Image Reconstruction

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Outline I

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Ray Transform I

- While the Radon transform R integrates over hyperplanes in $\mathbf{R}^n$, the `ray transform` P integrates over straight lines.
- For $n = 2$, R and P differ only in the notation.
- The treatment of P parallels the one of R.
  - backprojection;
  - inversion with riesz potentials.
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Ray Transform II

- A straight line in $\mathbf{R}^n$ can be determined by

$$\{x + t\theta : t \in \mathbf{R}\}, \quad (1)$$

  - a direction $\theta \in S^{n-1}$,
  - a point $x \in \theta^{\perp}$.
- The `ray transform` P is defined by

$$(Pf)(\theta, x) = \int_{\mathbf{R}} f(x + t\theta)\, dt. \quad (2)$$

- Pf is a function on

$$T^n = \{(\theta, x) : \theta \in S^{n-1}, x \in \theta^{\perp}\}. \quad (3)$$

- Let

$$\mathcal{S}(T^n) = \{g \in C^{\infty}(T^n) : x^{\alpha}\partial_x^{\beta} g(\theta, x) \text{ bounded for } \alpha, \beta \geq 0\}. \quad (4)$$

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$P: \mathcal{S}(\mathbf{R}^n \to \mathcal{S}(T^n)$ (I)

Theorem

If $f \in \mathcal{S}$, then $Pf \in \mathcal{S}(T^n)$.

Proof. For any $\alpha, \beta \geq 0$ and $k > 0$,

$$\left| x^\alpha \partial_x^\beta (Pf)(\theta, x) \right| = \left| \int_{\mathbf{R}} x^\alpha \partial_x^\beta f(x + t\theta)\, dt \right| \quad (5)$$

$$= \left| \int_{\mathbf{R}} x^\alpha \left(\partial_x^\beta f \right) (x + t\theta)\, dt \right| \quad (6)$$

$$\leq C(\alpha, \beta, k) \int_{\mathbf{R}} \frac{|x|^{|\alpha|}}{\left(1 + |x + t\theta|^2\right)^k}\, dt \quad (7)$$

$$(8)$$

$P: \mathcal{S}(\mathbf{R}^n \to \mathcal{S}(T^n)$ (II)

- Because $x \in \theta^\perp$,

$$|x + t\theta|^2 = |x|^2 + t^2 - 2t\langle x, \theta\rangle = |x|^2 + t^2.$$

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$$\int_{\mathbf{R}} \frac{|x|^{|\alpha|}}{\left(1 + |x + t\theta|^2\right)^k} dt \leq \int_{\mathbf{R}} \frac{|x|^{|\alpha|}}{\left(1 + |x|^2 + t^2\right)^k} dt \qquad (9)$$

$$\leq \max_{r \leq 0} \int_{\mathbf{R}} \frac{r^{|\alpha|}}{\left(1 + r^2 + t^2\right)^k} dt, \qquad (10)$$

which is bounded if $k >> 1$.

□

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Operators on T^n

- Fourier transform and convolution on T^n are defined with respect to the second variable,

$$\hat{g}(\theta, \xi) = \int_{\theta^\perp} g(\theta, x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx, \quad \xi \in \theta^\perp, \tag{11}$$

$$(g * h)(\theta, x) = \int_{\theta^\perp} g(\theta, x - y) h(\theta, y) \, dy, \quad x \in \theta^\perp. \tag{12}$$

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Fourier Slice Theorem

- **Theorem**

  ([Natterer and Wübbeling, 2001, Theorem 2.11]) Let $f \in \mathcal{S}$. Then for $\theta \in S^{n-1}$, $\xi \in \theta^{\perp}$,

$$(Pf)^{\wedge}(\theta, \xi) = \hat{f}(\xi). \tag{13}$$

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Proof of Fourier Slice Theorem

- $$(Pf)^{\wedge}(\theta,\xi) = \int_{\theta^{\perp}} (Pf)(\theta,x)\mathbf{e}^{-2\pi i\xi\cdot x}\,dx \tag{14}$$
$$= \int_{\theta^{\perp}} \int_{\mathbf{R}} f(x+t\theta)\mathbf{e}^{-2\pi i\xi\cdot x}\,dtdx. \tag{15}$$

- With $y = x + t\theta$ being the new variable of integration where $x \in \theta^{\perp}$ and $t \in \mathbf{R}$, we have $\xi \cdot x = \xi \cdot y$, $dtdx = dy$, hence,

$$(Pf)^{\wedge}(\theta,\xi) = \int_{\mathbf{R}^n} f(y)\mathbf{e}^{-2\pi i\xi\cdot y}\,dy \tag{16}$$
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Convolution Theorem

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  ([Natterer and Wübbeling, 2001, Theorem 2.12]) Let $f, g \in \mathcal{S}$. Then

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- By Theorem **??** and Theorem 1.2

$$((Pf) * (Pg))^{\wedge}(\theta, \xi) = \widehat{(Pf)}(\theta, \xi)\widehat{(Pg)}(\theta, \xi) \tag{19}$$
$$= \hat{f}(\xi)\hat{g}(\xi). \tag{20}$$

- On the other hand, we have

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Backprojection

- The backprojection operator $P^{\#}$ now is

$$(P^{\#}g)(x) = \int_{S^{n-1}} g(\theta, E_\theta x)\, d\theta \tag{23}$$

where E_θ is the orthogonal projection onto $\theta^\perp$, i.e.,

$$E_\theta x = x - \langle x, \theta \rangle \theta, \quad \forall x \in \mathbf{R}^n. \tag{24}$$

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$P^{\#}$ = the adjoint of P

- **Theorem**
  ([Natterer and Wübbeling, 2001, p. 18]) For $g \in \mathcal{S}(T^n)$, and $f \in \mathcal{S}(\mathbf{R}^n)$,

$$\int_{S^{n-1}} \int_{\theta^\perp} g(\theta, x)(Pf)(\theta, x)\, dx d\theta = \int_{\mathbf{R}^n} (P^{\#} g)(x) f(x)\, dx. \tag{25}$$

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Proof of $P^{\#}$ = the adjoint of P (I)

- For $g \in \mathcal{S}(T^n)$, and $f \in \mathcal{S}(\mathbf{R}^n)$, as in the proof of Theorem 1.2, for fixed $\theta \in S^{n-1}$,

$$\int_{\theta^\perp} g(\theta, x)(Pf)(\theta, x)\, dx \tag{26}$$

$$= \int_{\theta^\perp} g(\theta, x)\, dx \int_{\mathbf{R}} f(x + t\theta)\, dt \tag{27}$$

$$= \int_{\theta^\perp} \int_{\mathbf{R}} g(\theta, x) f(x + t\theta)\, dt dx \tag{28}$$

$$= \int_{\mathbf{R}^n} g(\theta, E_\theta y) f(y)\, dy, \tag{29}$$

where $y = x + t\theta$, $x \in \theta^\perp$.

Proof of $P^{\#}$ = the adjoint of P (II)

- Hence,

$$\int_{S^{n-1}} \int_{\theta^\perp} g(\theta, x)(Pf)(\theta, x)\, dx d\theta \tag{30}$$

$$= \int_{S^{n-1}} \int_{\mathbf{R}^n} g(\theta, E_\theta y) f(y)\, dy d\theta \tag{31}$$

$$= \int_{\mathbf{R}^n} f(x)\, dx \int_{S^{n-1}} g(\theta, E_\theta y)\, d\theta \tag{32}$$

$$= \int_{\mathbf{R}^n} (P^{\#} g)(y) f(y)\, dy. \tag{33}$$

Fitered Backprojection Theorem

- **Theorem**
  ([Natterer, 2001, Theorem II.1.3], [Natterer and Wübbeling, 2001, Theorem 2.13]) Let $f \in \mathcal{S}(\mathbf{R}^n)$ and $g \in \mathcal{S}(T^n)$. Then

$$(P^{\#}g) * f = P^{\#}(g * (Pf)). \tag{34}$$

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Proof of Fitered Backprojection Theorem I

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$$\left((P^{\#} g) * f \right)(x) \tag{35}$$

$$= \int_{\mathbf{R}^n} (P^{\#} g)(x-y) f(y)\, dy \tag{36}$$

$$= \int_{\mathbf{R}^n} \int_{S^{n-1}} g(\theta, E_\theta(x-y)) f(y)\, d\theta dy \tag{37}$$

$$= \int_{S^{n-1}} d\theta \int_{\mathbf{R}^n} g(\theta, E_\theta(x-y)) f(y)\, dy. \tag{38}$$

- Making the substitution $y = z + t\theta$, $z \in \theta^\perp$ in the inner integral, we obtain

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Proof of Fitered Backprojection Theorem II

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$$\left((P^{\#}g) * f\right)(x) \tag{39}$$

$$= \int_{S^{n-1}} d\theta \int_{\theta^\perp} dz \int_{\mathbf{R}} g(\theta, E_\theta x - z) f(z + t\theta)\, dt \tag{40}$$

$$= \int_{S^{n-1}} d\theta \int_{\theta^\perp} g(\theta, E_\theta x - z)(Pf)(\theta, z)\, dz \tag{41}$$

$$= \int_{S^{n-1}} (g * (Pf))(\theta, E_\theta x)\, d\theta \tag{42}$$

$$= P^{\#}(g * (Pf))(x). \tag{43}$$

Outline

Riesz transform to T^n

- The Riesz transform on T^n is defined on the second variable,

$$(I_\alpha g)^\wedge(\theta,\xi) = \frac{1}{(2\pi|\xi|)^\alpha}\hat{g}(\theta,\xi). \tag{44}$$

- Theorem

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Riesz transform to T^n

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([Natterer, 2001, Eq. (VII.2.8)])

$$\int_{\mathbf{R}^n} h(y)\,dy = \frac{1}{|S^{n-2}|}\int_{S^{n-1}}\int_{\theta^\perp} ||y|| h(y)\,dy d\theta. \qquad (46)$$

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Proof of Theorem 1.6 (I)

- We start with the Fourier inversion formula

$$(I_\alpha f)(x) = \int_{\mathbf{R}^n} \frac{1}{(2\pi||\xi||)^\alpha} \hat{f}(\xi) \mathbf{e}^{2\pi i x\cdot\xi}\, d\xi. \tag{47}$$

- By Lemma 1.7,

$$(I_\alpha f)(x) \tag{48}$$

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Proof of Theorem 1.6 (II)

- The inner integral can be expressed by the Riesz potential on T^n, hence,

$$(I_\alpha f)(x) = \frac{1}{2\pi|S^{n-2}|}\int_{S^{n-1}} (I_{\alpha-1}Pf)(\theta, E_\theta x)\, d\theta \quad (51)$$

$$= \frac{1}{2\pi|S^{n-2}|}[P^{\#} I_{\alpha-1}Pf](x) \quad (52)$$

The inversion formula follows by applying $I_{-\alpha}$.

Remark on Theorem 1.6

- The inversion formula for P in Theorem 1.6 is not as useful as the inversion formula for R of Theorem ??.
- To get the idea, consider the case $n = 3$ and put $\alpha = 0$. Then

$$f(x) = \frac{1}{4\pi^2} \int_{S^2} (I_{-1} g)(\theta, E_\theta x)\, d\theta. \qquad (53)$$

  Thus one needs $g(\theta, y)$ for all $\theta \in S^2$ and $y \in \theta^\perp$ in order to find the function f.
- In practice, it is rarely the case that g is known on all of S^2.
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Other Properties of Ray Transform

- Other properties such as the singular value decomposition and Sobolev space estimate of P can be found in [Natterer and Wübbeling, 2001, pp. 22 — 23] and references therein.

Outline

- Ray Transform
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  - Backprojection
  - Inversion with Riesz Potentials
- Cone Beam Transform
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  - Inversion Formulas
    - Grangeat's Formula
    - Smith's Formula

Cone Beam Transform I

- Let $\theta \in S^{n-1}$ and $a \in \mathbf{R}^n$.
- The `cone beam transform` D of $f \in \mathcal{S}(\mathbf{R}^n)$ is defined by

$$(Df)(a, \theta) = \int_0^\infty f(a + t\theta)\, dt. \tag{54}$$

  a is considered as the source of a ray with direction θ.
- It follows from the definitions that

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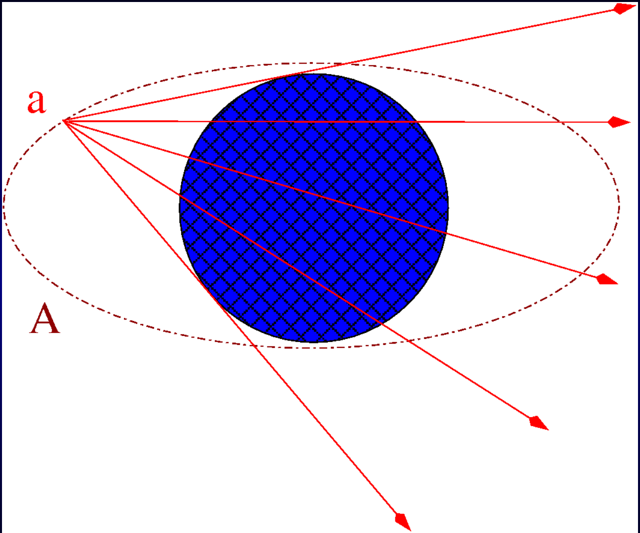

Figure: Cone beam transform.

Extension of Df

- Df can be extended to be a function on $\mathbf{R}^3 \times \mathbf{R}^3$ by putting

$$(Df)(a, y) = \int_0^\infty f(a + ty)\,dt = \frac{1}{|y|}\int_0^\infty f(a + t\frac{y}{|y|})\,dt. \quad (56)$$

- Df is extended as a function homogeneous of degree -1 in the second argument.
- Fourier transform on the second variable can also be defined,

$$(Df)^\wedge(a, \eta) = \int_{\mathbf{R}^n} (Df)(a, y)\mathbf{e}^{-2\pi i \eta \cdot y}\,dy. \quad (57)$$

- $(Df)^\wedge$ is a homogeneous function of order $1 - n$, by Theorem **??**.

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Outline

Motivation

- In a typical 3D tomographic setup,
  - the object f of compact support would be surrounded by a source curve A,
  - $(Df)(a, \theta)$ would be measured for $a \in A$ and $\theta \in S^2$.
- It is for this reason that we try to invert D.
- There relations between D and the Radon transform R, which in turn permit the inversion of D, provided the source curve A satisfies certain conditions.

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Hamaker's Formula

- All these relations are an outflow of a formula essentially obtained in [Hamaker et al., 1980].
- Theorem

Let h be a function on $\mathbf{R}$, homogeneous of degree $1-n$. Then $\forall\theta \in S^{n-1}$,

$$\int_{S^{n-1}} (Df)(a,\omega)h(\theta\cdot\omega)\,d\omega = \int_{\mathbf{R}} (Rf)(\theta,s)h(s-a\cdot\theta)\,ds. \tag{58}$$

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Proof of Hamaker's Formula

$$\int_{S^{n-1}} (Df)(a,\omega)h(\theta\cdot\omega)\,d\omega \tag{59}$$

$$= \int_{S^{n-1}}\int_0^\infty f(a+t\omega)h(\theta\cdot\omega)\,dtd\omega \tag{60}$$

$$= \int_{S^{n-1}}\int_0^\infty f(a+t\omega)h(t\theta\cdot\omega)\,t^{n-1}dtd\omega \quad \text{(by the homogeneity of } h) \tag{61}$$

$$= \int_{\mathbf{R}^n} f(a+x)h(x\cdot\theta)\,dx \tag{62}$$

$$= \int_{\mathbf{R}} ds \int_{\theta^\perp} f(a+s\theta+y)h(s)\,dy \quad (x=s\theta+y,\ y\in\theta^\perp) \tag{63}$$

$$= \int_{\mathbf{R}} h(s)\,ds \int_{\theta^\perp} f((a\cdot\theta+s)\theta+a^\perp+y)\,dy \quad (a=(a\cdot\theta)\theta+a^\perp,\ a^\perp\in\theta^\perp) \tag{64}$$

$$= \int_{\mathbf{R}} h(s)\,ds \int_{\theta^\perp} f((a\cdot\theta+s)\theta+y)\,dy \tag{65}$$

$$= \int_{\mathbf{R}} h(s)(Rf)(\theta, a\cdot\theta+s)\,ds \quad \text{(by (??))} \tag{66}$$

$$= \int_{\mathbf{R}} (Rf)(\theta,s)h(s-a\cdot\theta)\,ds. \tag{67}$$

Grangeat's Formula I

- For $n = 3$, the 1D δ' is of homogeneous $-1 - 1 = -2 = 1 - n$, see Eq. (**??**).
- Let $h = \delta'$. The right-hand side of Eq. (58) is

$$\int_{\mathbf{R}} (Rf)(\theta, s)\delta'(s - a \cdot \theta)\, ds \tag{68}$$

$$= -\int_{\mathbf{R}} \frac{\partial}{\partial s}(Rf)(\theta, s)\delta(s - a \cdot \theta)\, ds \tag{69}$$

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Grangeat's Formula II

- Hence, by Eq. (58),

$$\frac{\partial}{\partial s}(Rf)(\theta, s)\Big|_{s=a\cdot\theta} = -\int_{S^2} (Df)(a,\omega)\delta'(\theta\cdot\omega)\, d\omega. \tag{71}$$

- For any function $g \in \mathcal{S}$ and $\theta \in S^2$,

$$\sum_i \theta_i \frac{\partial}{\partial \omega_i}[g(\theta\cdot\omega)] = \sum_i \theta_i^2 g'(\theta\cdot\omega) = g'(\theta\cdot\omega). \tag{72}$$

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Grangeat's Formula III

- Therefore,

$$\int_{S^2} (Df)(a,\omega) g'(\theta\cdot\omega)\, d\omega \tag{73}$$

$$= \int_{S^2} (Df)(a,\omega) \sum_i \theta_i \frac{\partial}{\partial\omega_i} [g(\theta\cdot\omega)]\ d\omega \tag{74}$$

$$= -\int_{S^2} \sum_i \theta_i \frac{\partial}{\partial\omega_i} [(Df)(a,\omega)]\, g(\theta\cdot\omega)\, d\omega \tag{75}$$

$$= -\int_{S^2} \nabla_\theta [(Df)(a,\omega)]\, g(\theta\cdot\omega)\, d\omega \tag{76}$$

where ∇_θ is the directional derivative in the direction θ, acting on the second argument of Df.

Grangeat's Formula IV

- Hence, by Eq. (**??**),

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Grangeat's formula V

- **Theorem**
  ([Grangeat, 1991], [Natterer and Wübbeling, 2001, Theorem 2.2.19]) Let $f \in \mathcal{S}(\mathbf{R}^3)$. Then for $\theta \in S^2$, $a \in \mathbf{R}^3$,

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Smith's Formula

- Let

$$h(s) = \int_{\mathbf{R}} |\sigma| \mathbf{e}^{-2\pi i \sigma s} \, d\sigma. \tag{81}$$

- h is homogeneous of order $-1 - 1 = -2$.

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Smith's Formula II

- For the right-hand side of Eq. (58),

$$
\begin{aligned}
&((Rf) * h)^{\wedge}(\theta, \sigma) && (82)\\
=&(Rf)^{\wedge}(\theta, \sigma) \cdot \hat{h}(\sigma) && (83)\\
=&|\sigma|(Rf)^{\wedge}(\theta, \sigma) && (84)\\
=&\operatorname{sgn}(\sigma)\sigma(Rf)^{\wedge}(\theta, \sigma) && (85)\\
=&\frac{1}{2\pi i}\operatorname{sgn}(\sigma)\left(\frac{\partial}{\partial s}(Rf)\right)^{\wedge}(\theta, \sigma) && (86)\\
=&\frac{1}{2\pi}(-i\operatorname{sgn}(\sigma))\left(\frac{\partial}{\partial s}(Rf)\right)^{\wedge}(\theta, \sigma). && (87)
\end{aligned}
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Smith's Formula III

- By the definition of Hilbert transform Eq. (**??**),

$$Rf * h = \frac{1}{2\pi} H(\frac{\partial}{\partial s}(Rf)). \qquad (88)$$

- The right-hand side of Eq. (58) is then equal to

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Smith's Formula IV

- For the left-hand side of Eq. (58),

$$(Df)^\wedge(a,\theta) = \int_{\mathbf{R}^3} (Df)(a,x)\mathbf{e}^{-2\pi i\theta\cdot x}\,dx \tag{90}$$

$$= \int_0^\infty r^2\,dr \int_{S^2} \mathbf{e}^{-2\pi i r\theta\cdot\omega}(Df)(a,r\omega)\,d\omega \tag{91}$$

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- For the left-hand side of Eq. (58),

$$(Df)^\wedge(a,\theta) = \int_{\mathbf{R}^3} (Df)(a,x)\mathbf{e}^{-2\pi i\theta\cdot x}\, dx \tag{90}$$

$$= \int_0^\infty r^2\, dr \int_{S^2} \mathbf{e}^{-2\pi i r\theta\cdot\omega}(Df)(a,r\omega)\, d\omega \tag{91}$$

$$= \int_0^\infty r\, dr \int_{S^2} \mathbf{e}^{-2\pi i r\theta\cdot\omega}(Df)(a,\omega)\, d\omega \quad \text{(by Eq. (56))} \tag{92}$$

$$= \int_{S^2}\int_0^\infty (Df)(a,\omega) r\mathbf{e}^{-2\pi i r\theta\cdot\omega}\, dr\, d\omega \tag{93}$$

- and

$$(Df)^\wedge(a,-\theta) = \int_{S^2}\int_0^\infty (Df)(a,\omega) r\mathbf{e}^{2\pi i r\theta\cdot\omega}\, dr\, d\omega \tag{94}$$

$$= \int_{S^2}\int_{-\infty}^0 (Df)(a,\omega)|r|\mathbf{e}^{-2\pi i r\theta\cdot\omega}\, dr\, d\omega. \tag{95}$$

- Hence,

$$(Df)^\wedge(a,\theta) + (Df)^\wedge(a,-\theta) = \int_{S^2}\int_{\mathbf{R}} (Df)(a,\omega)|r|\mathbf{e}^{-2\pi i r\theta\cdot\omega}\, dr\, d\omega \tag{96}$$

$$= \int_{S^2} (Df)(a,\omega) h(\theta\cdot\omega). \tag{97}$$

Smith's Formula IV

- Therefore, by Eq. (58) and Eq. (89),

$$\frac{1}{2\pi} H(\frac{\partial}{\partial s}(Rf))(\theta, s)\Big|_{s=a\cdot\theta} = (Df)^\wedge(a,\theta) + (Df)^\wedge(a,-\theta). \tag{98}$$

- This is `Smith's formula` [Smith, 1985].

Smith's Formula IV

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