Based on the subsequence argument, there is a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and there exits a subsequence of the subdivisions, still denoted as Δ_n, such that for any rational pair $p < q$,

$$Q_{[p,q]}^{\Delta_n} \to q - p, \quad \text{on } \Omega_0.$$

The regularity of the Brownian motion

Theorem (Unbounded variation of the Wiener path)

The Wiener paths are a.s. of infinite variations on any interval.

Proof. Suppose the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. Based on the subsequence argument, there is a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and there exits a subsequence of the subdivisions, still denoted as Δ_n, such that for any rational pair $p < q$,

$$Q_{[p,q]}^{\Delta_n} \to q - p, \quad \text{on } \Omega_0.$$

Now for any rational interval $[p, q]$, we have

$$q - p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 \leq \sup_k |W_{t_{k+1}} - W_{t_k}| \cdot V(W(\omega), [p, q]).$$

The regularity of the Brownian motion

Theorem (Unbounded variation of the Wiener path)

The Wiener paths are a.s. of infinite variations on any interval.

Proof. Suppose the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. Based on the subsequence argument, there is a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and there exits a subsequence of the subdivisions, still denoted as Δ_n, such that for any rational pair $p < q$,

$$Q_{[p,q]}^{\Delta_n} \to q - p, \quad \text{on } \Omega_0.$$

Now for any rational interval $[p, q]$, we have

$$q - p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 \leq \sup_k |W_{t_{k+1}} - W_{t_k}| \cdot V(W(\omega), [p, q]).$$

From the uniform continuity of W on $[p, q]$, $\sup_k |W_{t_{k+1}} - W_{t_k}| \to 0$, thus we complete the proof. □

The regularity of the Brownian motion

Theorem (Smoothness of the Wiener path)

Consider the Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$*. Define* Ω_α *the set of functions that are Hölder continuous with exponent* α $(0 < \alpha < 1)$

$$\Omega_\alpha = \left\{ f \in C[0,1], \sup_{0 \le s,t \le 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < \infty \right\}.$$

Then if $0 \le \alpha < \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$*; if* $\alpha \ge \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 0$*.*

The regularity of the Brownian motion

Theorem (Smoothness of the Wiener path)

Consider the Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. *Define* Ω_α *the set of functions that are Hölder continuous with exponent* α $(0 < \alpha < 1)$

$$\Omega_\alpha = \left\{ f \in C[0,1], \sup_{0 \le s,t \le 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < \infty \right\}.$$

Then if $0 \le \alpha < \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$; *if* $\alpha \ge \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 0$.

- This result shows that the Brownian motion has very curious smoothness.

The regularity of the Brownian motion

Theorem (Smoothness of the Wiener path)

Consider the Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$*. Define* Ω_α *the set of functions that are Hölder continuous with exponent* α $(0 < \alpha < 1)$

$$\Omega_\alpha = \left\{ f \in C[0,1], \sup_{0 \le s,t \le 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < \infty \right\}.$$

Then if $0 \le \alpha < \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$*; if* $\alpha \ge \frac{1}{2}$, $\mathbb{P}(W_t \in \Omega_\alpha) = 0$*.*

- This result shows that the Brownian motion has very curious smoothness.
- Each trajectory is continuous and nowhere differentiable in a almost sure sense.

The regularity of the Brownian motion

Theorem (Generalized Kolmogorov's continuity theorem)

Let X_t $(t \in [0,1]^d)$ *be a Banach-valued process for which there exist three strictly positive constants* γ, c, ε *such that*

$$\mathbb{E}(|X_t - X_s|^\gamma) \leq c|t-s|^{d+\varepsilon},$$

then there is a modification $\tilde{X}$ *of* X *such that*

$$\mathbb{E}\Big(\sup_{s\neq t}(|\tilde{X}_t - \tilde{X}_s|/|t-s|^\alpha)\Big)^\gamma < \infty$$

for every $\alpha \in [0, \varepsilon/\gamma)$. *In particular, the paths of* $\tilde{X}$ *are Hölder continuous of order* α.

The regularity of the Brownian motion

Proof of the Theorem (Smoothness of the Wiener path).

- When $\alpha < 1/2$, according to the generalized Kolmogorov continuity theorem and the following identity

$$\mathbb{E}|W_t|^{2p} = Ct^p$$

for any $p \in \mathbb{N}$, we have $\epsilon/\gamma = (p-1)/2p = 1/2 - 1/2p$. Thus for $\alpha < 1/2$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$.

The regularity of the Brownian motion

Proof of the Theorem (Smoothness of the Wiener path).

- When $\alpha < 1/2$, according to the generalized Kolmogorov continuity theorem and the following identity

$$\mathbb{E}|W_t|^{2p} = Ct^p$$

for any $p \in \mathbb{N}$, we have $\epsilon/\gamma = (p-1)/2p = 1/2 - 1/2p$. Thus for $\alpha < 1/2$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$.

- When $\alpha > 1/2$, if there exists rational interval $[p, q]$ such that $|W_t - W_s| \le c|t-s|^\alpha$ for any $p \le s, t \le q$ then

$$\begin{aligned} q - p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 &\le c^2 \sum_k |t_{k+1} - t_k|^{2\alpha-1} |t_{k+1} - t_k| \\ &\le c^2 (q-p) \sup_k |t_{k+1} - t_k|^{2\alpha-1} \to 0, \end{aligned}$$

which is a contradiction.

The regularity of the Brownian motion

Proof of the Theorem (Smoothness of the Wiener path).

- When $\alpha < 1/2$, according to the generalized Kolmogorov continuity theorem and the following identity

$$\mathbb{E}|W_t|^{2p} = Ct^p$$

for any $p \in \mathbb{N}$, we have $\epsilon/\gamma = (p-1)/2p = 1/2 - 1/2p$. Thus for $\alpha < 1/2$, $\mathbb{P}(W_t \in \Omega_\alpha) = 1$.

- When $\alpha > 1/2$, if there exists rational interval $[p, q]$ such that $|W_t - W_s| \leq c|t-s|^\alpha$ for any $p \leq s, t \leq q$ then

$$\begin{aligned} q - p \leftarrow \sum_k (W_{t_{k+1}} - W_{t_k})^2 &\leq c^2 \sum_k |t_{k+1} - t_k|^{2\alpha - 1} |t_{k+1} - t_k| \\ &\leq c^2 (q-p) \sup_k |t_{k+1} - t_k|^{2\alpha - 1} \to 0, \end{aligned}$$

which is a contradiction.

- For the critical case $\alpha = 1/2$, one should apply the deep theorem on Lévy's modulus of continuity. □

More properties of Brownian motion

Theorem (Local law of the iterated logarithm)

For the standard Brownian motion, we have

$$\mathbb{P}\Big(\limsup_{t\to 0} \frac{W_t}{\sqrt{-2t\ln\ln t}} = 1\Big) = 1.$$

Correspondingly

$$\mathbb{P}\Big(\liminf_{t\to 0} \frac{W_t}{\sqrt{-2t\ln\ln t}} = -1\Big) = 1.$$

More properties of Brownian motion

Theorem (Local law of the iterated logarithm)

For the standard Brownian motion, we have

$$\mathbb{P}\left(\limsup_{t \to 0} \frac{W_t}{\sqrt{-2t \ln \ln t}} = 1 \right) = 1.$$

Correspondingly

$$\mathbb{P}\left(\liminf_{t \to 0} \frac{W_t}{\sqrt{-2t \ln \ln t}} = -1 \right) = 1.$$

Theorem (Strong Law of Large Numbers)

For the standard Brownian motion, we have

$$\lim_{t \to \infty} \frac{W_t}{t} = 0, \quad \textit{a.s.}$$

Summary

The Brownian motion is a very subtle and strange mathematical object.

- The Brownian path is always fluctuating and it is a very noisy curve.

Summary

The Brownian motion is a very subtle and strange mathematical object.

- The Brownian path is always fluctuating and it is a very noisy curve.
- Each trajectory is continuous and nowhere differentiable and it has unbounded variation in any finite interval.