Inequalities in Information Theory

A Brief Introduction

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Part I

Basic Concepts and Inequalities

Outline

The Entropy

- Definition
  1. The Shannon information content of an outcome x is defined to be

$$h(x) = \log_2 \frac{1}{P(x)}$$

  2. The entropy of an ensemble X is defined to be the average Shannon information content of an outcome:

$$H(X) = \sum_{x \in \mathcal{X}} P(X) \log_2 \frac{1}{P(X)} \quad (1)$$

  3. Conditional Entropy: the entropy of a r.v.,given another r.v.

$$H(X|Y) = -\sum_i \sum_j p(x_i, y_j) \log_2 p(x_i|y_j) \quad (2)$$

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The Joint Entropy

The joint entropy of X; Y is:

$$H(X, Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log_2 \frac{1}{p(x, y)} \tag{3}$$

Remarks

- The entropy H answers the question that what is the ultimate data compression.
- The entropy is a measure of the average uncertainty in the random variable.It is the number of bits on the average required to describe the random variable.

- Reference for [[2]Thomas and [4]David]

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The Mutual Information

Definition

The mutual information is the reduction in uncertainty when given another r.v., for two r.v. X and Y this reduction is

$$I(X;Y) = H(X) - H(X|Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \tag{4}$$

- The capacity of channel is

$$C = \max_{p(x)} I(X;Y)$$

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The relationships

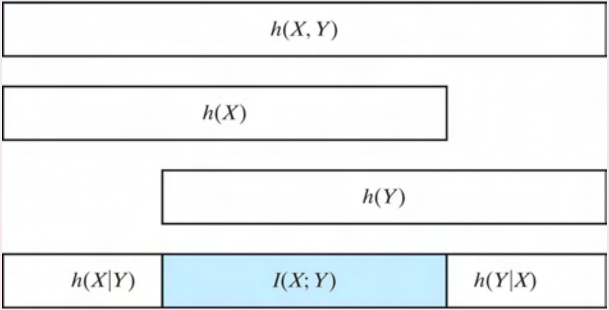

Figure: The relationships between Entropy and Mutual Information

- Graphic from [[3]Simon,2011].

The relative entropy

Definition

The relative entropy or Kullback Leibler distance between two probability mass functions $p(x)$ and $q(x)$ is defined as

$$D(p \parallel q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = E_p \log \frac{p(X)}{q(X)}. \tag{5}$$

1. The relative entropy and mutual information

$$I(X;Y) = D(p(x,y) \parallel p(x)p(y)) \tag{6}$$

2. Pythagorean decomposition: let $X = AU$, then

$$D(p_x \parallel p_u) = D(p_x \parallel \tilde{p}_x) + D(\tilde{p}_x \parallel p_u). \tag{7}$$

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Conditional definitions

Conditional mutual information

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$$= E_{p(x,y,z)} \log \frac{p(X,y|Z)}{p(X|Z)p(Y|Z)}. \tag{9}$$

Conditional relative entropy

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Differential entropy

Definition 1

The differential entropy $h(X_1, X_2, \ldots, X_n)$, some times written $h(f)$, is defined by

$$h(X_1, X_2, \ldots, X_n) = -\int f(x) \log f(x) dx \tag{12}$$

Definition 2

The relative entropy between probability densities f and g is

$$D(f \parallel g) = -\int f(x) \log(f(x)/g(x)) dx \tag{13}$$

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Chain Rules

1. Chain rule for entropy

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1). \quad (14)$$

2. Chain rule for information

$$I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_{i-1}, \ldots, X_1). \quad (15)$$

3. Chain rule for entropy

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Outline

Jensen's inequality

Definition

A function f is said to be convex if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad (17)$$

for all $0 \leq \lambda \leq 1$ and all x_1 and x_2 in the convex domain of f.

Theorem

If f is convex,then

$$f(EX) \leq Ef(x) \quad (18)$$

Proof

We consider discrete distributions only. The proof is given by induction. For a two mass point distribution, by definition. for k mass points, let $p'_i = p_i/(1-p_k)$ for $i \leq k-1$, the result can be derived easily.

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Log sum inequality

Theorem

For positive numbers, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq (\sum_{i=1}^{n} a_i) \log(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}) \tag{19}$$

with equality iff $\frac{a_i}{b_i}$ = constant.

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We substitute discrete distribution parameters in Jensen's Inequality by $\alpha_i = b_i / \sum_{j=1}^{n} b_j$ and the variables by $t_i = a_i / b_i$, we obtain the inequality.

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Inequalities in Entropy Theory

- By Jensen's inequality and Log Sum inequality, we can easily prove following basic conclusions:

$$0 \leq H(X) \leq \log \mid \mathcal{X} \mid \tag{20}$$

$$D(p \parallel q) \geq 0 \tag{21}$$

Further more,

$$I(X;Y) \geq 0 \tag{22}$$

- Note:the conditions when the equalities holds.

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Inequalities in Entropy Theory(cont.)

- Conditioning reduces entropy:

$$H(X|Y) \leq H(X)$$

- The chain rule and independence bound on entropy:

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots, X_1) \leq \sum_{i=1}^{n} H(X_i) \qquad (23)$$

- Note: the conclusions continue to hold for differential entropy.
- If X and Y are independent, then

$$h(X+Y) \geq h(Y)$$

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Convexity & concavity entropy theory

Theorem

$D(p \parallel q)$ is convex in the pair (p, q),i.e., if (p_1, q_1) and $(p_2, q_2)$are two pairs of probability mass functions, then

$$D(\lambda p_1 + (1-\lambda)p_2 \parallel \lambda q_1 + (1-\lambda)q_2) \leq \lambda D(p_1 \parallel q_1) + (1-\lambda)D(p_2 \parallel q_2) \tag{24}$$

for all $0 \leq \lambda \leq 1$.

- Apply the log sum inequality to the term on the left hand side of (24).

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Convexity & concavity in entropy theory(cont.)

Theorem

$H(p)$ is a concave function of p.

- Let u be the uniform distribution on $|\mathcal{X}|$ outcomes, then the concavity of H then follows directly from then convexity of D, since the following equality holds.

$$H(p) = \log|\mathcal{X}| - D(p \parallel u) \tag{25}$$

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Theorem

Let$(X, Y) \sim p(x, y) = p(x)p(y|x)$. The mutual information $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$ and a convex function of $p(y|x)$ for fixed $p(X)$.

- The detailed proof can be found in $[[2] Thomas, section 2.7]$. An alternative proof is given in [1],P51-52.

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Outline

$\mathcal{L}_1$ bound on entropy

Theorem

Let p and q be two probability mass functions on $\mathcal{X}$ such that

$$\| p - q \|_1 = \sum_{x \in \mathcal{X}} | p(x) - q(x) | \leq \frac{1}{2}.$$

Then

$$| H(p) - H(q) | \leq - \| p - q \|_1 \log \frac{\| p - q \|_1}{| \mathcal{X} |}. \tag{26}$$

Proof of $\mathcal{L}_1$ bound on entropy

Proof

Consider the function $f(t) = -t\log t$,it is concave and positive on $[0, 1]$, since $f(0) = f(1) = 0$.

1. Let $0 \le \nu \le \frac{1}{2}$, for any $0 \le t \le 1-\nu$,we have

$$| f(t) - f(t+\nu) | \le \max\{f(\nu), f(1-\nu)\} = -\nu\log\nu. \quad (27)$$

2. Let $r(x) = | p(x) - q(x) |$. Then

$$| H(p) - H(q) | = | \sum_{x\in\mathcal{X}} (-p(x)\log p(x) + q(x)\log q(x) | \quad (28)$$

$$\le \sum_{x\in\mathcal{X}} | (-p(x)\log p(x) + q(x)\log q(x) | \quad (29)$$

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Proof of $\mathcal{L}_1$ bound on entropy

Proof(cont.)

By using (27), we have

$$Left \le \sum_{x\in\mathcal{X}} -r(x)\log r(x) \tag{30}$$

$$= \| p-q \|_1 \sum_{x\in\mathcal{X}} -\frac{r(x)}{\| p-q \|_1} \log \frac{r(x)}{\| p-q \|_1} \| p-q \|_1 \tag{31}$$

$$= - \| p-q \|_1 \log \| p-q \|_1 + \| p-q \|_1 H\left(\frac{r(x)}{\| p-q \|_1}\right) \tag{32}$$

$$\le - \| p-q \|_1 \log \| p-q \|_1 + \| p-q \|_1 \log | \mathcal{X} | . \tag{33}$$

The lower bound of relative entropy

Theorem

$$D(P_1 \parallel P_2) \geq \frac{1}{2\ln 2} \parallel P_1 - P_2 \parallel_1^2 . \tag{34}$$

Proof

(1)Binary case. Consider two binary distribution with parameter p and q with $p \leq q$. We will show that

$$p \log \frac{p}{q} + (1-p)\log\frac{1-p}{1-q} \geq \frac{4}{2\ln 2}(p-q)^2.$$

Let

$$g(p,q) = p \log \frac{p}{q} + (1-p)\log\frac{1-p}{1-q} - \frac{4}{2\ln 2}(p-q)^2.$$

The lower bound of relative entropy

Proof(cont.)

Then

$$\frac{\partial g(p,q)}{\partial q} \leq 0$$

since $q(1-q) \leq \frac{1}{4}$ and $q \leq p$. For $q = p$, $g(p,q) = 0$, and hence $g(p,q) \geq 0$ for $q \leq p$, which proves the binary case.

The lower bound of relative entropy

Proof(cont.)

(2)For the general case, for any two distribution P_1 and P_2,let $A=\{x:P_1(x)>P_2(x)\}$. Define $Y=\phi(X)$, the indicator of the set A,and let $\hat{P}_1$ and $\hat{P}_2$ be the distribution of Y. By the data processing inequality([2]Thomas,section 2.8) applied to relative entropy, we have

$$D(P_1 \parallel P_2) \geq D(\hat{P}_1 \parallel \hat{P}_2) \geq \frac{4}{2\ln 2}(P_1(A)-P_2(A))^2 = \frac{1}{2\ln 2}\parallel P_1 - P_2 \parallel_1^2 .$$

Part II

Entropy in Statistics

Outline

Data processing inequality and its corollaries

Data processing inequality

If $X \to Y \to Z$, then

$$I(X;Y) \geq I(X;Z). \quad (35)$$

Corollary

In particular, if $Z = g(Y)$,we have

$$I(X;Y) \geq I(X;g(Y)). \quad (36)$$

Corollary

If $X \to Y \to Z$, then

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Data processing inequality

If $X \to Y \to Z$, then

$$I(X;Y) \geq I(X;Z). \tag{35}$$

Corollary

In particular, if $Z = g(Y)$,we have

$$I(X;Y) \geq I(X;g(Y)). \tag{36}$$

Corollary

If $X \to Y \to Z$, then

$$I(X;Y|Z) \geq I(X;Y). \tag{37}$$

Entropy in Markov chain

Theorem

For a *Markov Chain*:

1. Relative entropy $D(\mu_n \parallel \mu'_n)$ decreases with time.
2. Relative entropy $D(\mu_n \parallel \mu)$ between a distribution and the stationary distribution decreases with time.
3. Entropy $H(X_n)$ increases if the stationary distribution is uniform.
4. The conditional entropy $H(X_n|X_1)$ increases with time for a stationary Markov chain.
5. Shuffles increase entropy.

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Proof for item 1

Let μ_n and μ'_n be two probability distributions on the state space of a Markov chain at time n, corresponding to p and q as joint mass functions. By the chain rule:

$$
\begin{aligned}
&D(p(x_n, x_{n+1}) \parallel q(x_n, x_{n+1})) \\
&\quad = D(p(x_n) \parallel q(x_n)) + D(p(x_{n+1}|x_n) \parallel q(x_{n+1}|x_n)) \\
&\quad = D(p(x_{n+1}) \parallel q(x_{n+1})) + D(p(x_n|x_{n+1}) \parallel q(x_n|x_{n+1}))
\end{aligned}
$$

Proof for item 1(cont.)

Since the probability transition function $p(x_{n+1}|x_n) = q(x_{n+1}|x_n)$ from the Markov chain, hence $D(p(x_{n+1}|x_n) \parallel q(x_{n+1}|x_n)) = 0$, and also$D(p(x_n|x_{n+1}) \parallel q(x_n|x_{n+1})) \geq 0$, we have

$$D(p(x_n) \parallel q(x_n)) \geq D(p(x_{n+1}) \parallel q(x_{n+1}))$$

or

$$D(\mu_n \parallel \mu_n') \geq D(\mu_{n+1} \parallel \mu_{n+1}').$$

Proof for item 2

Let $\mu'_n = \mu$, and $\mu'_{n+1} = \mu$, μ can be any stationary distribution. By item 1, the inequality holds.

Remarks

The monotonically non-increasing non-negative sequence $D(\mu_n \parallel \mu)$ has 0 as its limit if the stationary distribution is unique.

Remark on item 3

Let the stationary distribution μ be uniform, then by

$$D(\mu_n \parallel \mu) = \log|\mathcal{X}| - H(\mu_n) = \log|\mathcal{X}| - H(X_n)$$

we know the conclusion holds.

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Proof for item 4

$$H(X_n|X_1) \geq H(X_n|X_1, X_2) = H(X_n|X_2) = H(X_{n-1}|X_1)$$

Remarks on item 5

If T is a shuffle *permutation*of cards and X is the initial *random* position, and if T is independent of X, then

$$H(TX) \geq H(X)$$

where TX is the permutation by the shuffle T on X.

- Proof

$$H(TX) \geq H(TX|T) = H(T^{-1}TX|T) = H(X|T) = H(X)$$

- Reference for [[2]Thomas, section 4.4.]

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Entropy in Markov chain

Theorem(Fano's inequality)

For any estimator $\hat{X}$ such that $X \to Y \to \hat{X}$,with $P_e = Pr(X \neq \hat{X})$, we have

$$H(P_e) + P_e \log(|\mathcal{X}|) \geq H(X|\hat{X}) \geq H(X|Y) \tag{38}$$

this inequality can be weakened to

$$1 + P_e \log \mid \mathcal{X} \mid \geq H(X|Y) \tag{39}$$

or

$$P_e \geq \frac{H(X|Y) - 1}{\log \mid \mathcal{X} \mid}. \tag{40}$$

Proof of Fano's inequality

Proof

Define an error random varible,

$$E = \begin{cases} 1, & \text{if } \hat{X} \neq X \\ 0, & \text{if } \hat{X} = X \end{cases}$$

Then,

$$H(E, X|\hat{X}) = H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} = \underbrace{H(E|\hat{X})}_{\leq H(E) = H(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log(|\mathcal{X}|)}.$$

since

$$\begin{aligned} H(X|E, \hat{X}) &= Pr(E = 0)H(X|\hat{X}, E = 0) + Pr(E = 1)H(X|\hat{X}, E = 1) \\ &\leq (1 - P_e)0 + P_e \log | \mathcal{X} | . \end{aligned}$$

Proof of Fano's inequality

Proof(cont.)

By the data-processing inequality, we have $I(X;\hat{X}) \geq I(X;Y)$ since $X \to Y \to \hat{X}$ is a Markov chain, and therefore $H(X|\hat{X}) \geq H(X|Y)$. Thus we have (38) holds.

- For any two random variables X and Y, if the estimator $g(Y)$ takes values in the set X, we can strengthen the inequality slightly by replacing $\log|\mathcal{X}|$ with $\log(|\mathcal{X}|-1)$.

Proof of Fano's inequality

Proof(cont.)

By the data-processing inequality, we have $I(X;\hat{X}) \geq I(X;Y)$ since $X \to Y \to \hat{X}$ is a Markov chain, and therefore $H(X|\hat{X}) \geq H(X|Y)$. Thus we have (38) holds.

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Empirical probability mass function

Theorem

Let $X_1, X_2, \ldots, X_n$ be i.i.d $\sim p(x)$. Let $\tilde{p}_n$ be the empirical probability mass function of $X_1, X_2, \ldots, X_n$. Then

$$ED(\hat{p}_n \parallel p) \leq ED(\hat{p}_{n-1} \parallel p) \tag{41}$$

Proof

Use $D(\hat{p}_n \parallel p) = E_{\hat{p}_n} \log \frac{\hat{p}_n}{p(x)} = E_{\hat{p}_n} \log \hat{p}_n - \log p(x)$, we have$E_p D(\hat{p}_n \parallel p) = H(p) - H(\hat{p}_n)$,then by item 3 in Markov Chain.

Outline

4 Entropy in Markov chain

5 Bounds on entropy on distributions

Entropy of a multivariate normal distribution

Lemma

Let$X_1, X_2, \ldots, X_n$ have a multivariate normal distribution with mean μ and covariance matrix $\mathbf{K}$. Then

$$h(X_1, X_2, \ldots, X_n) = h(\mathcal{N}(\mu, \mathbf{K})) = \frac{1}{2}\log(2\pi e)^n \mid \mathbf{K} \mid \text{ bits}, \tag{42}$$

where $\mid \mathbf{K} \mid$ denotes the determinant of K.

Bounds on differential entropies

Theorem

Let the random vector $\mathbf{X} \in \mathbf{R}^n$ have zero mean and covariance $\mathbf{K} = E\mathbf{X}\mathbf{X}^t$, i.e., $K_{ij} = EX_iX_j, 1 \leq j, j \leq n$. Then

$$h(\mathbf{X}) \leq \frac{1}{2} \log (2\pi e)^n |\mathbf{K}|, \tag{43}$$

with equality *iff* $\mathbf{X} \sim \mathcal{N}(0, \mathbf{K})$.

Bounds on differential entropies

Proof

Let $g(\mathbf{x})$ be any density satisfying $\int g(\mathbf{x}) x_i x_j d\mathbf{x} = K_{ij}$ for all i, j. Let $\phi_K \sim \mathcal{N}(0, K)$. Note that $\log \phi_K(x)$ is a quadratic form and $\int x_i x_j \phi_K(\mathbf{x}) d\mathbf{x} = K_{ij}$. Then

$$\begin{aligned}
0 &\leq D(g \parallel \phi_K) \\
&= \int g \log(g/\phi_K) \\
&= -h(g) - \int g \log \phi_K \\
&= -h(g) - \int \phi_K \log \phi_K \\
&= -h(g) + h(\phi_K)
\end{aligned}$$

since $h(\phi_K) = \frac{1}{2} \log (2\pi e)^n |\mathbf{K}|$, the conclusion holds.

Bounds on discrete entropies

Theorem

$$H(p_1, p_2, \ldots) \le \frac{1}{2}\log(2\pi e)\left(\sum_{i=1}^{\infty} p_i i^2 - \left(\sum_{i=1}^{\infty} i p_i\right)^2 + \frac{1}{12}\right) \qquad (44)$$

Proof

Define new r.v. X, with the distribution $Pr(X = i) = p_i$, $U \sim \mathcal{U}(0,1)$, define $\tilde{X}$ by $\tilde{X} = X + U$. Then

$$\begin{aligned} H(X) &= -\sum_{i=1}^{\infty} p_i \log p_i \\ &= -\sum_{i=1}^{\infty}\left(\int_i^{i+1} f_{\tilde{X}}(x)dx\right)\log\left(\int_i^{i+1} f_{\tilde{X}}(x)dx\right) \end{aligned}$$

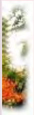
Bounds on discrete entropies

Proof(cont.)

$$H(X) = -\sum_{i=1}^{\infty} \int_{i}^{i+1} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) dx$$
$$= -\int_{1}^{\infty} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) dx$$
$$= h(\tilde{X})$$

since $f_{\tilde{X}}(x) = p_i$ for $i \leq x < i+1$. Hence

$$h(\tilde{X}) \leq \frac{1}{2} \log(2\pi e) Var(\tilde{X}) = \frac{1}{2} \log(2\pi e)(Var(X) + Var(U))$$
$$= \frac{1}{2} \log(2\pi e) \left(\sum_{i=1}^{\infty} p_i i^2 - \left(\sum_{i=1}^{\infty} i p_i \right)^2 + \frac{1}{12} \right).$$



Entropy and fisher information

- The Fisher information matrix is a measure of the minimum error in estimating a parameter vector of a distribution.
- The Fisher information matrix of the distribution of X with a parameter vector θ is defined as

$$J(\theta) = E\{\left[\frac{\partial}{\partial\theta}\log f_\theta(X)\right]\left[\frac{\partial}{\partial\theta}\log f_\theta(X)\right]^T\} \tag{45}$$

for any $\theta \in \Theta$.

- If f_θ is twice differentiable in θ, and alternative expression is

$$J(\theta) = -E\left[\frac{\partial^2}{\partial\theta\partial\theta^T}\log f_\theta(X)\right]. \tag{46}$$

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Fisher information of a distribution

- Let X be any r.v. with density $f(x)$, for a location parameter θ, the fisher information w.r.t. θ is given by

$$J(\theta) = \int_{-\infty}^{\infty} f(x-\theta)\left[\frac{\partial}{\partial\theta}\ln f(x-\theta)\right]^2 dx.$$

- As the differentiation w.r.t. x is equivalent to θ, so we can rewrite the Fisher information as

$$\begin{aligned} J(X) = J(\theta) &= \int_{-\infty}^{\infty} f(x)\left[\frac{\partial}{\partial x}\ln f(x)\right]^2 dx \\ &= \int_{-\infty}^{\infty} f(x)\left[\frac{\frac{\partial}{\partial x} f(x)}{f(x)}\right]^2 dx. \end{aligned}$$

Cramér-Rao inequality

Theorem

The mean-squared error of any unbiased estimator $T(X)$ of the parameter θ is lower bounded by the reciprocal of the Fisher information:

$$Var[T(X)] \geq [J(\theta)]^{-1}. \tag{47}$$

Proof

By Cauchy-Schwarz inequality,

$$Var[T(X)]Var\left(\frac{\partial \log f}{\partial \theta}\right) \geq Cov^2\left(T(X), \frac{\partial \log f}{\partial \theta}\right)$$

Then

$$Cov^2\left(T(X), \frac{\partial \log f}{\partial \theta}\right) = E\left(T(X)\frac{\partial \log f}{\partial \theta}\right) = \frac{\partial}{\partial \theta}E_\theta(T(X)) = 1.$$

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Entropy and Fisher information

Theorem

Let X be any random variable with a finite variance with a density $f(x)$. Let Z be an independent normally distributed random variable with zero mean and unit variance. Then

$$\frac{\partial}{\partial t} h_e(X + \sqrt{t}Z) = \frac{1}{2} J(X + \sqrt{t}Z), \tag{48}$$

where h_e is the differential entropy to base e. In particular, if the limit exists as $t \to 0$,

$$\frac{\partial}{\partial t} h_e(X + \sqrt{t}Z) \mid_{t=0} = \frac{1}{2} J(X). \tag{49}$$

Proof

- Let $Y_t = X + \sqrt{t}Z$. Then the density of Y_t is

$$g_t(y) = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dx.$$

- It's easy to verify that

$$\frac{\partial}{\partial t} g_t(y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} g_t(y). \tag{50}$$

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Proof

- Since $h_e(Y_t) = -\int_{-\infty}^{\infty} g_t(y) \ln g_t(y) dy$ Differentiating, by $\int g_t(y)dy = 1$ and (50), then integrate by parts, we obtain

$$\frac{\partial}{\partial t} h_e(Y_t) = -\frac{1}{2}\left[\frac{\partial g_t(y)}{\partial y} \ln g_t(y)\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial y} g_t(y)\right]^2 \frac{1}{g_t(y)} dy.$$

- The first term above goes to 0 at both limit, and by definition, the first term is $\frac{1}{2}J(Y_t)$. Thus the theorem is prove.

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Part III

Some important theories deduced from entropy

Outline

6. Entropy rates of subsets

7. The Entropy power inequality

Entropy on subsets

Definition: Average Entropy Rate

Let $(X_1, X_2, \ldots, X_n)$ have a density, and for every $S \subseteq \{1, 2, \ldots, n\}$,denote by $X(S)$ the subset $\{X_i : i \in S\}$. Let

$$h_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{h(X(S))}{k}. \tag{51}$$

Here $h_k^{(n)}$ is the average entropy in bits per symbol of a randomly drawn k-element subset of $(X_1, X_2, \ldots, X_n)$.

- The average conditional entropy rate and average mutual information rate can be defined similarly on $h(X(S)|X(S^c))$ and$I(X(S); X(S^c))$.

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Entropy on subsets

Theorem

1. For average entropy rate,

$$h_1^{(n)} \geq h_2^{(n)} \geq \ldots \geq h_n^{(n)}. \tag{52}$$

2. For average conditional entropy rate,

$$g_1^{(n)} \leq g_2^{(n)} \leq \ldots \leq g_n^{(n)}. \tag{53}$$

3. For average mutual information,

$$f_1^{(n)} \geq f_2^{(n)} \geq \ldots \geq f_n^{(n)}. \tag{54}$$

Proof for Theorem, item 1

- We first proof $h_n^{(n)} \le h_{n-1}^{(n)}$. Since for $i = 1, 2, \ldots, n$,

$$\begin{aligned} h(X_1, X_2, \ldots, X_n) &= h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \\ &\quad + h(X_i | X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \\ &\le h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \\ &\quad + h(X_i | X_1, X_2, \ldots, X_{i-1}) \end{aligned}$$

- Adding these n inequalities and using the chain rule, we obtain

$$\frac{1}{n} h(X_1, X_2, \ldots, X_n) \le \frac{1}{n} \sum_{i=1}^{n} \frac{h(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)}{n-1}$$

Thus $h_n^{(n)} \le h_{n-1}^{(n)}$ holds.

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Proof for Theorem, item 1(cont.)

- For each k-element subset, $h_k^{(k)} \leq h_{k-1}^{(k)}$,
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- and hence the inequality remains true after taking the expectation over all k-element subsets chosen uniformly from the n elements.

Entropy on subsets

Proof for Theorem,item 2 and 3

(1) We prove $g_n^{(n)} \leq g_{n-1}^{(n)}$ first.By

$$h(X_1, X_2, \ldots, X_n) \leq \sum_{i-1}^{n} h(X_i)$$

$$(n-1)h(X_1, X_2, \ldots, X_n) \geq \sum_{i=1}^{n} (h(X_1, X_2, \ldots, X_n) - h(X_i))$$
$$= \sum_{i=1}^{n} h(X_1, X_2, \ldots, X_{i-1}, X_i, \ldots, X_n | X_i).$$

Similar as the proof of item 1, we have $g_k^{(k)} \leq g_{k-1}^{(k)}$.
(2) Since $I(X(S); X(S^c) = h(X(S)) - h(X(S)|X(S^c))$, item 3 holds.

Outline

The Entropy power inequality

Theorem

If $\mathbf{X}$ and $\mathbf{Y}$ are independent random n-vectors with densities, then

$$2^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \geq 2^{\frac{2}{n}h(\mathbf{X})} + 2^{\frac{2}{n}h(\mathbf{Y})}. \tag{55}$$

Remarks

For normal distributions, since $2^{2h(X)} = (2\pi e)\sigma_X^2$, we have a new statement of the entropy power inequality.

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The entropy power inequality

Theorem: the entropy power inequality

For two independent random variables X and Y,

$$h(X+Y) \geq h(X'+Y')$$

where X' and Y' are independent normal random variables with $h(X') = h(X)$ and $h(Y') = h(Y)$.

Definitions

- The set sum $A + B$ of two sets $A, B \subset \mathcal{R}^n$ is defined as the set $\{x + y : x \in A, y \in B\}$.
- Example: The set sum of two spheres of radius 1 at the origins is a sphere of radius 2 at the origin.
- Let the $\mathcal{L}_r$ norm of the density be defined by $\| f \|_r = \left(\int f^r(x)dx\right)^{\frac{1}{r}}$.
- The Rényi entropy $h_r(X)$ of order r is defined as

$$h_r(X) = \frac{1}{1-r} \log \left[\int f^r(x)dx \right] \tag{56}$$

for $0 < r < \infty, r \neq 1$.

Remarks on definition

Remarks

- If we take the limit as $r \to 1$, we obtain the Shannon entropy function

$$h(X) = h_1(x) = -\int f(x) \log f(x) dx.$$

- If we take the limit as $r \to 0$, we obtain the logarithm of the support set,

$$h_0 = \log(\mu\{x : f(x) > 0\}).$$

- Thus the zeroth order Rényi entropy gives the measure of the support set of the density of f.

The Brunn-Minkowski inequality

Theorem: Brunn-Minkowski inequality

The volume of the set sum of two sets A and B is greater than the volume of the set sum of two spheres A' and B' with the same volume as A and B, respectively, i.e.,

$$V(A+B) \geq V(A'+B')$$

where A' and B' are spheres with $V(A') = V(A)$ and $V(B') = V(B)$.

The Rényi Entropy Power

Definition

The Rényi entropy power $V_r(X)$ of order r is defined as

$$V_r(X) = \begin{cases} \left[\int f^r(x)dx\right]^{\frac{2}{n}\frac{r'}{r}}, & 0 < r \leq \infty, r \neq 1, \frac{1}{r} + \frac{1}{r'} = 1 \\ \exp[\frac{2}{n}h(X)], & r = 1 \\ \mu(\{x : f(x) > 0\})^{\frac{2}{n}}, & r = 0 \end{cases}$$

Theorem

For two independent random variables X and Y and any $0 \leq r < \infty$ and any $0 \leq \lambda \leq 1$, let $p = \frac{r}{r+\lambda(1-r)}$, $q = \frac{r}{r+(1-\lambda)(1-r)}$, we have

$$\log V_r(X+Y) \geq \lambda \log V_p(X) + (1-\lambda) \log V_q(Y) + H(\lambda) \quad (57)$$

$$+ \left(\frac{1+r}{1-r}\right)\left[H\left(\frac{r+\lambda(1-r)}{1+r}\right) - H\left(\frac{r}{1+r}\right)\right]. \quad (58)$$

Remarks on the Rényi Entropy Power

- The Entropy power inequality. Taking the limit of (58) as $r \to 1$ and setting $\lambda = \frac{V_1(X)}{V_1(X)+V_1(Y)}$, we obtain

$$V_1(X+Y) \geq V_1(X) + V_1(Y).$$

- The Brunn-Minkowski inequality. Similarly letting $r \to 0$ and choosing $\lambda = \frac{\sqrt{V_0(X)}}{\sqrt{V_0(X)}+\sqrt{V_0(Y)}}$, we obtain

$$\sqrt{V_0(X+Y)} \geq \sqrt{V_0(X)} + \sqrt{V_0(Y)}$$

Now let A and B be the support set of X and Y. Then $A+B$ is the support set of $X+Y$, and the equation above reduces to

$$[\mu(A+B)]^{1/n} \geq [\mu(A)]^{1/n} + [\mu(B)]^{1/n},$$

which is the Brunn-Minkowski inequality.

Part IV

Important applications

Outline

Basic concepts

Definition

1. The type P_x of a sequence $x_1, x_2, \ldots, x_n$ is the relative proportion of occurrences in $\mathcal{X}$,i.e., $P_{\mathbf{x}}(a) = N(a|\mathbf{x})/n$ for all $a \in \mathcal{X}$.
2. Let $\mathcal{P}_n$ denote the set of types with a sequence of n symbols.
3. If $P \in \mathcal{P}_n$, then the type class of P, denoted $T(P)$ is defined as:

$$T(P) = \{\mathbf{x} \in \mathcal{X}^n : P_{\mathbf{x}} = P\}$$

Bound on number of types

Theorem: the probability of x

If $X_1, X_2, \ldots, X_n$ are drawn i.i.d.$\sim Q(x)$, then the probability of $\mathbf{x}$ depends only on its type and is given by

$$Q^{(n)}(\mathbf{x}) = 2^{-n(H(P_{\mathbf{x}})+D(P_{\mathbf{x}}\|Q))} \tag{59}$$

Proof

$$\begin{aligned} Q^{(n)}(\mathbf{x}) &= \prod_{i=1}^{n} Q(X_i) = \prod_{a\in\mathcal{X}} Q(a)^{N(a|\mathbf{x})} \\ &= \prod_{a\in\mathcal{X}} Q(a)^{nP_{\mathbf{x}}(a)} = \prod_{a\in\mathcal{X}} 2^{nP_{\mathbf{x}}\log Q(a)} \\ &= 2^{n\sum_{a\in\mathcal{X}}(-P_{\mathbf{x}}(a)\log\frac{P_{\mathbf{x}}(a)}{Q(a)}+P_{\mathbf{x}}(a)\log P_{\mathbf{x}}(a))}. \end{aligned}$$

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Size of type class $T(P)$

Theorem

$$| \mathcal{P}_n | \leq (n+1)^{|\mathcal{X}|}. \tag{60}$$

Theorem

For any type of $P \in \mathcal{P}_n$,

$$\frac{1}{(n+1)^{|\mathcal{X}|}} 2^{nH(P)} \leq | T(P) | \leq 2^{nH(P)}. \tag{61}$$

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For any type of $P \in \mathcal{P}_n$,

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Size of type class $T(P)$

Proof

By (59), if $\mathbf{x} \in T(P)$, then $P^{(n)}(\mathbf{x}) = 2^{-nH(P)}$, we have

$$1 \geq P^{(n)}(T(P)) = \sum_{\mathbf{x} \in T(P)} P^{(n)}(\mathbf{x}) = \sum_{\mathbf{x} \in T(P)} 2^{-nH(P)} = \mid T(P) \mid 2^{-nH(P)}.$$

For the lower bound, we use the fact $P^{(n)}(T(P)) \geq P^{(n)}(T(\hat{P}))$, for all $\hat{P} \in \mathcal{P}_n$ without proof.

$$\begin{aligned} 1 = \sum_{Q \in \mathcal{P}_n} P^{(n)}(T(Q)) &\leq \sum_{Q \in \mathcal{P}_n} P^{(n)}(T(P)) \\ &\leq (n+1)^{|\mathcal{X}|} P^{(n)}(T(P)) = (n+1)^{|\mathcal{X}|} \mid T(P) \mid 2^{-nH(P)}. \end{aligned}$$

Probability of type class

Theorem

for any $P \in P_n$ and any distribution Q, the probability of the type class $T(P)$ under $Q^{(n)}$ is

$$\frac{1}{(n+1)^{|\mathcal{X}|}} 2^{-nD(P\|Q)} \leq |\ Q^{(n)}(T(P))\ | \leq 2^{-nD(P\|Q)}. \tag{62}$$

Proof

$$Q^{(n)}(T(P)) = \sum_{\mathbf{x} \in T(P)} Q^{(n)}(\mathbf{x}) = \sum_{\mathbf{x} \in T(P)} 2^{-n(D(P\|Q)+H(P))}$$
$$= |\ T(P)\ |\ 2^{-n(D(P\|Q)+H(P))}$$

Then use the bounds on $|\ T(P)\ |$ derived in last theorem.

Probability of type class

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for any $P \in P_n$ and any distribution Q, the probability of the type class $T(P)$ under $Q^{(n)}$ is

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$$Q^{(n)}(T(P)) = \sum_{\mathbf{x} \in T(P)} Q^{(n)}(\mathbf{x}) = \sum_{\mathbf{x} \in T(P)} 2^{-n(D(P\|Q)+H(P))}$$
$$= \mid T(P) \mid 2^{-n(D(P\|Q)+H(P))}$$

Then use the bounds on $\mid T(P) \mid$ derived in last theorem.

Summarize

- We can summarize the basic theorems concerning types in four equations:

$$| \mathcal{P}_n | \leq (n+1)^{|\mathcal{X}|}, \tag{63}$$

$$Q^{(n)}(\mathbf{x}) = 2^{-n(H(P_{\mathbf{x}})+D(P_{\mathbf{x}}\|Q))}, \tag{64}$$

$$| T(P) | \doteq 2^{nH(P)}, \tag{65}$$

$$Q^{(n)}(T(P)) \doteq 2^{-nD(P\|Q)}. \tag{66}$$

- There are only a polynomial number of types and an exponential number of sequences of each type.
- We can calculate the behavior of long sequences based on the properties of the type of the sequence.

Outline

Tight bounds on the size of $\binom{n}{k}$

Lemma

For $0 < p < 1$, $q = 1 - p$, such that np is an integer,

$$\frac{1}{\sqrt{8npq}} \le \binom{n}{np} 2^{-nH(p)} \le \frac{1}{\sqrt{\pi npq}}. \tag{67}$$

Tight bounds on the size of $\binom{n}{k}$

Proof of Lemma

Applying a strong form of Stirling's approximation, which states that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}. \tag{68}$$

we obtain

$$\begin{aligned}\binom{n}{np} &\le \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}}{\sqrt{2\pi np}\left(\frac{np}{e}\right)^{np}\sqrt{2\pi nq}\left(\frac{nq}{e}\right)^{nq}} \\ &= \frac{1}{\sqrt{2\pi npq}}\frac{1}{p^{np}q^{nq}}e^{\frac{1}{12n}} \\ &< \frac{1}{\sqrt{\pi npq}}2^{nH(p)}\end{aligned}$$

Since $e^{\frac{1}{12n}} < e^{\frac{1}{12}} < \sqrt{2}$. The lower bound is obtained similarly.

Tight bounds on the size of $\binom{n}{k}$

Proof of Lemma(cont.)

$$\begin{aligned}\binom{n}{np} &\geq \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-\left(\frac{1}{12np}+\frac{1}{12nq}\right)}}{\sqrt{2\pi np}\left(\frac{np}{e}\right)^{np}\sqrt{2\pi nq}\left(\frac{nq}{e}\right)^{nq}} \\ &= \frac{1}{\sqrt{2\pi npq}}\frac{1}{p^{np}q^{nq}}e^{-\left(\frac{1}{12np}+\frac{1}{12nq}\right)} \\ &< \frac{1}{\sqrt{2\pi npq}}2^{nH(p)}e^{-\left(\frac{1}{12np}+\frac{1}{12nq}\right)}\end{aligned}$$

If $np \geq 1$, and $nq \geq 3$,then $e^{-\left(\frac{1}{12np}+\frac{1}{12nq}\right)} \geq e^{-\frac{1}{9}} = 0.8948 > \frac{\sqrt{\pi}}{2} = 0.8862$. For $np = 1$, $nq = 1$or 2, and $np = 2$, $nq = 2$ can easily be verified that the inequality still holds. Thus we proved the Lemma.

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