A generalization of Morse lemma and its applications

Jiang Ming

Department of Information Science, School of Mathematics, Peking University, Beijing 100871, People’s Republic of China

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1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary and let

\[
E(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(x,u),
\]

(1)

where \( F(x,u) = \int_0^u f(x,\tau)\, d\tau \) and \( f \) is a Caratheodory function satisfying

\[
|f(x,\tau)| \leq C(1 + |\tau|^q) \quad \text{with} \quad 1 \leq q \leq \frac{n+2}{n-2}.
\]

(2)

Assume that \( u_0 \in H^1_0(\Omega) \) is a critical point of \( E \) in the \( C^1 \) topology. Then \( u_0 \) is also a critical point of \( E \) in the \( H^1_0 \) topology. It is a natural question if the types of critical point \( u_0 \) in different topologies coincide. Some recent results in this direction are as follows:

In [3], Brezis and Nirenberg proved that if \( u_0 \) is a local minimizer of \( E \) in the \( C^1 \) topology, then \( u_0 \) is also a local minimizer of \( E \) in the \( H^1_0 \) topology.

For isolated critical points, one classifies them by critical groups \( C_q(E,u_0) \), \( q = 0, 1, \ldots \). In [5], Chang presented a detailed study on this problem via critical groups with the additional assumption that \( f \in C^1(\bar{\Omega} \times \mathbb{R}) \) and

\[
|f_u(x,\tau)| \leq C(1 + |\tau|^{q-1}).
\]

(3)

* E-mail: jiangm@sxx0.math.pku.edu.cn.
Chang proved that if $u_0$ is an isolated critical point of $E$, then
\[ C_*(E|_{C^1(M,N)}; u_0) = C_*(E|_{H^1(M,N) \cap C^0(M,N)}; u_0). \]

Combining with a result in [4]: an isolated critical point $u_0$ is a local minimizer of $E$ if and only if \( \text{rank} \ C_q(E, u_0) = \delta_{q0} \), it follows that an isolated local minimizer of $E$ in the $C^1$ topology is also a local minimizer in the $H^1$ topology. The proof of [5] depends on the Morse splitting lemma (Theorem I.5.1 in [4]) and some regularity arguments. We remark that the Morse splitting lemma depends strongly on the underlying Hilbert space structure.

Introduced by Chang and inspired by the above results, the author considered the following similar question. Let $M$ and $N$ be two compact Riemannian manifolds. Let $E$ be the energy functional for mappings from $M$ to $N$. Assume that $u$ is a smooth harmonic mapping. Then it is a critical point of $E$ in $C^1(M,N)$ and $H^1(M,N) \cap C^0(M,N)$. If it is an isolated one in both spaces, then do we have
\[ C_*(E|_{C^1(M,N)}; u) = C_*(E|_{H^1(M,N) \cap C^0(M,N)}; u)? \] (4)

The feature of this problem is that we cannot use the Morse splitting lemma now, since the underlying space $H^1(M,N) \cap C^0(M,N)$ is not a Hilbert–Riemannian manifold and does not admit a Hilbertian structure. Although the Morse splitting lemma fails in this case, both the spaces are contained in a larger Hilbert–Riemannian manifold $H^1(M,N)$ and the argument in the proof of the Morse splitting lemma ([4], pp. 44–46.) can be revised to yield a similar Morse splitting of the energy functional at $u$. This splitting enables us to get an affirmative answer to the above problem (4) concerning harmonic mappings.

Later on, the author realized that it was possible to put the above result in an abstract form. The first step is to get a generalization of Morse splitting lemma to the non-Hilbertian case. The relation between $C^1(M,N), H^1(M,N) \cap C^0(M,N)$ and $H^1(M,N)$ is generalized to assumptions about the underlying spaces. Note that in dealing with the local behavior of the functional at an isolated critical point, only the tangent space at that point is involved. Hence, the final assumptions SP1 SP2 SP3 (cf. Section 2) are about the tangent spaces, i.e., about some Banach or Hilbert spaces. A crucial condition to yield the conclusion is about the behavior of the Hessian $E''(u)$. In the case of semilinear elliptic equation or harmonic mapping, the speciality of the problems hinders us from getting insight into the generality of the problem. The present condition in this paper is a generalization of the condition about $E''(u)$ in [4].

In the Hilbertian case, the critical group at non-degenerate critical points is determined by the negative spaces of $E''$. In dealing with a critical group calculated in different spaces, the negative spaces of Hessian in different spaces should be assumed to be the same to ensure the similarity of critical groups. In addition, since we are studying a problem about one functional in different spaces, we need a certain regularity condition CP2 (cf. Section 2) to relate these different things together.

The present result Theorem 2.5 still needs an auxiliary underlying Hilbert space structure. However, this structure is just a subsidiary. Once the generalization of Morse splitting lemma, i.e., Theorem 2.5, is established, we proceed to get a result on critical
groups in different topologies similar to the above results about semilinear elliptic equation and harmonic mapping. The final result is Corollary 2.8. A similar result, Corollary 2.7, as that of Brezis and Nirenberg in [3] about a local minimizer follows as a corollary of the generalized splitting lemma, too.

The contents of this paper are as follows: Section 2 contains various assumptions used in this paper and some of their relations and deductions. The results obtained in this paper are presented in this section as Theorem 2.5, Corollaries 2.7 and 2.8. Section 3 contains the proof of the main result, Theorem 2.5 and Section 4 the proofs of both Corollaries 2.7 and 2.8. In Section 5, some interesting applications are examined, which include the semilinear elliptic equation, harmonic mappings, and the equation of constant mean curvature.

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2. Preliminaries, assumptions and main results

In this section, we list all the assumptions that we need and present the main results that we get. First come the assumptions about spaces, which are generalization of the relationship between the tangent spaces of $C^1(M,N)$, $H^1(M,N) \cap C^0(M,N)$ and $H^1(M,N)$.

SP1 $X$ and $Y$ are Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively.

SP2 $H$ is a Hilbert space with inner product $(\cdot , \cdot )$. The induced norms on $H$ is $\| \cdot \|$.

SP3 $X$ is dense in $(Y, \| \cdot \|_Y)$.

The assumptions about the functional are as follows. We need the Hilbertian structure of $H$ to express $E[\cdot ]_Y$ and $E''[\cdot ]_Y$ as in many applications.

FN1 $E: H \mapsto R$, $E[\cdot ]_Y$ is twice continuously differentiable.

FN2 $\exists$ a continuously differentiable map $A: X \mapsto X$ such that

$$E[\cdot ]_X(\phi ) = (A(\cdot , \phi ), \forall \phi \in X$$

FN3 $\exists$ a continuous map $B: Y \mapsto L(H,H)$ such that

$$E''[\cdot ]_Y(\phi , \psi ) = (B(\cdot , \phi )\psi , \forall \phi , \psi \in Y$$

In some applications, the derivatives in the smooth space $X$ is more convenient to calculate than those in the non-smooth space $Y$. In the case of a harmonic mapping, e.g., assumption FN3 is verified through a complicated and tedious calculation. The following stronger assumption FN3* is easy to verify in an application and implies FN3 together with FN1 and FN2.

FN3* (a) $\forall \phi \in X, \exists c(x) \geq 0$ such that

$$|E''[\cdot ]_X(\phi , \psi )| \leq c(\cdot )\| \phi \|\| \psi \| \forall \phi , \psi \in X$$

(b) $\forall \epsilon > 0, \exists \delta > 0$, such that if $x, \bar{x} \in X$ with $\| x - \bar{x} \|_Y < \delta$, then

$$|E''[\cdot ]_X(\phi , \psi ) - E''[\cdot ]_X(\bar{\phi} , \bar{\psi})| \leq \epsilon \| \phi \|\| \psi \| \forall \phi , \psi \in X$$
**Proposition 2.1.** If SP1, SP2, SP3, FN1, FN2 hold, then FN3$^*$ implies FN3.

**Proof.** Form (a), there is a map $B: X \rightarrow L(H,H)$ such that

$$E_{\|\cdot\|}(\phi, \psi) = (B(\phi), \psi).$$

From (b), we know that $B$ is uniformly continuous as a map form $X \subset Y$ to $L(H,H)$. Since $X$ is dense in $Y$, $B$ can be extended to a continuous map from $Y$ to $L(H,H)$. We still denote it as $B$. For all $y \in Y$, there exists $x_n \rightarrow y$. Since $\forall \phi, \psi \in X$,

$$E'(x_n)(\phi, \psi) = E_{\|\cdot\|}(x_n)(\phi, \psi) = (B(x_n)\phi, \psi),$$

and $B$ is continuous on $Y$ and $E_{\|\cdot\|} \in C^2$ by FN1, we have

$$E'(y)(\phi, \psi) = (B(y)\phi, \psi), \forall \phi, \psi \in X.$$

By the density of $X$ in $Y$ again and the continuity of the inclusion of $X \subset Y \subset H$, the above equality holds $\forall \phi, \psi \in Y$, i.e., FN3 holds. □

**Remark 2.2.** Under the above assumptions, it is easy to see that

$$A'(x) = B(x)|_X \quad \text{and} \quad B(x)(X) \subset X, \forall x \in X. \quad (5)$$

Since $\|\cdot\|_X$ is stronger than the norm $\|\cdot\|_Y$ restricted on $X$, the above assumptions FN2 and FN3 also hold when $Y$ is replaced by $X$. It is easy to see that $E_{\|\cdot\|} \in C^2$.

The assumptions at the critical point are as follows. Without loss of generality, we assume that the critical point under consideration is the origin $0$.

CP1 $0$ is a critical point of $E_{\|\cdot\|}$. $0$ is either not in the spectrum $\sigma(B(0))$ or is an isolated point of $\sigma(B(0))$.

CP2 (regularity) if $\phi \in H$ such that $B(0)(\phi) = \psi$ for some $\psi \in X$, then $\phi \in X$.

**Remark 2.3.** If $0$ is a critical point of $E_{\|\cdot\|}$, then $\forall \phi \in X$, $E'_{\|\cdot\|}(0, \phi) = (A(0), \phi) = 0$. Since $X$ is dense in $Y$ by SP3, we have $E'_{\|\cdot\|}(0, \phi) = (A(0), \phi) = 0$, $\forall \phi \in Y$. Therefore, $0$ is also a critical point of $E_{\|\cdot\|}$.

Let $N = \ker(B(0))$. By CP2, $N \subset X$. Since $N$ is a closed linear subspace of $H$, it is also closed in $X$ and $Y$, respectively. Let $N^\perp$ be the orthogonal complement of $N$ in $H$. Let $P$ be the orthogonal projection $P: H \rightarrow N$. Let

$$X^\perp = X \cap N^\perp = (I - P)(X) \quad \text{and} \quad Y^\perp = Y \cap N^\perp = (I - P)(Y),$$

where $I$ is the identity operator of $H$. Note that $X^\perp$ and $Y^\perp$ are closed linear subspaces in $X$ and $Y$ with the corresponding topologies, respectively. Then we have the following topological direct sum decomposition:

$$H = N \oplus N^\perp, \quad Y = N \oplus Y^\perp \quad \text{and} \quad X = N \oplus X^\perp.$$

Since $(N, \|\cdot\|_N)$, $(N, \|\cdot\|_N)$ and $(N, \|\cdot\|_Y|_N)$ are complete, we have
Proposition 2.4. The norms $\|\cdot\|_X, \|\cdot\|_Y$ and $\|\cdot\|$ when restricted to $N$ are equivalent.

We assume throughout that there is a constant $K > 0$ such that

$$\|z\|_X \leq K\|z\|, \quad \forall z \in N.$$  \hspace{1cm} (6)

By SP2, we may assume, with loss of generality, that

$$\|x\|_Y \leq \|x\|_X, \quad \forall x \in X \quad \text{and} \quad \|y\| \leq \|y\|_Y, \quad \forall y \in Y.$$  

We shall use $B^r_Y$ and $B^r_X$ to denote balls of radius $r$ centered at $\theta$ in $X$ and $Y$ according to the corresponding topologies respectively.

The main result is a generalization of Morse splitting lemma in [4] (p. 44) to Banach spaces.

Theorem 2.5. Under the above assumptions SP1, SP2, SP3, FN1, FN2, FN3, CP1 and CP2, there exists a ball $B^r_Y$ of $(Y, \|\cdot\|_Y)$, an origin-preserving local homeomorphism $\varphi$ defined on $B^r_Y$ and a $C^1$ map $h : B^r_Y \cap N \to X^\perp$ such that $\forall y \in B^r_Y$

$$E \circ \varphi(y) = \frac{1}{2}(B(\theta)y^\perp, y^\perp) + E(h(z) + z),$$

where $z = P(y)$ and $y^\perp = (I - P)(y)$. Moreover, $\varphi(B^r_Y \cap X) \subset X$ and $\varphi : B^r_Y \to X$ is also an origin-preserving local homeomorphism in the topology of $X$.

Remark 2.6. When the three spaces $X = Y = H$ are the same Hilbert space, the above result is the Morse splitting lemma, Theorem 5.1 of [4], p. 44.

The following corollary is a generalization of Brezis and Nirenberg’s result in [3].

Corollary 2.7. Under the assumptions of Theorem 1, if $\theta$ is a local minimizer of $E|_X$, then $\theta$ is also a local minimizer of $E|_Y$.

To state the generalization of Chang’s result of [5], we need some other notations. Since $B(\theta)$ is a self-adjoint operator on $H$, let $E(t)$ be its spectral resolution. Then

$$B(\theta) = \int_{(-\infty, \infty)} t \, dE(t).$$

Let

$$P_- = \int_{(-\infty, 0]} dE(t) \quad \text{and} \quad P_+ = \int_{[0, \infty)} dE(t)$$

and

$$H_- = P_-(H) \quad \text{and} \quad H_+ = P_+(H).$$
Then
\[ H = H_- \oplus H_+. \]

Note that \( N \subset H_- \).

**Corollary 2.8.** Under the same assumptions as in the Theorem, if \( \theta \) is an isolated critical point of \( E|_Y \), then \( \theta \) is also an isolated critical point of \( E|_X \). Moreover, if \( H_- \subset X \), then the qth critical groups with coefficient group \( G \) of \( E|_X \) and \( E|_Y \) at \( \theta \) are equal, i.e.,
\[ C_q(E|_X, \theta, G) = C_q(E|_Y, \theta, G). \]

### 3. Proof of the splitting theorem

**Step 1:** By FN2, define \( T : N \times X^\perp \to X^\perp \) as
\[ T(z, x) = (I - P)A(z + x), \quad \forall (z, x) \in N \times X^\perp. \]

\( T \) is a \( C^1 \) map with \( T(\theta, \theta) = 0 \). For \( \phi \in X^\perp \),
\[ T'_s(\theta, \theta)\phi = (I - P)B(\theta)\phi. \]

Since \( \forall \phi \in X \) and \( \psi \in N \),
\[ (B(\theta)\phi, \psi) = (\phi, B(\theta)\psi) = (\phi, \theta) = 0, \]
so \( B(\theta)\phi \in N^\perp \). By Eq. (5), if \( \phi \in X^\perp \), \( B(\theta)\phi \in X^\perp \). Therefore,
\[ T'_s(\theta, \theta)\phi = B(\theta)\phi, \quad \forall \phi \in X^\perp. \]

We shall show that \( T'_s(\theta, \theta) : X^\perp \to X^\perp \) is an isomorphism.

*It is one-to-one:* If \( T'_s(\theta, \theta)(\phi) = 0 \) for some \( \phi \in X^\perp \), then \( \phi \in X^\perp \cap N = \{0\} \). So \( T'_s(\theta, \theta) \) is one-to-one.

*It is onto:* For \( \psi \in X^\perp \),
\[ T'_s(\theta, \theta)(\phi) = \psi \text{ has a solution } \Leftrightarrow B(\theta)\phi = \psi \text{ has a solution } \phi \in X^\perp. \]

If 0 is not in \( \sigma(B(\theta)) \), the required solvability is obvious from CP2. If 0 is an isolated point of \( \sigma(B(\theta)) \), consider the self-adjoint operator \( B(\theta) : H \to H \). The range \( R(B(\theta)) \) is closed ([6], p. 359). Therefore, the range \( R(B(\theta)) = \ker(B(\theta)) = N^\perp \). For \( \psi \in X^\perp \subset N^\perp \), \( B(\theta)\phi = \psi \) has a solution \( \phi \) in \( H \). By CP2, this solution is in \( X \). Then \( (I - P)\phi \) is the required solution in \( X^\perp \).

By the implicit function theorem, \( \exists \delta_1, \delta_2 > 0 \), and a \( C^1 \) map \( h : B^X_{\delta_1} \cap N \to B^X_{\delta_2} \cap X^\perp \) with \( h(\theta) = \theta \) such that
\[ T(z, h(z)) = (I - P)A(z + h(z)) = \theta. \]

Define \( F : (B^X_{\delta_1} \cap N) \times Y^\perp \to \mathbb{R} \) as
\[ F(z, u) = E(z + h(z) + u) - E(z + h(z)), \quad \forall z \in B^X_{\delta_1} \cap N \text{ and } u \in Y^\perp. \]
By FN2, \( \forall \phi \in X^\perp \),
\[
F_u''(z, \theta)_{|_{\phi}} = E [F_{\phi}^\prime(\phi)] = (A(z + h(z)), \phi) \\
= (A(z + h(z)), (I - P)\phi) = ((I - P)A(z + h(z)), \phi) = 0.
\]

Since \( X \) is dense in \( Y \), we have
\[
F_u''(z, \theta)_{|_{\phi}} = 0, \quad \forall \phi \in Y^\perp.
\]

Therefore,
\[
F_u'(z, \theta)_{|_{\phi}} = 0, \quad \forall z \in B_{\delta_0}^X \cap N.
\]

\( \forall \phi, \psi \in Y^\perp \), we have
\[
F_{w\phi}''(\theta, \phi)_{|_{\psi}} = \frac{\partial}{\partial s} E(s\phi + t\psi) = E [F_{\psi}''(\theta)_{|_{\phi,\psi}}] = (B(\theta)\phi, \psi).
\]

As in [4], we define
\[
F_2(\phi) = \frac{1}{2} (B(\theta)\phi, \phi), \quad \forall \phi \in Y^\perp.
\]

**Step 2:** Since \( B(\theta) : N^\perp \rightarrow N^\perp \) is an isomorphism, there is a constant \( \lambda > 0 \) such that
\[
\|B(\theta)\phi\| \geq \lambda \|\phi\|, \quad \forall \phi \in N^\perp. \tag{7}
\]

**Step 3:** Consider the flow \( \eta(s) = \eta(s, u) \) defined by the following ODE:
\[
\dot{\eta}(s) = -\frac{B(\theta)\eta(s)}{\|B(\theta)\eta(s)\|^2}, \quad \eta(0) = u, \tag{8}
\]
on \( Y^\perp \setminus \{\theta\} \). As in [4], we claim that \( \eta \) is well defined for \( |s| < \|u\| \). Since \( \|\eta(s) - u\| \leq |s| \), we have \( \|\eta(s)\| \geq \|u\| - |s| \). From this, together with \( \eta(s) \in Y^\perp \subset N^\perp \), it follows that the denominator of the vector field is not zero for \( |s| < \|u\| \). We now establish some inequalities.

(a) \( \forall 0 < e < \lambda, \exists \delta_3 > 0 \) such that
\[
|F(z, u) - F_2(u)| < e\|u\|^2, \quad \forall (z, u) \in (B_{\delta_3}^Y \cap N) \times (B_{\delta_3}^Y \cap Y^\perp).
\]

Since
\[
|F(z, u) - F_2(u)| = |F(z, u) - F(z, \theta) - F'_u(z, \theta)(u) - F_2(u)| \\
= \left| \int_0^1 (1 - t)[F_{w\phi}(z, tu)(u, u) - 2F_2(u)] \, dt \right| \\
= \left| \int_0^1 (1 - t)[E(z + h(z) + tu)(u, u) - 2F_2(u)] \, dt \right|
\]
\[
\int_0^1 (1-t)(B(z + h(z) + tu) - B(\theta) ;u) dt \leq \int_0^1 (1-t)\|B(z + h(z) + tu) - B(\theta)\| \cdot \|u\|^2 dt,
\]

the required inequality follows if we choose \( \delta_3 \) small enough such that \( B_{\delta_3}^+ \cap N \subset B_{\delta_3}^+ \cap N \) and
\[
\|B(z + h(z) + u) - B(\theta)\| < \varepsilon, \quad \forall (z,u) \in (B_{\delta_3}^+ \cap N) \times (B_{\delta_3}^+ \cap Y^\perp).
\] (9)

(b) \( \forall t \in (-\|u\|, \|u\|) \) and \( u \in Y^\perp \setminus \{\theta\} \), we have
\[
|F_2(\eta(t,u)) - F_2(u)| = \left| \int_0^t \frac{d}{ds} F_2(\eta(s,u)) ds \right| = \left| \int_0^t ((B(\theta)\eta(s,u)), \eta) ds \right|
\]
\[
= \left| \int_0^{|t|} \|B(\theta)\eta(s,u)\| ds \right| \geq \lambda \left| \int_0^{|t|} \|\eta(s,u)\| ds \right|
\]
\[
\geq \lambda \left( \|u\|-|t| - \frac{t^2}{2} \right),
\]

where \( \lambda \) is the constant obtained in Eq. (7). For \( t \in (-\|u\|, \|u\|) \), \( \|\eta(t,u)\| \leq |t| + \|u\| \leq 2\|u\| \). Hence, for \( u \in B_{\delta_3}^+ \setminus \{\theta\} \), \( \eta(t,u) \in B_{\delta_3}^+ \setminus \{\theta\} \). We conclude from (a) and (b) that \( \forall z \in B_{\delta_3}^+ \cap N \) and \( \forall u \in B_{\delta_3}^+ \cap Y^\perp \setminus \{\theta\} \),

- \( F_2(\eta(t,u)) \) as a function of \( t \) is strictly decreasing on \( (-\|u\|, \|u\|) \);
- \( F_2(\eta(-t,u)) > F(z,u) > F_2(\eta(t,u)) \) holds for
\[
\left( 1 - \sqrt{1 - \frac{2\varepsilon}{\lambda}} \right) \|u\| \leq t \leq \|u\|.
\]

Therefore, there exists a unique \( \tilde{\eta}(z,u) \) with
\[
|\tilde{\eta}(z,u)| \leq \left( 1 - \sqrt{1 - \frac{2\varepsilon}{\lambda}} \right) \|u\|,
\] (10)
such that
\[
F_2(\eta(\tilde{\eta}(z,u),u)) = F(z,u).
\] (11)

Define a map \( \xi : (B_{\delta_3}^+ \cap N) \times (B_{\delta_3}^+ \cap Y^\perp) \rightarrow Y^\perp \) as follows:
\[
\xi(z,u) = \begin{cases} 
\theta & \text{if } u = \theta, \\
\eta(\tilde{\eta}(z,u),u) & \text{if } u \neq \theta.
\end{cases}
\]

Define a map \( \Phi : (B_{\delta_3}^+ \cap N) \times (B_{\delta_3}^+ \cap Y^\perp) \rightarrow N \times Y^\perp \) as follows:
\[
\Phi(z,u) = (z, \xi(z,u)).
\]
We shall verify that \( \Phi \) is an origin-preserving local homeomorphism. That \( \tilde{\eta}(z,u) \) is continuous on \((B^T_1 \cap N) \times ((B^T_1 \cap Y^+) \setminus \{\theta\})\) follows from the implicit function theorem,
\[
\frac{\partial}{\partial t} F_2(\eta(t,u)) = -\|B(\theta)\eta(t,u)\| \neq 0
\]
and for \( u = \theta \), the continuity follows from Eq. (10). Therefore, \( \Phi \) is continuous.

**Step 4:** We have used the path \( \eta(t,u) \) to carry a point \((z,u)\) to the point \((z,u)\); the same path can be used for the opposite purpose, i.e., to define the inverse map \( \Psi = \Phi^{-1} \).

A similar argument verifies the continuity of \( \Psi \). Therefore, \( \Phi \) is a homeomorphism.

The details are as follows.

Given \((z;u) \in (B^T_1 \cap N) \times (B^T_1 \cap Y^+) \setminus \{\theta\})\), consider the flow \( \eta(s,u) \) defined in Eq. (8), which exists for \( s \in (-\|\tilde{u}\|, \|\tilde{u}\|) \).

From the inequalities in (a) and (b) we have
\[
1 - \sqrt{1 - \frac{2e}{\lambda}} \|u\| \leq s \leq \|u\|
\]

A different step from the proof in Step 3 is the following estimation, using FN3 and Eq. (9),
\[
-\frac{\partial}{\partial s} F(z, \zeta(s)) = \frac{1}{\|B(\theta)\zeta(s)\|} F'_u(z, \zeta(s)) \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\|
\]
\[
= \frac{1}{\|B(\theta)\zeta(s)\|} (F'_u(z, \zeta(s)) - F'_u(z, \theta)) \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\|
\]
\[
\geq \frac{1}{\|B(\theta)\zeta(s)\|} \int_0^1 F''_{uu}(z, \zeta(s)) \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\|
\]
\[
\geq \frac{1}{\|B(\theta)\zeta(s)\|} \int_0^1 |(F''_{uu}(z, \zeta(s)) - F''_{uu}(\theta, \theta))\|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| \|B(\theta)\zeta(s)\| > 0.
\]

Therefore, we conclude that
\[
F(z, \zeta(s)) \text{ as a function of } s \text{ is strictly decreasing on } (-\|\tilde{u}\|, \|\tilde{u}\|).
\]

Therefore, there exists a unique \( \tilde{s}(z, \tilde{u}) \) with
\[
|\tilde{s}(z, \tilde{u})| \leq \left(1 - \sqrt{1 - \frac{2e}{\lambda}} \right) \|\tilde{u}\| \tag{12}
\]
such that
\[
F(z, \zeta(\tilde{s}(z, \tilde{u}))) = F_2(\tilde{u}) \tag{13}
\]
Then we define a map \( \chi: (B_{\delta_3}^T \cap N) \times (B_{\delta_3}^T \cap Y^\perp) \mapsto Y^\perp \) as follows:

\[
\chi(\bar{z}, \bar{u}) = \begin{cases} 
\theta & \text{if } \bar{u} = 0, \\
\eta(\bar{s}(\bar{z}, \bar{u}), \bar{u}) & \text{if } \bar{u} \neq 0.
\end{cases}
\]

Define a map \( \Psi: (B_{\delta_3}^T \cap N) \times (B_{\delta_3}^T \cap Y^\perp) \mapsto N \times Y^\perp \) as follows:

\[
\Psi(\bar{z}, \bar{u}) = (\bar{z}, \chi(\bar{z}, \bar{u})),
\]

which is origin-preserving. The continuity of \( \Psi \) follows in a similar way as that of \( \Phi \) in Step 3.

To prove that \( \Phi \) is an origin-preserving local homeomorphism, we need only to show that \( \Psi \) is the inverse map of \( \Phi \) in a neighborhood of 0. Let \((z, u) \in (B_{\delta_3}^T \cap N) \times (B_{\delta_3}^T \cap Y^\perp)\), let \((\bar{z}, \bar{u}) = \Phi(z, u) \in (B_{3\delta}^T \cap N) \times (B_{3\delta}^T \cap Y^\perp)\). Let \((z^*, u^*) = \Psi(\bar{z}, \bar{u})\). Then \(z^* = z\), \(u \) and \(u^* \) are on the same flow defined as in Eq. (8). Since \( F(z, u) = F_2(\bar{u}) = F(z, u^*) \), then it follows that \( u = u^* \) from the monotonicity of \( F(z, \eta(c)) \). Hence, \( \Phi \circ \Phi(z, u) = (z, u) \). Similarly, we have \( \Phi \circ \Psi(z, u) = (z, u) \).

Note that when \( u \in X^\perp \setminus \{0\} \), the flow \( \eta(\cdot, u) \in X^\perp \). Therefore,

\[
\Phi, \Psi: (B_{\delta_3}^T \cap N) \times (B_{\delta_3}^T \cap X^\perp) \mapsto N \times X^\perp
\]

are origin-preserving local homeomorphism in the topology of \( X \).

Step 5: By the inverse function theorem, \( \Theta: (z, u) \mapsto (z, u - h(z)) \) is an origin-preserving local diffeomorphism. Then

\[
\tilde{\Phi}(z, u) = \Phi \circ \Theta(z, u) = (z, \tilde{\eta}(z, u - h(z)) )
\]

is an origin-preserving local homeomorphism. Let \( \varphi = \tilde{\Phi}^{-1} \), which is the required origin-preserving local homeomorphism. We may assume that \( \varphi \) is a local homeomorphism defined on \( B_{\delta_4}^T \cap N \times B_{\delta_4}^T \cap Y^\perp \) as well as on \( B_{\delta_4}^T \) for some \( \delta_4 > 0 \), since \( Y = N \oplus Y^\perp \) is a topological direct sum. Choose \( \delta \) such that

\[
0 < \delta < \delta_4/(K+1),
\]

where \( K \) is the constant in Eq. (6). If \( y \in B_{\delta_4}^T \), let \( z = P(y) \in N \) and \( y^\perp = y - z = (I - P)(z) \), then \( \|z\| \leq \|y\| < \delta \). Note \( \|z\|_x \leq K \|z\| \leq K \|y\| < K\delta < \delta_4 \) and

\[
\|y^\perp\|_Y \leq \|y\|_Y + \|z\|_Y \leq \|y\|_Y + \|z\|_X \leq (K + 1)\|y\|_Y < \delta_4.
\]

Let \((z, u) = \varphi(y) = \varphi(z, y^\perp)\), i.e., \((z, y^\perp) = \tilde{\Phi}(z, u) = (z, \tilde{\eta}(z, u - h(z)))\). Then

\[
E \circ \varphi(y) = E(z + u) = F(z, u - h(z)) + E(z + h(z)) = F_2(\tilde{\eta}(z, u - h(z)) + E(z + h(z)) = \frac{1}{2}(B(\theta)y^\perp, y^\perp) + E(h(z) + z).
\]

Note that by our construction, \( \varphi \) maps \( B_{\delta_4}^T \cap X \) to \( X \) and \( \varphi: B_{\delta_4}^T \mapsto X \) is also an origin-preserving local homeomorphism in the topology of \( X \). The proof is completed. \( \square \)
4. Proof of the Corollaries

Proof of Corollary 2.7. If \( \theta \) is a local minimizer of \( E|_X \), we have
\[
F_2(\phi) \geq 0, \quad \forall \phi \in X.
\]
By the density of \( X \) in \( H \) and the fact that \( \| \cdot \|_Y \) is stronger than \( \| \cdot \| \), we have
\[
F_2(\phi) \geq 0, \quad \forall \phi \in Y.
\]
For \( y \) sufficiently close to \( \theta \) in \( \| \cdot \|_Y \), let \( z = P(y) \in N \subset X \). Then \( h(z) \in X \). Let \( y^\perp = (I - P)(y) \in y^\perp \). We have
\[
E(\phi) = F_2(y^\perp) + E(z + h(z)) \geq E(z + h(z)) \geq E(\theta).
\]
Since \( \phi \) is an origin-preserving local homeomorphism, the conclusion follows at once. \( \square \)

Proof of Corollary 2. The first part is obvious. Under the assumptions, we claim that

Claim 1. \( \| y \|_D = \| P_\perp(y) \|_Y + \| P_\parallel(y) \|_Y \) is a norm on \( Y \) equivalent to \( \| \cdot \|_Y \).

The claim will be proved at the end of this proof. We shall use this norm on \( Y \) in the subsequent proof instead of the original norm \( \| \cdot \|_Y \) of \( Y \). Let \( \phi \) be the homeomorphism defined on \( B_Y^\delta \) as in Theorem 2.5. Note that balls of \( Y \) are measured in the new norm \( \| \cdot \|_D \) now. Let \( U^X = \phi(B_Y^\delta \cap X) \) and \( U^Y = \phi(B_Y^\delta) \). Choose \( \delta \) small enough such that \( U^Y \) is a neighborhood of \( \theta \) in \( Y \) and \( \theta \) is the only critical point of \( E|_Y \) on \( U^Y \) and such that \( U^X \) is also a neighborhood of \( \theta \) in \( X \) and \( \theta \) is the only critical point of \( E|_X \) on \( U^X \). Let \( c = E(\theta) \) and
\[
Y_c = \{ y \in U^Y: E(y) \leq c \} \quad \text{and} \quad X_c = Y_c \cap X.
\]
Then we have
\[
C_q(E|_Y, \theta, G) = H_q(Y_c, Y_c \setminus \{ \theta \}, G)
\]
and
\[
C_q(E|_X, \theta, G) = H_q(X_c, X_c \setminus \{ \theta \}, G).
\]
Now since \( \phi \) is an origin-preserving local homeomorphism defined on \( B_Y^\delta \) and \( B_Y^\delta \cap X \), we have
\[
C_q(E|_Y, \theta, G) = H_q(\phi^{-1}(Y_c), \phi^{-1}(Y_c) \setminus \{ \theta \}, G)
\]
and
\[
C_q(E|_X, \theta, G) = H_q(\phi^{-1}(Y_c) \cap X, \phi^{-1}(Y_c) \cap X \setminus \{ \theta \}, G).
\]
By Theorem 2.5,
\[
\phi^{-1}(Y_c) = \{ y \in B_Y^\delta: F_2(y^\perp) + E(z + h(z)) \leq c \}.
\]
By the definition of the new norm \(|\cdot|_{D}\) of \(Y\), define \(q : [0, 1] \times B^{Y}_{\theta} \rightarrow Y\) as
\[
q(t, y) = P_{-}(y) + (1 - t)P_{+}(y).
\]
We claim that

**Claim 2.** \(q\) is continuous. When \(q\) is restricted to \([0, 1] \times (B^{Y}_{\theta} \cap X)\) as a map to \(X\), it is also continuous in the topology of \(X\).

The proof of the claim is given at the end of the proof. Now note that

(i) \(q(0, \cdot) = \text{id}\);

(ii) Let \(V^{Y} = \{ y \in B^{Y}_{\theta} \cap H_{-} : F_{2}(y^{+}) + E(z + h(z)) \leq c \}\). Then
\[
q(1, \varphi^{-1}(Y_{c})) \subset V^{Y}.
\]

(iii) \(q(1, \varphi^{-1}(Y_{c}) \setminus \{ \theta \}) \subset V^{Y} \setminus \{ \theta \} \).

It is easy to see that we only need to show that if \(y \in \varphi^{-1}(Y_{c})\) and \(y \neq \theta\), then \(q(1, y) \neq \theta\). We prove this by contradiction. If \(q(1, y) = \theta\), we have \(P_{-}(y) = \theta\). Then \(z = \theta\) and \(y = P_{+}(y) = y^{+}\) by the definition of \(y^{+}\). Now that \(y \in \varphi^{-1}(Y_{c})\) implies that
\[
F_{2}(y^{+}) + E(\theta + h(\theta)) \leq c,
\]
from it we have
\[
(B\theta)P_{+}(y), P_{+}(y)) \leq 0.
\]
Since
\[
(B\theta)P_{+}(y), P_{+}(y)) = (P_{+}B\theta)P_{+}(y), y) = \int_{(0, \infty)} t\varphi^{2}(y, t) d(E(t)y, y) = \int_{(0, \infty)} t d(E(t)y, y) \geq 0,
\]
we have
\[
\int_{(0, \infty)} t d(E(t)y, y) = 0.
\]
Then it follows that the measure \(E(0, \infty))/y, y) = 0\). Thus
\[
(y, y) = (P_{+}(y), y) = \int_{(0, \infty)} d(E(t)y, y) = (E(0, \infty)/y, y) = 0.
\]
Hence \(y = \theta\), which is a contradiction.

(iv) \(q(t, \varphi^{-1}(Y_{c}) \setminus \{ \theta \}) \subset \varphi^{-1}(Y_{c}) \setminus \{ \theta \}\) for \(t \in [0, 1]\).

We need to show that

(a) if \(y \in \varphi^{-1}(Y_{c})\), then \(q(t, y) \in \varphi^{-1}(Y_{c})\)

and that

(b) if \(t \in [0, 1]\) and if \(y \in \varphi^{-1}(Y_{c})\) and \(y \neq \theta\), then \(q(t, y) \neq \theta\).

For the proof of (b), if \(t \neq 1\), it is obvious and if \(t = 1\), it is proved in (iii). For the proof of (a), note that we use the norm \(|\cdot|_{D}\) on \(Y\) now. If \(y \in \varphi^{-1}(Y_{c})\), we have
\(q(t, y) \in B_Y^T\). That is why we use the norm \(\|\cdot\|_D\) instead of \(\|\cdot\|_Y\). By assumption, we have

\[
\frac{1}{2}(B(\theta)(y^\perp), y^\perp) + E(z + h(z)) \leq E(\theta),
\]

where \(z = P(y)\) and \(y^\perp = y - z\). Since \(P = \int_{\{0\}} dE(t) = E(\{0\})\), we have \(P(q(t, y)) = z\).

Hence

\[
(q(t, y))^\perp = q(t, y) - z = y - z - tP_+(y) = y^\perp - tP_+(y).
\]

Therefore, we only need to show that

\[
(B(\theta)(y^\perp - tP_+(y)), y^\perp - tP_+(y)) \leq (B(\theta)(y^\perp), y^\perp),
\]

i.e.

\[
\frac{1}{2}(B(\theta)(P_+(y)), P_+(y)) - 2t(B(\theta)(y^\perp), P_+(y)) \leq 0.
\]

Since

\[
(B(\theta)(y^\perp), P_+(y)) = (P_+B(\theta)(y^\perp), P_+(y)) = (B(\theta)P_+(y^\perp), P_+(y)) = (B(\theta)P_+(y), P_+(y)) \geq 0,
\]

thus (a) is proved.

(v) \(q(t, \cdot)|_{V^Y} = \text{id}\). This is obvious.

Then we have

\[
C_q(E|_Y, \theta, G) = H_q(V^Y, V^Y \setminus \{\theta\}, G).
\]

A similar proof shows that

\[
C_q(E|_X, \theta, G) = H_q(V^Y \cap X, V^Y \cap X \setminus \{\theta\}, G).
\]

Since \(V^Y = V^Y \cap X\), the conclusion follows at once. \(\square\)

**Proof of the Claim 1.** It is easy to see that \(\|\cdot\|_D\) is a norm on \(Y\) and is stronger than \(\|\cdot\|_Y\) since \(\forall y \in Y\)

\[
\|y\|_Y = \|P_-(y) + P_+(y)\|_Y \leq \|P_-(y)\|_Y + \|P_+(y)\|_Y = \|y\|_D.
\]

Since

\[
\|y\|_D = \|P_-(y)\|_Y + \|y - P_-(y)\|_Y \leq 2\|P_-(y)\|_Y + \|y\|_Y,
\]

the claim is true if \(P_-\) as a map from \(Y\) to \(Y\) is continuous in the topology of \(Y\). This is proved via the closed-graph theorem. Assume that \(\|y_n - y\|_Y \to 0\) and \(\|P_-(y_n) - v\|_Y \to 0\). Then \(\|y_n - y\| \to 0\) and \(\|P_-(y_n) - v\| \to 0\). Since \(P_-: H \to H\) is continuous, we have \(\|P_-(y_n) - P_-(y)\| \to 0\). Then \(v = P_-(y)\). Hence \(P_-\) as a map from \(Y\) to \(Y\) is closed. The claim is proved. \(\square\)
Proof of the Claim 2. Assume \( t_n, t \in [0, 1] \) and \( y_n, y \in B^Y_\delta \) such that \( t_n \to t \) and \( \| y_n - y \|_Y \to 0 \). Since \( P_- \) is continuous in the topology of \( Y \), we have \( P_- (y_n) \to P_- (y) \). Therefore

\[
P_+ (y_n) = y_n - P_- (y_n) \to y - P_- (y) = P_+ (y).
\]

Then it is easy to see that \( \varphi (t_n, y_n) \to \varphi (t, y) \).

A similar proof proves that \( \varphi \) restricted to \([0, 1] \times B^Y_\delta \cap X \) as a map to \( X \) is also continuous in the topology of \( X \).

5. Examples and applications

Example 1. Let \( \Omega \) be bounded in \( \mathbb{R}^n \) with a smooth boundary and consider the functional \( E \) defined in Eq. (1). Assume that \( f \) satisfies conditions (2) and (3). Let \( X = C^1_0 (\Omega) \) and \( Y = H = H^1_0 (\Omega) \). The inner product on \( H \) is

\[
(\phi, \psi) = \int_\Omega \nabla \phi \cdot \nabla \psi.
\]

Our assumptions about spaces SP1, SP2, SP3 hold. We assume \( u \) is a critical point of \( E \) in \( H \). By regularity, it is also a critical point of \( E|_X \). We have \( A(u) = \text{id} - (-\Delta)^{-1} f (\cdot, u) \) and \( B(u) = \text{id} - (-\Delta)^{-1} f (\cdot, u) \), where \((-\Delta)^{-1} : H \to H \) is the inverse of \( \Delta \) under zero boundary value condition. The assumptions about functional FN1, FN2 and FN3 are satisfied. By the properties of elliptic operators, the assumptions at critical point CP1 and CP2 are also satisfied. Hence, if \( u \) is a local minima of \( E|_X \) in \( X \), then it is also a local minima of \( E \) in \( Y = H \). This is the result in [3]. If \( \theta \) is an isolated critical point of \( E|_Y \) in \( Y \), since \( H_- \subset C^\infty (\Omega) \cap C^1_0 (\Omega) \subset X \), then we have

\[
C_q (E|_X, u, G) = C_q (E|_Y, u, G).
\]

This is the result in [5].

Example 2. Let \( M \) be a smooth compact Riemannian manifold. Let \( N \) be a smooth complete Riemannian manifold. We first assume that \( M \) is without a boundary. We shall indicate how to deal with the case \( \partial M \neq \emptyset \). For \( u \in L^2_1 (M, N) \), the energy of it is

\[
F(u) = \frac{1}{2} \int_M |d_{\mathcal{M}}|^2 d_{\mathcal{K}_M}.
\]

Assume that \( u \in C^1 (M, N) \) is a harmonic map, i.e., \( u \) is a critical point of \( F|_{C^1_0 (M, N)} \) \( u \) is also a critical point of \( F \) restricted to the Banach manifold \( L^2_1 (M, N) \cap C^0 (M, N) \). Let \( X = T_u (C^1_0 (M, N)) \) be the tangent space of \( C^1(M, N) \) at \( u \) and \( Y = T_u (L^2_1 (M, N) \cap C^0 (M, N)) \) be the tangent space of \( L^2_1(M, N) \cap C^0(M, N) \) at \( u \). Let \( u^{-1} TN \) be the induced bundle over \( M \), where \( TN \) is the tangent bundle of \( N \). Then we have

\[
X = C^1 (u^{-1} TN) \quad \text{and} \quad Y = L^2_1 (u^{-1} TN) \cap C^0 (u^{-1} TN).
\]
Let $d$ be the exterior differential operator $d : \Lambda^p(u^{-1}TN) \to \Lambda^{p+1}(u^{-1}TN)$. Let $d^*$ be the codifferential operator. The inner product $(\cdot, \cdot)$ on $L^2_2(\Lambda^p T^*M \otimes u^{-1}TN)$ is defined as follows. For $\phi$ and $\psi \in L^2_2(\Lambda^p T^*M \otimes u^{-1}TN)$, define

$$(\phi, \psi) = (\phi, \psi)_0 + (d\phi, d\psi)_0 + (d^*\phi, d^*\psi)_0,$$

where $(\phi, \psi)_0 = \int_M \langle \phi, \psi \rangle dV_M,$

where $\langle \cdot, \cdot \rangle$ is the induced metric on $\Lambda^p T^*M \otimes u^{-1}TN$. Define $\Delta = d^*d + dd^*$. $\Delta$ is a strongly elliptic, self-adjoint and positive operator on $H = L^2_1(u^{-1}TN) = L^2_1(\Lambda^0 T^*M \otimes u^{-1}TN)$. Let $B = \Delta + id$. Then, by regularity theory, there exists $B^{-1} : H \to H$ such that $B^{-1} : X \to X$ and $B^{-1} : Y \to Y$. We have for $\phi$ and $\chi \in H$,

$$(\phi, \psi) = (B\phi, \psi)_0 = (\phi, B\psi)_0$$

and

$$(B^{-1}\phi, \psi) = (\phi, \psi)_0 = (\phi, B^{-1}\psi).$$

$X$ and $Y$ are normed in a natural way. The assumptions about spaces SP1, SP2 and SP3 are satisfied.

For $v \in H$, define

$$E(v) = F(\exp_u v),$$

where $\forall x \in M, \exp_u v(x) = \exp_{u(x)} v(x)$, and $\exp_{u(x)} : T_{u(x)}N \to N$ is the exponential map at $u(x) \in N$. Then the zero section $\theta$ is a critical point of $E|_{X}$ and $E|_{Y}$. Since $F|_{L^2_2(M,N) \cap C\infty(M,N)}$ is the restriction of a quadratic form to the Banach manifold $L^2_2(M,N) \cap C\infty(M,N), E|_{Y}$ is $C\infty$. Given $v$ and $\phi \in C\infty(u^{-1}TN)$, let $\xi \mapsto \exp_u (v + \xi)$ be the map from $T_0N$ to $N$. Let $(\exp_u v)_* : T_0N \to T\exp_u vN$ be its tangent map and $(\exp_u v)^*$ be the adjoint map. We have, by [6],

$$E|_{X}(v)_*(\phi) = \frac{d}{dt}E(v + t\phi)|_{t=0} = \frac{d}{dt}F(\exp_u (v + t\phi))|_{t=0} = \int_M \langle \exp_u v, \phi, \tau(\exp_u v) \rangle dV_M = \int_M \langle \phi, (\exp_u v)^* \tau(\exp_u v) \rangle dV_M.$$

where $\tau$ denotes the tension field. Let $A(v) = B^{-1}(\exp_u v)^* \tau(\exp_u v)$. Then

$$E|_{X}(v)_*(\phi) = (A(v), \phi).$$

To see that $A : X \to X$ is continuously differentiable, one way is to rewrite $A(v)$ in a local coordinate system. We omit the tedious calculation here. To show that $E$ satisfies FN3, we verify that FN3* holds. It is easy to see that FN3* (a) holds. To verify FN3* (b), note that for $v \in X, \phi, \psi \in X, E|_X'(v)(\phi, \psi)$ is the integral of a bilinear form of $\phi, D\phi, \psi, D\psi$ with coefficients being smooth functions of $x \in M$ and $v(x)$. Since the
topology of $Y$ is stronger than the uniform $C^0$ topology, FN3* (b) follows from the uniform continuity of those coefficients. By [7],

$$E_{10}^0(\phi, \psi) = \int_M (\Delta \phi - \text{Trace } R^N(\mathbf{d}u, \phi) \mathbf{d}u, \psi) \mathbf{d}V_M$$

$$= \int_M (J_u \phi, \psi) \mathbf{d}V_M,$$

where $J_u = \Delta - \text{Trace } R^N(\mathbf{d}u, \cdot) \mathbf{d}u$ is the Jacobi operator. Therefore,

$$E_{10}^0(\phi, \psi) = \int_M (B\phi - \phi - \text{Trace } R^N(\mathbf{d}u, \phi) \mathbf{d}u, \psi) \mathbf{d}V_M$$

$$= (B\phi - \phi - \text{Trace } R^N(\mathbf{d}u, \phi) \mathbf{d}u, \psi)$$

Therefore $B(\phi)\phi = \phi - B^{-1}(\phi + \text{Trace } R^N(\mathbf{d}u, \phi) \mathbf{d}u)$. Since $B^{-1}: H \mapsto H$ is compact, $0$ is either not in $\sigma(B(\phi))$ or is an isolated point of $\sigma(B(\phi))$. Hence CP1 holds. If $B(\phi)(\phi) = \psi$,

where $\phi \in H$ and $\psi \in X$. Then

$$\Delta \phi - \text{Trace } R^N(\mathbf{d}u, \phi) \mathbf{d}u = B\psi.$$ 

We have

$$\Delta[\phi - \psi] - \text{Trace } R^N(\mathbf{d}u, \phi - \psi) \mathbf{d}u = \psi + \text{Trace } R^N(\mathbf{d}u, \psi) \mathbf{d}u \in X.$$

Therefore, $\phi - \psi \in C^2(u^{-1}TN)$. Thus $\phi \in X$. Hence CP2 holds. Applying the results in the above sections implies that if $u$ is a local minimizer in $C^1(M, N)$, then it is also a local minimizer in $L^2_1(M, N) \cap C^0(M, N)$; if $u$ is an isolated critical point in $L^2_1(M, N) \cap C^0(M, N)$, we have

$$C_q(E_{10}(C^1(M, N)), u, G) = C_q(E_{10}(L^2_1(M, N) \cap C^0(M, N)), u, G),$$

noting that the condition $H_{\phi} \subset X$ is valid.

If $M$ is a manifold with a boundary, we consider the Dirichelet problem about harmonic mappings. Given $\phi \in C^\infty(M, N)$, let

$$C^1_\phi(M, N) = \{ v \in C^1(M, N): v|_{\partial M} = \phi|_{\partial M} \}$$

and let

$$L^2_{1, \phi}(M, N) = \{ v \in L^2_1(M, N): v|_{\partial M} = \phi|_{\partial M} \}.$$

Let $X = T_u(C^1_\phi(M, N))$ and $Y = T_u(L^2_{1, \phi}(M, N) \cap C^0(M, N))$. In this case, we have

$$X = C^0(u^{-1}TN) \quad \text{and} \quad Y = L^2_{1, \phi}(u^{-1}TN) \cap C^0(u^{-1}TN).$$

Another modification is that $B = \Delta$. With these modifications, the same conclusions also hold.
If \( \dim(M) = 2 \) and we replace \( L^\infty(M, N) \) with \( C^0(M, N) \) in the above, similar conclusions also hold with corresponding modifications.

**Example 3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain and \( h \in \mathbb{R} \) be a given constant. Given \( u_0 \in C^0(\Omega, \mathbb{R}^3) \), the Dirichlet problem for surfaces of prescribed mean curvature is to find \( u \in C^2(\Omega, \mathbb{R}^3) \cap C^0(\Omega, \mathbb{R}^3) \) such that

\[
\Delta u = 2h \cdot u_x \wedge u_y \quad \text{in} \, \Omega \text{ and } u|_{\partial \Omega} = u_0.
\]

This problem is of variational type. Given \( h \), let

\[
E_h(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + 2hV(u),
\]

where \( V(u) = \frac{1}{3} \int_\Omega u \cdot u_x \wedge u_y \). For simplicity, we assume \( u_0 \in C^2(\Omega, \mathbb{R}^3) \cap C^0(\Omega, \mathbb{R}^3) \).

Then \( E_h \) is well defined on \( H^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3) \). \( E_h \) can be extended to an analytic functional on \( u_0 + H^1_0(\Omega, \mathbb{R}^3) \), c.f. \([10, 11]\). Critical points of \( E_h \) in \( H^1(\Omega, \mathbb{R}^3) \) are solutions to Eq. \((14)\) by regularity results in \([1, 8]\). Let \( H = H^1_0(\Omega, \mathbb{R}^3) \). The inner product on \( H \) is

\[
(\phi, \psi) = \int_\Omega \nabla \phi \cdot \nabla \psi.
\]

Define \( E : H \mapsto \mathbb{R} \) as

\[
E(u) = E_h(u + u_0).
\]

Let \( X = H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3) \) and \( Y = H^1_0(\Omega, \mathbb{R}^3) \) with the natural topology. It is easy to see that SP1, SP2, SP3, FN1 are satisfied. From \([10]\), for \( u, \phi \) in \( X \),

\[
E'(u, \phi) = \int_\Omega \nabla(u + u_0) \cdot \nabla \phi + 2h\phi \cdot (u + u_0)_x \wedge (u + u_0)_y = (A(u), \phi),
\]

where \( A(u) = u + 2h(-\Delta)^{-1}[(u + u_0)_x \wedge (u + u_0)_y] \). It is to see that \( A : X \mapsto X \) is continuously differentiable by regularity results in \([1]\). To verify that FN3 hold, we verify still FN3* hold. For \( u, v, \phi \) and \( \psi \in X \), by Eq. \((1.8)\) of \([10, \text{p. 92}]\),

\[
E''(u)(\phi, \psi) - E''(u)(\phi, \psi) = \frac{2h}{3} \left[ \int_\Omega (\phi_x \wedge \psi_y + \psi_x \wedge \phi_y)(u - v)ight.
\]

\[
+ \int_\Omega (\phi_x \wedge (u_y - v_y) + (u_x - v_x) \wedge \phi_y) \cdot \psi
\]

\[
+ \int_\Omega (\psi_x \wedge (u_y - v_y) + (u_x - v_x) \wedge \psi_y) \cdot \phi \right].
\]

These three terms are estimated via Corollary 1.1 in \([1]\), by which we have

\[
|E''(u)(\phi, \psi) - E''(u)(\phi, \psi)| \leq C(\Omega)\|\phi\| \|\psi\| \|u - v\|.
\]

Hence FN3* is satisfied. Therefore FN3 is satisfied, too.
If \( u \in X \) is a critical point of \( E, u \in C^2 \) by regularity. By (1.10”) in [10], p. 93, we have for \( \phi, \psi \in C^\infty_0 \),

\[ E''(u)(\phi, \psi) = (B(u)\phi \cdot \psi) = \int_{\Omega} \nabla \phi \nabla \psi + 2h(\phi_x \wedge \psi_y + \psi_x \wedge \phi_y) \cdot u. \]

By approximation, it also holds for \( \phi, \psi \in Y \). By the theory of elliptic operators, CP1 is satisfied. If \( B(u)\psi = \psi \) for some \( \psi \in X \), by (1.9) in [10], p. 92, we have

\[ (B(u)\phi, \eta) = \int_{\Omega} \nabla \phi \nabla \eta + 2h(\phi_x \wedge u_y + u_x \wedge \phi_y) \cdot \eta. \]

Therefore

\[ \phi + 2h(-\Delta)^{-1}[\phi_x \wedge u_y + u_x \wedge \phi_y] = \psi, \]

i.e.,

\[ (-\Delta)[\psi - \phi] = \phi_x \wedge u_y + u_x \wedge \phi_y. \]

By the regularity result in [1] again, \( \psi - \phi \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3) \). Therefore \( \phi \in X \).

Hence CP2 holds.

If \( u \) is a local minimizer of \( E \) in \( X = H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3) \), then it is also a local minimizer of \( E \) in \( Y = H^1_0(\Omega, \mathbb{R}^3) \). This is just Lemma 2 in [2] by Brézis and Coron. The discussion on critical groups by applying Corollary 2.8 is left to the reader.

References