Lecture 11. Random Walk and Brownian motion

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1D Random Walk — Connection with Markov Chains

- 1D Symmetric Random Walk: Suppose a particle suffers displacements along a straight line from the origin, denote its position $X_n \in \mathbb{Z}$. Let ξ_i are $i.i.d.$ random moves such that $\xi_i = \pm 1$ with probability $\frac{1}{2}$, and let

$$X_n = \xi_1 + \xi_2 + \ldots + \xi_n \quad (i.e. \quad X_0 = 0)$$

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- $\{X_n\}$ is called a unconstrained symmetric random walk on $\mathbb{Z}$. Given $X_n = i$, we have

$$\begin{aligned} P\{X_{n+1} = i \pm 1 |\; X_n = i\} &= \frac{1}{2}, \\ P\{X_{n+1} = \text{anything else} |\; X_n = i\} &= 0. \end{aligned}$$

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  It is a typical example of the simplest Markov chains.
- After taking N steps, the particle could be at any of the points

$$-N, -N+2, \ldots, \ldots, N-2, N.$$

1D Random Walk — Distribution of X_N

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$$W(m, N) = \frac{N!}{(\frac{N+m}{2})!(\frac{N-m}{2})!}\left(\frac{1}{2}\right)^N,$$

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  and it is easy to note that m can be odd or even only according as N is odd or even.
- The expectation position and mean square deviation are

$$\mathbb{E}X_N = 0, \quad \mathbb{E}X_N^2 = N,$$

  then the root mean square displacement is $\sqrt{N}$.

Diffusion Coefficient

Definition (Diffusion coefficient)

The 1D diffusion coefficient D is defined as

$$D = \frac{\langle (X_N - X_0)^2 \rangle}{2N}.$$

It is assumed $\mathbb{E}X_N = X_0$ here. In general continuous case, it is defined as

$$D = \lim_{t \to \infty} \frac{\langle (X_t - X_0)^2 \rangle}{2dt},$$

where d is the space dimension.

For this simplest random walk, $D = \frac{1}{2}$.

Scaling limit of 1D random walk

- Now suppose we rescale the random walk with the spatial steplength l and the time spacing τ for each movement, we take the limit in the following sense when considering the point (x, t)

$$N, m \to \infty, \quad l, \tau \to 0, \text{ and } N\tau = t, \; ml = x.$$

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- To make the continuum limit physically reasonable, we also ask to fix the diffusion coefficient

$$D = \frac{\langle (X_{N\tau} - X_0)^2 \rangle}{2N\tau} = \frac{l^2}{2\tau}$$

in the limit. That is, we take the scaling $l \sim O(\sqrt{\tau})$.

1D Random Walk: Distribution

- So we consider the case $N, m \gg 1$, and $m \ll N$ since $m/N = x/t \cdot \tau/l \to 0$ for any fixed x, t when $l \sim O(\sqrt{\tau})$.

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- So we consider the case $N, m \gg 1$, and $m \ll N$ since $m/N = x/t \cdot \tau/l \to 0$ for any fixed x, t when $l \sim O(\sqrt{\tau})$.
- By Stirling's formula

$$\log n! = (n + \frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + O(n^{-1}) \quad (n \to +\infty),$$

we have

$$\begin{aligned} \log W(m, N) \quad \approx \quad & (N + \frac{1}{2}) \log N - \frac{1}{2}(N + m + 1) \log \left[\frac{N}{2}(1 + \frac{m}{N})\right] \\ & - \frac{1}{2}(N - m + 1) \log \left[\frac{N}{2}(1 - \frac{m}{N})\right] - \frac{1}{2} \log 2\pi - N \log 2. \end{aligned}$$

1D Random Walk: Distribution

- Since $m \ll N$ we have Taylor series expansion for $x \ll 1$

$$\log(1+x) = x - \frac{1}{2}x^2 + O(x^3),$$

thus

$$\log W(m,N) \approx -\frac{1}{2}\log N + \log 2 - \frac{1}{2}\log 2\pi - \frac{m^2}{2N} + O\left(\left(\frac{m}{N}\right)^2\right).$$

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- In other words, one obtains the the asymptotic formula

$$W(m,N) \approx \Big(\frac{2}{\pi N}\Big)^{\frac{1}{2}} \exp(-\frac{m^2}{2N}).$$

Scaling limit of 1D random walk

- Consider the intervals Δx which are large compared with the length l. The probability that $y \in (x - \Delta x/2, x + \Delta x/2)$ for the continuous probability density $W(x,t)$ satisfies

$$W(x,t)\Delta x \approx \int_{x-\Delta x/2}^{x+\Delta x/2} W(y,t)dy \approx \sum_{\substack{m' \in \{m, m\pm 2, m\pm 4, \dots\} \\ m'l \in (x-\Delta x/2, x+\Delta x/2)}} W(m',N)$$

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$$\approx W(m,N)\frac{\Delta x}{2l}, \quad (x = ml)$$

- since m can take only even or odd values depending on whether N is even or odd. Combining the results above one has

$$W(x,t)\Delta x = \frac{1}{\sqrt{2\pi t \frac{l^2}{\tau}}} \exp\left(-\frac{x^2}{2t\frac{l^2}{\tau}}\right)\Delta x,$$

thus the limiting probability density at time t

$$W(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right).$$

Random walk with reflecting Barriers

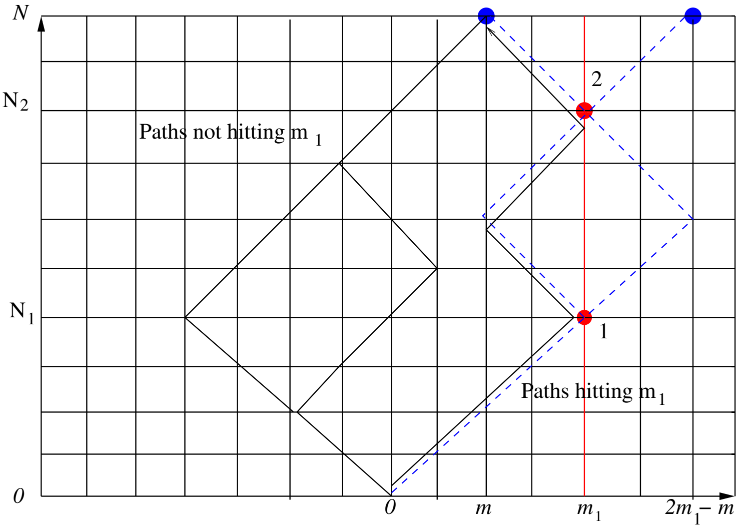

Figure: Schematics of reflection principle.

Random walk with reflecting Barriers

Case 1. A reflecting barrier at $m = m_1$:

- Suppose $m_1 > 0$. We now ask the probability $W_r(m, N; m_1)$ that the particle will arrive at $m(\leq m_1)$ after N steps.

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  1. One class only contains the paths not hitting m_1 and finally reaching m;
  2. the other class contains the paths hitting m_1 before time N and finally reaching m_1 or $2m_1 - m$.

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$$1^k = \left(\frac{1}{2} + \frac{1}{2}\right)^k \to (R + L)^k = R \dots RR + \cdots + LL \dots LL.$$

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  Each term on the RHS corresponding a free path.)
- The number of the paths hitting m_1 and hitting m finally is equal to that of the paths hitting $2m_1 - m$ finally.

Random walk with reflecting Barriers

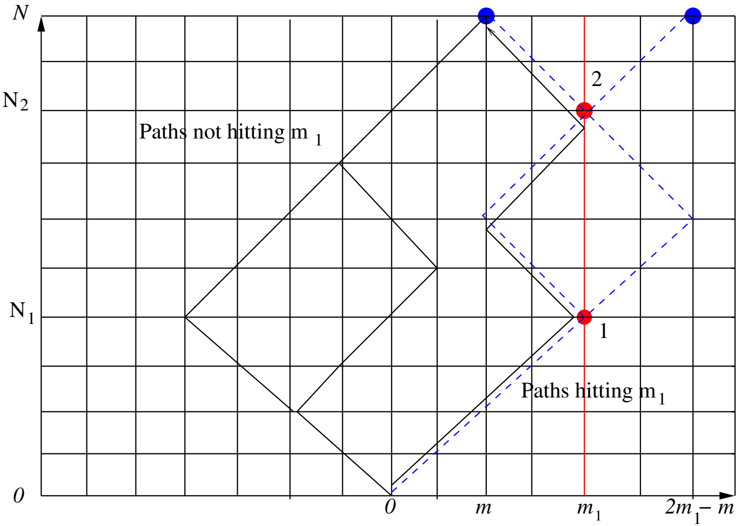

Figure: Schematics of reflection principle.

RW with reflecting Barriers: Distribution and Scaling Limit

These assertions are called the reflection principle, which is the basis of the following calculations for reflection and absorbing barrier problem.

- So we have the following identity

$$W_r(m, N; m_1) = W(m, N) + W(2m_1 - m, N).$$

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- Then passing to the continuum limit we have

$$W_r(x, t; x_1) = \frac{1}{\sqrt{4\pi Dt}} \left[\exp(-\frac{x^2}{4Dt}) + \exp(-\frac{(2x_1 - x)^2}{4Dt}) \right],$$

and we may note in this case

$$\left.\frac{\partial W_r}{\partial x}\right|_{x=x_1} = 0.$$

RW with absorbing Barriers: Distribution and Scaling Limit

Case 2. Absorbing wall at $m = m_1$:

- Similarly as before we easily deduce that

$$W_a(m, N; m_1) = W(m, N) - W(2m_1 - m, N).$$

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and we may note in this case

$$W_a(x, t; x_1) = 0.$$

RW with absorbing Barriers: First Hitting Probability

- Define the first hitting probability

$$a(m_1, N) = \mathrm{Prob}\{X_N = m_1, \text{ and } X_n < m_1, \forall n < N\}$$

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$$a(m_1, N) = \frac{1}{2} W_a(m_1 - 1, N - 1; m_1) = \frac{m_1}{N} W(m_1, N)$$

by the relation

$$W(m-1, N-1) = \frac{N+m}{N} W(m, N), \quad W(m+1, N-1) = \frac{N-m}{N} W(m, N).$$

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- In the large N limit

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- In the continuum limit one obtains

$$a(x_1, t) = \frac{x_1}{t}\frac{1}{\sqrt{4\pi D t}}\exp(-\frac{x_1^2}{4Dt}) = -D\frac{\partial W}{\partial x}\Big|_{x=x_1}.$$

First Return Time and Related Probabilities

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and $\sigma_{2n} := +\infty$ if $X_k \neq 0$ for $1 \leq k \leq 2n$.

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- For $0 \le k \le n$ we define

$$u_{2k} = \mathbb{P}(X_{2k} = 0), \quad f_{2k} = \mathbb{P}(\sigma_{2n} = 2k).$$

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- It is clear that $u_{2k} = C_{2k}^{k} \cdot 2^{-2k}$, and we have

$$f_{2k} = 2\frac{1}{2} \cdot \frac{1}{2k-1} W(1, 2k-1)$$

by the reflection principle.

Lead Probability

- Now define $P_{2k,2n}$ be the probability that during the interval $[0, 2n]$ the particle spends $2k$ units of time on the positive side (We say that the particle is on the positive side in the interval $[m-1, m]$ if one, at least, of the value X_{m-1} and X_m is positive).

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Lemma

Let $u_0 = 1$ *and* $0 \leq k \leq n$. *Then*

$$P_{2k,2n} = u_{2k} \cdot u_{2n-2k}.$$

Proof of Lemma: 3 Steps

The proof of the lemma is divided into three main steps.

- Step 1. Show that it holds for $k = 0$ and $k = n$.
- Step 2. Prove the relation

$$u_{2k} = \sum_{r=1}^{k} f_{2r} \cdot u_{2(k-r)}.$$

- Step 3. Show that

$$P_{2k,2n} = u_{2k} \cdot u_{2n-2k}.$$

  holds for general k.

We will do this step by step.

Proof of Lemma: Step 1

Step 1. Proof for the case $k = 0$. The idea is to establish a one-one correspondence between the set $\{X_{2n} = 0\}$ and the paths always on positive side.

- Suppose we have a path with $X_{2n} = 0$ and

$$\min_{0 \le k \le 2n} X_k = -m,$$

  where $m > 0$. Denote $l = \max\{k | X_k = -m\}$.

- We can map this path into a path only in the positive side. Take a reflection of the path $\{X_k\}_{0 \le k \le l}$ with respect to the axis $t = l$ and denote the new path by $\{\tilde{X}_k\}_{0 \le k \le l}$ such that $\tilde{X}_k = X_{l-k}$.
- Concatenate $\tilde{X}_0$ to the point $(2n, 0)$ and translate the left endpoint of the new path into the origin.
- With such manipulation, we get a path on the positive side and the right endpoint is $(2n, 2m)$.

Proof of Lemma: Step 1

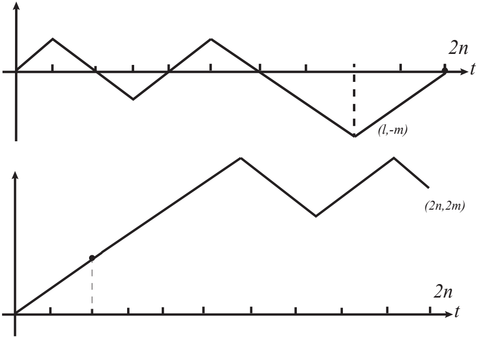

Figure: Schematics of construction from a path with $X_{2n} = 0$ to a new path on the positive side with 3 steps: 1. Reflection, 2. Concatenation, 3. Shift.

Proof of Lemma: Step 1

Conversely,

- For each path on the positive side with the right endpoint is $(2n, 2m)$, we take $l = \min\{k|X_k = m\}$.
- We can cut the part beyond $t = l$, make a reflection with respect to $t = l$, concatenate it to the left endpoint of the rest part and translate the whole path into the origin, we then get a new path with $X_{2n} = 0$.
- These manipulations are illustrate in the Figure.

The case for $k = n$ is trivially true by reflection symmetry and the case $k = 0$.

Proof of Lemma: Step 2

Recall $u_{2k} = \mathbb{P}(X_{2k} = 0)$, $f_{2k} = \mathbb{P}(\sigma_{2n} = 2k)$. Now let us prove the following relation

$$u_{2k} = \sum_{r=1}^{k} f_{2r} \cdot u_{2(k-r)}.$$

- Since $\{X_{2k} = 0\} \subset \{\sigma_{2n} \leq 2k\}$, we have

$$\{X_{2k} = 0\} = \{X_{2k} = 0\} \cap \{\sigma_{2n} \leq 2k\} = \sum_{r=1}^{k} \{X_{2k} = 0\} \cap \{\sigma_{2n} = 2r\}$$

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- Consequently

$$\begin{aligned} u_{2k} = & \quad \mathbb{P}(X_{2k} = 0) = \sum_{r=1}^{k} \mathbb{P}(X_{2k} = 0, \sigma_{2n} = 2r) \\ = & \quad \sum_{r=1}^{k} \mathbb{P}(X_{2k} = 0 | \sigma_{2n} = 2r) \mathbb{P}(\sigma_{2n} = 2r). \end{aligned}$$

Proof of Lemma: Step 2

- But

$$
\begin{aligned}
&\mathbb{P}(X_{2k} = 0 | \sigma_{2n} = 2r) = \mathbb{P}(X_{2k} = 0 | X_1 \neq 0, \ldots, X_{2r-1} \neq 0, X_{2r} = 0) \\
&\quad = \mathbb{P}(X_{2r} + (\xi_{2r+1} + \cdots + \xi_{2k}) = 0 | X_1 \neq 0, \ldots, X_{2r-1} \neq 0, X_{2r} = 0) \\
&\quad = \mathbb{P}(X_{2r} + (\xi_{2r+1} + \cdots + \xi_{2k}) = 0 | X_{2r} = 0) \\
&\quad = \mathbb{P}(\xi_{2r+1} + \cdots + \xi_{2k} = 0) = \mathbb{P}(X_{2(k-r)} = 0) = u_{2(k-r)}.
\end{aligned}
$$

Proof of Lemma: Step 2

- But

$$\begin{aligned}\mathbb{P}(X_{2k}=0|\sigma_{2n}=2r) &= \mathbb{P}(X_{2k}=0|X_1\neq 0,\ldots,X_{2r-1}\neq 0, X_{2r}=0)\\ &= \mathbb{P}(X_{2r}+(\xi_{2r+1}+\cdots+\xi_{2k})=0|X_1\neq 0,\ldots,X_{2r-1}\neq 0,X_{2r}=0)\\ &= \mathbb{P}(X_{2r}+(\xi_{2r+1}+\cdots+\xi_{2k})=0|X_{2r}=0)\\ &= \mathbb{P}(\xi_{2r+1}+\cdots+\xi_{2k}=0)=\mathbb{P}(X_{2(k-r)}=0)=u_{2(k-r)}.\end{aligned}$$

- According to the definition

$$u_{2k}=\mathbb{P}(X_{2k}=0), \quad f_{2k}=\mathbb{P}(\sigma_{2n}=2k),$$

we get the desired result

$$u_{2k}=\sum_{r=1}^{k} f_{2r}\cdot u_{2(k-r)}.$$

Proof of Lemma: Step 3

Now let us prove the Lemma when $1 \leq k \leq n-1$.

- If the particle is on the positive side for exactly $2k$ instants, it must pass through zero. Let $2r$ be the time of first passage through zero. There are two possibilities: either $X_k \geq 0$ for all $k \leq 2r$, or $X_k \leq 0$ for all $k \leq 2r$.

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- The number of paths of the first kind is

$$(2^{2r} \cdot \frac{1}{2} f_{2r}) \cdot (2^{2(n-r)} \cdot P_{2(k-r),2(n-r)}) = \frac{1}{2} 2^{2n} f_{2r} P_{2(k-r),2(n-r)}.$$

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- The number of paths of the second kind is

$$\frac{1}{2} 2^{2n} f_{2r} P_{2k,2(n-r)}.$$

Proof of Lemma: Step 3

- Consequently, for $1 \leq k \leq n-1$,

$$P_{2k,2n} = \frac{1}{2}\sum_{r=1}^{k} f_{2r}P_{2(k-r),2(n-r)} + \frac{1}{2}\sum_{r=1}^{k} f_{2r}P_{2k,2(n-r)}.$$

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- Suppose that $P_{2k,2m} = u_{2k} \cdot u_{2m-2k}$ holds for $m = k, k+1, \ldots, n-1$. we have (Question: How is the induction applied here?)

$$\begin{aligned} P_{2k,2n} &= \frac{1}{2}u_{2n-2k}\sum_{r=1}^{k} f_{2r}u_{2k-2r} + \frac{1}{2}u_{2k}\sum_{r=1}^{k} f_{2r}u_{2n-2k-2r} \\ &= \frac{1}{2}u_{2n-2k}u_{2k} + \frac{1}{2}u_{2k}u_{2n-2k} = u_{2k}u_{2n-2k}. \end{aligned}$$

This completes the proof.

Probability of the fraction of time spending on positive side

- Now let $\gamma(2n)$ be the number of time units that the particle spends on the positive axis in the interval $[0, 2n]$. Then when $x < 1$,

$$\mathbb{P}\Big\{\frac{1}{2} < \frac{\gamma(2n)}{2n} \leq x\Big\} = \sum_{k, 1/2 < 2k/2n \leq x} P_{2k,2n}.$$

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- Since

$$u_{2k} \sim \frac{1}{\sqrt{\pi k}} \qquad \left(\text{Note } \ln W(0, 2k) \sim \ln \frac{1}{\sqrt{\pi k}}\right)$$

by Stirling's formula as $k \to \infty$, we have

$$P_{2k,2n} \sim \frac{1}{\pi\sqrt{k(n-k)}}$$

as $k, n - k \to \infty$.

Probability of the fraction of time spending on positive side

- Therefore

$$\sum_{\{k,1/2<2k/2n\leq x\}} P_{2k,2n} - \sum_{k,1/2<2k/2n\leq x} \frac{1}{\pi n}\cdot\left[\frac{k}{n}\left(1-\frac{k}{n}\right)\right]^{-\frac{1}{2}} \to 0, \quad n\to\infty,$$

Whence

$$\sum_{\{k,1/2<2k/2n\leq x\}} P_{2k,2n} \to \frac{1}{\pi}\int_{\frac{1}{2}}^{x}\frac{dt}{\sqrt{t(1-t)}}, \quad n\to\infty.$$

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- From the symmetry,

$$\sum_{\{k,2k/2n\le 1/2\}} P_{2k,2n} \to \frac{1}{2}$$

and

$$\frac{1}{\pi} \int_{\frac{1}{2}}^{x} \frac{dt}{\sqrt{t(1-t)}} = \frac{2}{\pi} \arcsin \sqrt{x} - \frac{1}{2}.$$

Arcsine law

Thus we obtain the following well-known theorem:

- Arcsine Law: The probability that the fraction of the time spent by the particle on the positive side is at most x tends to $\frac{2}{\pi}\arcsin\sqrt{x}$: (Rescaling limit yields the Arcsine law for the occupation time of Brownian motion)

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$$\sum_{\{k, k/n \leq x\}} P_{2k,2n} \to \frac{2}{\pi}\arcsin\sqrt{x}.$$

- Counter-intuitive result: One player will win in most time even in a fair game!

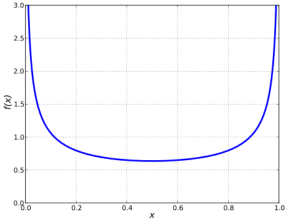

Table of Contents

Einstein's work on the theory of Brownian motion

In 1905, A. Einstein published a seminal paper on the theory of Brownian motion (he also publishes two other seminal papers on Special Relativity and photoemission in this year). Two major points in Einstein's solution to Brownian motion are

1. The motion is caused by the exceedingly frequent impacts on the pollen grain of the incessantly moving molecules of liquid in which it is suspended;

Einstein's work on the theory of Brownian motion

In 1905, A. Einstein published a seminal paper on the theory of Brownian motion (he also publishes two other seminal papers on Special Relativity and photoemission in this year). Two major points in Einstein's solution to Brownian motion are

1. The motion is caused by the exceedingly frequent impacts on the pollen grain of the incessantly moving molecules of liquid in which it is suspended;
2. The motion of these molecules is so complicated that its effect on the pollen grain can only be described probabilistically in terms of exceedingly frequent statistically independent impacts.

Einstein's work on the theory of Brownian motion

His mathematical interpretation is as follows (1D version).

- In a small time interval τ, the X-coordinates of an individual particle will increase by an amount Δ. There will be a certain "frequency law" for Δ

$$dn = n\phi(\Delta)d\Delta$$

where

$$\int_{-\infty}^{+\infty} \phi(\Delta)d\Delta = 1, \quad \phi(-\Delta) = \phi(\Delta),$$

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  and ϕ is only different from 0 for very small values of Δ.
- Let $f(x,t)$ be the number of particles per unit volume, then

  $$f(x,t+\tau)dx = \int_{-\infty}^{+\infty} f(x-\Delta,t)dx\phi(\Delta)d\Delta.$$

  Since τ is small

  $$f(x,t+\tau) = f(x,t) + \frac{\partial f}{\partial t}\tau,$$

Einstein's work on the theory of Brownian motion

- furthermore

$$f(x-\Delta,t) = f(x,t) - \Delta\frac{\partial f}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 f}{\partial x^2} + \cdots .$$

Einstein's work on the theory of Brownian motion

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$$f(x,t)+\frac{\partial f}{\partial t}\tau = f\int_{-\infty}^{+\infty}\phi(\Delta)d\Delta+\frac{\partial f}{\partial x}\int_{-\infty}^{+\infty}\Delta\phi(\Delta)d\Delta+\frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{+\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta+\cdots .$$

Einstein's work on the theory of Brownian motion

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- Set

$$\frac{1}{\tau}\int_{-\infty}^{+\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta = D$$

throwing h.o.t., we have

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}.$$

Einstein's work on the theory of Brownian motion

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But the overall philosophy and treatment is highly original!