Image Reconstruction

Ming Jiang

School of Mathematical Sciences
Peking University

ming-jiang@pku.edu.cn

March 21, 2011

Outline I

Introduction

- This lecture is a review on functional analsysis and Fourier analysis for L^1 and L^2 functions and tempered distributions.
- Various function spaces will be briefly reviewed.
- References are [Stein, 1970, Stein and Weiss, 1971, Yosida, 1980, Hömander, 1990, Rudin, 1991, Natterer, 2001, Natterer and Wübbeling, 2001].

Introduction

- This lecture is a review on functional analsysis and Fourier analysis for L^1 and L^2 functions and tempered distributions.
- Various function spaces will be briefly reviewed.
- References are [Stein, 1970, Stein and Weiss, 1971, Yosida, 1980, Hömander, 1990, Rudin, 1991, Natterer, 2001, Natterer and Wübbeling, 2001].

Introduction

- This lecture is a review on functional analsysis and Fourier analysis for L^1 and L^2 functions and tempered distributions.
- Various function spaces will be briefly reviewed.
- References are [Stein, 1970, Stein and Weiss, 1971, Yosida, 1980, Hömander, 1990, Rudin, 1991, Natterer, 2001, Natterer and Wübbeling, 2001].

Outline

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field $\mathbf{K}$ if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  - i (associativity of addition)
  $$(x+y)+z = x+(y+z), \qquad \forall x,y,z \in X; \tag{1}$$
  - ii (commutativity of addition)
  $$x+y = y+x, \qquad \forall x,y \in X; \tag{2}$$
  - iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that
  $$x+\theta = x, \qquad \forall x \in X; \tag{3}$$
  - iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that
  $$x+u = \theta. \tag{4}$$

  The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field $\mathbf{K}$ if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  i (associativity of addition)
  $$(x+y)+z = x+(y+z), \qquad \forall x,y,z \in X; \tag{1}$$
  ii (commutativity of addition)
  $$x+y = y+x, \qquad \forall x,y \in X; \tag{2}$$
  iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that
  $$x+\theta = x, \qquad \forall x \in X; \tag{3}$$
  iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that
  $$x+u = \theta. \tag{4}$$
  The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field **K** if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  - i (associativity of addition)

$$(x + y) + z = x + (y + z), \qquad \forall x, y, z \in X; \tag{1}$$

  - ii (commutativity of addition)

$$x + y = y + x, \qquad \forall x, y \in X; \tag{2}$$

  - iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that

$$x + \theta = x, \qquad \forall x \in X; \tag{3}$$

  - iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that

$$x + u = \theta. \tag{4}$$

The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field $\mathbf{K}$ if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  i (associativity of addition)

$$(x+y)+z = x+(y+z), \qquad \forall x, y, z \in X; \tag{1}$$

  ii (commutativity of addition)

$$x+y = y+x, \qquad \forall x, y \in X; \tag{2}$$

  iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that

$$x+\theta = x, \qquad \forall x \in X; \tag{3}$$

  iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that

$$x+u = \theta. \tag{4}$$

  The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field **K** if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  - i (associativity of addition)
  $$(x+y)+z = x+(y+z), \qquad \forall x,y,z \in X; \tag{1}$$
  - ii (commutativity of addition)
  $$x+y = y+x, \qquad \forall x,y \in X; \tag{2}$$
  - iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that
  $$x+\theta = x, \qquad \forall x \in X; \tag{3}$$
  - iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that
  $$x+u = \theta. \tag{4}$$
  The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces I

- A set X is called a `vector space` or `liner space` over a field $\mathbf{K}$ if the following conditions are satisfied.
- An addition $+$ is defined on X such that X is an `abelian group`,
  - i (associativity of addition)
  $$(x + y) + z = x + (y + z), \qquad \forall x, y, z \in X; \qquad (1)$$
  - ii (commutativity of addition)
  $$x + y = y + x, \qquad \forall x, y \in X; \qquad (2)$$
  - iii (identity element of addition) There exists an element $\theta \in X$, called the zero vector, such that
  $$x + \theta = x, \qquad \forall x \in X; \qquad (3)$$
  - iv (inverse elements of addition) for all $x \in X$, there exists an element $u \in X$, called the inverse of x with repect to the addition $+$, such that
  $$x + u = \theta. \qquad (4)$$
  The inverse is denoted by $-x$, $\forall x \in X$.

Vector Spaces II

- A scalar multiplication is defined as a map from $\mathbf{K} \times X \to X$ denoted as $(\alpha, x) \in \mathbf{K} \times \to \alpha x \in X$ for such that

  v (distributivity of scalar multiplication with respect to vector addition)

$$\alpha(x + y) = \alpha x + \alpha y, \qquad \forall \alpha \in \mathbf{K}, \forall x, y \in X; \tag{5}$$

  vi (distributivity of scalar multiplication with respect to field addition)

$$(\alpha + \beta)x = \alpha x + \beta x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{6}$$

  vii (compatibility of scalar multiplication with field multiplication)

$$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{7}$$

  viii (identity element of scalar multiplication)

$$1x = x, \qquad \forall x \in X, \tag{8}$$

  where 1 is the multiplicative identity in the field **K**.

Vector Spaces II

- A `scalar multiplication` is defined as a map from $\mathbf{K} \times X \to X$ denoted as $(\alpha, x) \in \mathbf{K} \times \to \alpha x \in X$ for such that

  v (distributivity of scalar multiplication with respect to vector addition)

$$\alpha(x+y) = \alpha x + \alpha y, \qquad \forall \alpha \in \mathbf{K}, \forall x, y \in X; \tag{5}$$

  vi (distributivity of scalar multiplication with respect to field addition)

$$(\alpha+\beta)x = \alpha x + \beta x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{6}$$

  vii (compatibility of scalar multiplication with field multiplication)

$$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{7}$$

  viii (identity element of scalar multiplication)

$$1x = x, \qquad \forall x \in X, \tag{8}$$

  where 1 is the multiplicative identity in the field $\mathbf{K}$.

Vector Spaces II

- A `scalar multiplication` is defined as a map from $\mathbf{K} \times X \to X$ denoted as $(\alpha, x) \in \mathbf{K} \times \to \alpha x \in X$ for such that

  v (distributivity of scalar multiplication with respect to vector addition)

  $$\alpha(x+y) = \alpha x + \alpha y, \qquad \forall \alpha \in \mathbf{K}, \forall x, y \in X; \qquad (5)$$

  vi (distributivity of scalar multiplication with respect to field addition)

  $$(\alpha+\beta)x = \alpha x + \beta x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \qquad (6)$$

  vii (compatibility of scalar multiplication with field multiplication)

  $$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \qquad (7)$$

  viii (identity element of scalar multiplication)

  $$1x = x, \qquad \forall x \in X, \qquad (8)$$

  where 1 is the multiplicative identity in the field $\mathbf{K}$.

Vector Spaces II

- A `scalar multiplication` is defined as a map from $\mathbf{K} \times X \to X$ denoted as $(\alpha, x) \in \mathbf{K} \times \to \alpha x \in X$ for such that

  v (distributivity of scalar multiplication with respect to vector addition)

$$\alpha(x+y) = \alpha x + \alpha y, \qquad \forall \alpha \in \mathbf{K}, \forall x, y \in X; \tag{5}$$

  vi (distributivity of scalar multiplication with respect to field addition)

$$(\alpha+\beta)x = \alpha x + \beta x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{6}$$

  vii (compatibility of scalar multiplication with field multiplication)

$$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{7}$$

  viii (identity element of scalar multiplication)

$$1x = x, \qquad \forall x \in X, \tag{8}$$

  where 1 is the multiplicative identity in the field **K**.

Vector Spaces II

- A `scalar multiplication` is defined as a map from $\mathbf{K} \times X \to X$ denoted as $(\alpha, x) \in \mathbf{K} \times \to \alpha x \in X$ for such that

v (distributivity of scalar multiplication with respect to vector addition)

$$\alpha(x + y) = \alpha x + \alpha y, \qquad \forall \alpha \in \mathbf{K}, \forall x, y \in X; \tag{5}$$

vi (distributivity of scalar multiplication with respect to field addition)

$$(\alpha + \beta)x = \alpha x + \beta x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{6}$$

vii (compatibility of scalar multiplication with field multiplication)

$$\alpha(\beta x) = (\alpha\beta)x, \qquad \forall \alpha, \beta \in \mathbf{K}, \forall x \in X; \tag{7}$$

viii (identity element of scalar multiplication)

$$1x = x, \qquad \forall x \in X, \tag{8}$$

where 1 is the multiplicative identity in the field **K**.

Vector Spaces III

- References
  - i [Yosida, 1980, p. 20];
  - ii [Rudin, 1991, Chapter 1];
  - ii Vector space at Wikipedia.

Vector Spaces III

- References
  i [Yosida, 1980, p. 20];
  ii [Rudin, 1991, Chapter 1];
  ii Vector space at Wikipedia.

Vector Spaces III

- References
  i [Yosida, 1980, p. 20];
  ii [Rudin, 1991, Chapter 1];
  ii Vector space at Wikipedia.

Vector Spaces III

- References
  i [Yosida, 1980, p. 20];
  ii [Rudin, 1991, Chapter 1];
  ii Vector space at Wikipedia.

Inner Product Spaces I

- A vector space X is called a `inner product space` if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y \rangle$, called the `inner product` of x and y, such that the following rules hold,

  i
  $$\langle x, y \rangle = \overline{\langle y, x \rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;

  ii
  $$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \forall x, y, z \in X, \tag{10}$$

  iii
  $$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$

  iv
  $$\langle x, x \rangle \geq 0, \qquad \forall x \in X; \tag{12}$$

  v
  $$\langle x, x \rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces I

- A vector space X is called a inner product space if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y \rangle$, called the inner product of x and y, such that the following rules hold,

  i
  $$\langle x, y \rangle = \overline{\langle y, x \rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;

  ii
  $$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \forall x, y, z \in X, \tag{10}$$

  iii
  $$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$

  iv
  $$\langle x, x \rangle \geq 0, \qquad \forall x \in X; \tag{12}$$

  v
  $$\langle x, x \rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces I

- A vector space X is called a `inner product space` if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y\rangle$, called the `inner product` of x and y, such that the following rules hold,

  i
  $$\langle x, y\rangle = \overline{\langle y, x\rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;

  ii
  $$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \qquad \forall x, y, z \in X, \tag{10}$$

  iii
  $$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$

  iv
  $$\langle x, x\rangle \geq 0, \qquad \forall x \in X; \tag{12}$$

  v
  $$\langle x, x\rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces I

- A vector space X is called a `inner product space` if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y\rangle$, called the `inner product` of x and y, such that the following rules hold,
  i
  $$\langle x, y\rangle = \overline{\langle y, x\rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;
  ii
  $$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \qquad \forall x, y, z \in X, \tag{10}$$
  iii
  $$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$
  iv
  $$\langle x, x\rangle \geq 0, \qquad \forall x \in X; \tag{12}$$
  v
  $$\langle x, x\rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces I

- A vector space X is called a inner product space if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y \rangle$, called the inner product of x and y, such that the following rules hold,
  i
  $$\langle x, y \rangle = \overline{\langle y, x \rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;
  ii
  $$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \forall x, y, z \in X, \tag{10}$$
  iii
  $$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$
  iv
  $$\langle x, x \rangle \geq 0, \qquad \forall x \in X; \tag{12}$$
  v
  $$\langle x, x \rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces I

- A vector space X is called a `inner product space` if to each pair of vectors x and $y \in X$ is associated a number $\langle x, y \rangle$, called the `inner product` of x and y, such that the following rules hold,

  i
  $$\langle x, y \rangle = \overline{\langle y, x \rangle}, \qquad \forall x, y \in X, \tag{9}$$
  where the overline denotes complex conjugation;

  ii
  $$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \forall x, y, z \in X, \tag{10}$$

  iii
  $$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{11}$$

  iv
  $$\langle x, x \rangle \geq 0, \qquad \forall x \in X; \tag{12}$$

  v
  $$\langle x, x \rangle = 0, \qquad \text{only if } x = \theta. \tag{13}$$

Inner Product Spaces II

- References
  - [Rudin, 1991, Chapter 12].

Inner Product Spaces II

- References
  - [Rudin, 1991, Chapter 12].

Normed Spaces I

- A vector space X is said to be a normed space if to every $x \in X$ is associated a nonnegative real number $||x||$, called the norm of x, such that the following rules hold,
  i
  $$||x + y|| \leq ||x|| + ||y||, \qquad \forall x, y \in X; \tag{14}$$
  ii
  $$||\alpha x|| = |\alpha| ||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{15}$$
  iii
  $$||x|| \geq 0, \qquad \text{if } x \neq \theta. \tag{16}$$
- References
  i [Rudin, 1991, Chapter 1].

Normed Spaces I

- A vector space X is said to be a `normed space` if to every $x \in X$ is associated a nonnegative real number $||x||$, called the `norm` of x, such that the following rules hold,
  i
  $$||x + y|| \leq ||x|| + ||y||, \qquad \forall x, y \in X; \qquad (14)$$
  ii
  $$||\alpha x|| = |\alpha| ||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \qquad (15)$$
  iii
  $$||x|| \geq 0, \qquad \text{if } x \neq \theta. \qquad (16)$$

- References
  i [Rudin, 1991, Chapter 1].

Normed Spaces I

- A vector space X is said to be a `normed space` if to every $x \in X$ is associated a nonnegative real number $||x||$, called the `norm` of x, such that the following rules hold,

  i
  $$||x+y|| \leq ||x|| + ||y||, \qquad \forall x, y \in X; \tag{14}$$

  ii
  $$||\alpha x|| = |\alpha| ||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{15}$$

  iii
  $$||x|| \geq 0, \qquad \text{if } x \neq \theta. \tag{16}$$

- References
  i [Rudin, 1991, Chapter 1].

Normed Spaces I

- A vector space X is said to be a `normed space` if to every $x \in X$ is associated a nonnegative real number $||x||$, called the `norm` of x, such that the following rules hold,

  i
  $$||x+y|| \leq ||x|| + ||y||, \qquad \forall x, y \in X; \tag{14}$$
  ii
  $$||\alpha x|| = |\alpha| ||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{15}$$
  iii
  $$||x|| \geq 0, \qquad \text{if } x \neq \theta. \tag{16}$$

- References

  i [Rudin, 1991, Chapter 1].

Normed Spaces I

- A vector space X is said to be a `normed space` if to every $x \in X$ is associated a nonnegative real number $||x||$, called the `norm` of x, such that the following rules hold,
  i
  $$||x + y|| \leq ||x|| + ||y||, \qquad \forall x, y \in X; \tag{14}$$
  ii
  $$||\alpha x|| = |\alpha|||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{15}$$
  iii
  $$||x|| \geq 0, \qquad \text{if } x \neq \theta. \tag{16}$$
- References
  i [Rudin, 1991, Chapter 1].

Normed Spaces I

- A vector space X is said to be a `normed space` if to every $x \in X$ is associated a nonnegative real number $||x||$, called the `norm` of x, such that the following rules hold,

  i
  $$||x+y|| \le ||x|| + ||y||, \qquad \forall x, y \in X; \tag{14}$$
  ii
  $$||\alpha x|| = |\alpha| ||x||, \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{15}$$
  iii
  $$||x|| \ge 0, \qquad \text{if } x \ne \theta. \tag{16}$$

- References

  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,

  i
  $$0 \leq d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  ii
  $$d(x, y) = 0, \text{ if and only if } x = y; \tag{18}$$
  iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$
  iv
  $$d(x, y) \leq d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$

- References

  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,

  i
  $$0 \leq d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  ii
  $$d(x, y) = 0, \text{ if and only if } x = y; \tag{18}$$
  iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$
  iv
  $$d(x, y) \leq d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$

- References

  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,

  i
  $$0 \leq d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$

  ii
  $$d(x, y) = 0, \text{if and only if } x = y; \tag{18}$$

  iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$

  iv
  $$d(x, y) \leq d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$

- References

  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,
  i
  $$0 \le d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  ii
  $$d(x, y) = 0, \text{ if and only if } x = y; \tag{18}$$
  iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$
  iv
  $$d(x, y) \le d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$
- References
  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,
  i
  $$0 \leq d(x,y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  ii
  $$d(x,y) = 0, \text{ if and only if } x = y; \tag{18}$$
  iii
  $$d(x,y) = d(y,x), \qquad \forall x, y \in X; \tag{19}$$
  iv
  $$d(x,y) \leq d(x,z) + d(z,y), \qquad \forall x, y, z \in X; \tag{20}$$
- References
  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,
  i
  $$0 \leq d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  ii
  $$d(x, y) = 0, \text{if and only if } x = y; \tag{18}$$
  iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$
  iv
  $$d(x, y) \leq d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$
- References
  i [Rudin, 1991, Chapter 1].

Metric Spaces

- A set X is said to be a `metric space` if to every pair $x, y \in X$ is associated a nonnegative real number $d(x, y)$, called the `distance` between x and y, such that the following rules hold,
  - i
  $$0 \le d(x, y) < \infty, \qquad \forall x, y \in X; \tag{17}$$
  - ii
  $$d(x, y) = 0, \text{if and only if } x = y; \tag{18}$$
  - iii
  $$d(x, y) = d(y, x), \qquad \forall x, y \in X; \tag{19}$$
  - iv
  $$d(x, y) \le d(x, z) + d(z, y), \qquad \forall x, y, z \in X; \tag{20}$$
- References
  - i [Rudin, 1991, Chapter 1].

Inner Product Spaces, Noremed Spaces, and Metric Spaces

- Every inner product space in slide 5 is a normed space, by defining its norm as

$$||x|| = \sqrt{\langle x, x \rangle}. \tag{21}$$

- Every normed space in slide 7 is a metric space, by defining its metric as

$$d(x, y) = ||x - y||. \tag{22}$$

Inner Product Spaces, Noremed Spaces, and Metric Spaces

- Every inner product space in slide 5 is a normed space, by defining its norm as

$$||x|| = \sqrt{\langle x, x \rangle}. \tag{21}$$

- Every normed space in slide 7 is a metric space, by defining its metric as

$$d(x, y) = ||x - y||. \tag{22}$$

Topological Spaces

- A topological space is a set X in which a collection τ of subsets (called open sets) has been specified, with the following properties,
  i $X \in \tau$, $\emptyset \in \tau$;
  ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  i [Rudin, 1991, Chapter 1].

Topological Spaces

- A `topological space` is a set X in which a collection τ of subsets (called `open sets`) has been specified, with the following properties,
  - i $X \in \tau$, $\emptyset \in \tau$;
  - ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  - iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  - i [Rudin, 1991, Chapter 1].

Topological Spaces

- A topological space is a set X in which a collection τ of subsets (called open sets) has been specified, with the following properties,
  - i $X \in \tau$, $\emptyset \in \tau$;
  - ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  - iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  - i [Rudin, 1991, Chapter 1].

Topological Spaces

- A topological space is a set X in which a collection τ of subsets (called open sets) has been specified, with the following properties,
  i $X \in \tau$, $\emptyset \in \tau$;
  ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  i [Rudin, 1991, Chapter 1].

Topological Spaces

- A `topological space` is a set X in which a collection τ of subsets (called `open sets`) has been specified, with the following properties,
  - i $X \in \tau$, $\emptyset \in \tau$;
  - ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  - iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  - i [Rudin, 1991, Chapter 1].

Topological Spaces

- A `topological space` is a set X in which a collection τ of subsets (called `open sets`) has been specified, with the following properties,
  i $X \in \tau$, $\emptyset \in \tau$;
  ii (finite intersection) $A \bigcap B \in \tau$ if A and $B \in \tau$;
  iii (arbitrary union) if $A_\lambda \in \tau$, with $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.
- References
  i [Rudin, 1991, Chapter 1].

Topology in Metric Spaces

- In a metric space (X, d), the `open ball` with center at $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\} . \qquad (23)$$

- A subset $A \subset X$ is defined to be open if for every $a \in A$, there exists a ball with center at a and radius $\epsilon > 0$ such that $B_\epsilon(a) \subset A$.
- Metrics are topological spaces in this way.
- References
  - [Rudin, 1991, Chapter 1].

Topology in Metric Spaces

- In a metric space (X, d), the `open ball` with center at $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\}. \tag{23}$$

- A subset $A \subset X$ is defined to be open if for every $a \in A$, there exists a ball with center at a and radius $\epsilon > 0$ such that $B_\epsilon(a) \subset A$.
- Metrics are topological spaces in this way.
- References
  - [Rudin, 1991, Chapter 1].

Topology in Metric Spaces

- In a metric space (X, d), the `open ball` with center at $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\} . \tag{23}$$

- A subset $A \subset X$ is defined to be open if for every $a \in A$, there exists a ball with center at a and radius $\epsilon > 0$ such that $B_\epsilon(a) \subset A$.
- Metrics are topological spaces in this way.
- References
  - [Rudin, 1991, Chapter 1].

Topology in Metric Spaces

- In a metric space (X, d), the `open ball` with center at $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\} . \tag{23}$$

- A subset $A \subset X$ is defined to be open if for every $a \in A$, there exists a ball with center at a and radius $\epsilon > 0$ such that $B_\epsilon(a) \subset A$.
- Metrics are topological spaces in this way.
- References
  - [Rudin, 1991, Chapter 1].

Topology in Metric Spaces

- In a metric space (X, d), the `open ball` with center at $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\}. \tag{23}$$

- A subset $A \subset X$ is defined to be open if for every $a \in A$, there exists a ball with center at a and radius $\epsilon > 0$ such that $B_\epsilon(a) \subset A$.
- Metrics are topological spaces in this way.
- References
  i [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is closed if and only if its complement is open;
- the closure $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the interior of a subset E is the union of all open sets that are subsests of E;
- a neighborhood of a point $x \in X$ is any open set that contains x;
- τ is a Hausdorff topology if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Topological Definitions

In a topology space (X, τ),

- a subset $E \subset X$ is `closed` if and only if its complement is open;
- the `closure` $\overline{E}$ of a subset E is the intersection of all closed sets that contain E;
- the `interior` of a subset E is the union of all open sets that are subsests of E;
- a `neighborhood` of a point $x \in X$ is any open set that contains x;
- τ is a `Hausdorff topology` if distinct points of X have disjoint neighborhoods.
- References
  - [Rudin, 1991, Chapter 1].

Convergence Definitions

- In a Hausdorff topology space (X, τ), a sequence $\{x_n\}$ converges to a point $x \in X$,
$$\lim_n x_n = x, \tag{24}$$
if every neighborhood of x contains all but finitely many of the points x_n.
- In a metric space (X, d), a sequence $\{x_n\}$ converges to a point $x \in X$, if and only if
$$\lim_n d(x_n, x) = 0. \tag{25}$$
- In a normed space $(X, ||\cdot||)$, a sequence $\{x_n\}$ converges to a point $x \in X$, if and only if
$$\lim_n ||x_n - x|| = 0. \tag{26}$$
- References
  - [Rudin, 1991, Chapter 1].

Convergence Definitions

- In a Hausdorff topology space (X, τ), a sequence $\{x_n\}$ `converges` to a point $x \in X$,

$$\lim_n x_n = x, \tag{24}$$

  if every neighborhood of x contains all but finitely many of the points x_n.

- In a metric space (X, d), a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n d(x_n, x) = 0. \tag{25}$$

- In a normed space $(X, ||\cdot||)$, a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n ||x_n - x|| = 0. \tag{26}$$

- References
  - [Rudin, 1991, Chapter 1].

Convergence Definitions

- In a Hausdorff topology space (X, τ), a sequence $\{x_n\}$ converges to a point $x \in X$,

$$\lim_n x_n = x, \tag{24}$$

  if every neighborhood of x contains all but finitely many of the points x_n.
- In a metric space (X, d), a sequence $\{x_n\}$ converges to a point $x \in X$, if and only if

$$\lim_n d(x_n, x) = 0. \tag{25}$$

- In a normed space $(X, ||\cdot||)$, a sequence $\{x_n\}$ converges to a point $x \in X$, if and only if

$$\lim_n ||x_n - x|| = 0. \tag{26}$$

- References
  - [Rudin, 1991, Chapter 1].

Convergence Definitions

- In a Hausdorff topology space (X, τ), a sequence $\{x_n\}$ `converges` to a point $x \in X$,

$$\lim_n x_n = x, \tag{24}$$

  if every neighborhood of x contains all but finitely many of the points x_n.
- In a metric space (X, d), a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n d(x_n, x) = 0. \tag{25}$$

- In a normed space $(X, ||\cdot||)$, a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n ||x_n - x|| = 0. \tag{26}$$

- References
  - [Rudin, 1991, Chapter 1].

Convergence Definitions

- In a Hausdorff topology space (X, τ), a sequence $\{x_n\}$ `converges` to a point $x \in X$,

$$\lim_n x_n = x, \quad (24)$$

if every neighborhood of x contains all but finitely many of the points x_n.

- In a metric space (X, d), a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n d(x_n, x) = 0. \quad (25)$$

- In a normed space $(X, ||\cdot||)$, a sequence $\{x_n\}$ `converges` to a point $x \in X$, if and only if

$$\lim_n ||x_n - x|| = 0. \quad (26)$$

- References
  - i [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \quad (27)$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

  whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Cauchy Sequences and Completeness

- In a metric space (X, d), a sequence $\{x_n\}$ is a `Cauchy sequence`, if to every $\varepsilon > 0$, there exists an integer N, such that

$$d(x_m, x_n) < \varepsilon, \tag{27}$$

whenever $m > N$ and $n > N$.

- If every Cauchy sequence in (X, d) converges to a point of X, then d is said to be a `complete metric` on X.
- (X, d) is said to be a `complete metric space`.
- Complete normed spaces are called `Banach spaces`.
- Complete inner product spaces are called `Hilbert spaces`.
- References
  - [Rudin, 1991, Chapter 1].

Continuity Definitions

Let f be a map from a topology space X to another topology space Y.

- For $x_0 \in X$, f is `continuous` at x_0, if for every neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V_{f(x_0)}$.
- f is `continuous` if f is continuous at every $x \in X$.
- If X and Y are metric spaces with metrics d and ρ, respectively, f is `continuous` at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that

$$\rho(f(x), f(x_0)) < \varepsilon, \tag{28}$$

whenever $d(x, x_0) < \delta$.
- References
  - [Dugundji, 1966, Chapter III and IX].

Continuity Definitions

Let f be a map from a topology space X to another topology space Y.

- For $x_0 \in X$, f is `continuous` at x_0, if for every neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V_{f(x_0)}$.
- f is `continuous` if f is continuous at every $x \in X$.

- If X and Y are metric spaces with metrics d and ρ, respectively, f is `continuous` at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that

$$\rho(f(x), f(x_0)) < \varepsilon, \tag{28}$$

whenever $d(x, x_0) < \delta$.
- References
  - [Dugundji, 1966, Chapter III and IX].

Continuity Definitions

Let f be a map from a topology space X to another topology space Y.

- For $x_0 \in X$, f is `continuous` at x_0, if for every neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V_{f(x_0)}$.
- f is `continuous` if f is continuous at every $x \in X$.
- If X and Y are metric spaces with metrics d and ρ, respectively, f is `continuous` at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that

$$\rho(f(x), f(x_0)) < \varepsilon, \tag{28}$$

whenever $d(x, x_0) < \delta$.

- References
  - [Dugundji, 1966, Chapter III and IX].

Continuity Definitions

Let f be a map from a topology space X to another topology space Y.

- For $x_0 \in X$, f is `continuous` at x_0, if for every neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V_{f(x_0)}$.
- f is `continuous` if f is continuous at every $x \in X$.

- If X and Y are metric spaces with metrics d and ρ, respectively, f is `continuous` at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that

$$\rho(f(x), f(x_0)) < \varepsilon, \tag{28}$$

whenever $d(x, x_0) < \delta$.
- References
  - [Dugundji, 1966, Chapter III and IX].

Continuity Definitions

Let f be a map from a topology space X to another topology space Y.

- For $x_0 \in X$, f is `continuous` at x_0, if for every neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset V_{f(x_0)}$.
- f is `continuous` if f is continuous at every $x \in X$.

- If X and Y are metric spaces with metrics d and ρ, respectively, f is `continuous` at x_0, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that

$$\rho(f(x), f(x_0)) < \varepsilon, \tag{28}$$

whenever $d(x, x_0) < \delta$.
- References
  - [Dugundji, 1966, Chapter III and IX].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  - i every point of X is a closed set;
  - ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- **Theorem**

  Every topological vector space is a Hausdorff space.

  - References
    - i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  i every point of X is a closed set;
  ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- Theorem
  Every topological vector space is a Hausdorff space.
  - References
    i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  - i every point of X is a closed set;
  - ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a vector topology on X and X is a topological vector space.

- **Theorem**

  Every topological vector space is a Hausdorff space.

- References
  - i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  i every point of X is a closed set;
  ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- **Theorem**
  Every topological vector space is a Hausdorff space.
  - References
    i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  - i every point of X is a closed set;
  - ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- Theorem
  Every topological vector space is a Hausdorff space.
  - References
    - i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  - i every point of X is a closed set;
  - ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- **Theorem**

  Every topological vector space is a Hausdorff space.
  - References
    - i [Rudin, 1991, Chapter 1].

Topological Vector Spaces

- Suppose τ is a topology on a vector space such that
  i every point of X is a closed set;
  ii the vector space operations, $+$ and $\cdot$, are continuous with respect to τ;

  τ is said to be a `vector topology` on X and X is a `topological vector space`.

- Theorem
  Every topological vector space is a Hausdorff space.
  - References
    i [Rudin, 1991, Chapter 1].

Seminorms

- A seminorm on a vector space X is a real valued function p such that
  i
  $$p(x+y) \leq p(x) + p(y), \qquad \forall x, y \in X; \qquad (29)$$
  ii
  $$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \qquad (30)$$
- A family $\mathcal{P}$ of seminorms on X is said to be separating if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References
  i [Rudin, 1991, Chapter 1].

Seminorms

- A seminorm on a vector space X is a real valued function p such that
  i
  $$p(x+y) \le p(x) + p(y), \qquad \forall x, y \in X; \qquad (29)$$
  ii
  $$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \qquad (30)$$
- A family $\mathcal{P}$ of seminorms on X is said to be separating if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References
  i [Rudin, 1991, Chapter 1].

Seminorms

- A `seminorm` on a vector space X is a real valued function p such that
  - i
$$p(x+y) \le p(x) + p(y), \qquad \forall x, y \in X; \tag{29}$$
  - ii
$$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{30}$$
- A family $\mathcal{P}$ of seminorms on X is said to be `separating` if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References
  - i [Rudin, 1991, Chapter 1].

Seminorms

- A seminorm on a vector space X is a real valued function p such that
  - i
  $$p(x+y) \leq p(x) + p(y), \qquad \forall x, y \in X; \qquad (29)$$
  - ii
  $$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \qquad (30)$$
- A family $\mathcal{P}$ of seminorms on X is said to be separating if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References
  - i [Rudin, 1991, Chapter 1].

Seminorms

- A `seminorm` on a vector space X is a real valued function p such that

  i
  $$p(x+y) \le p(x) + p(y), \qquad \forall x, y \in X; \qquad (29)$$

  ii
  $$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \qquad (30)$$

- A family $\mathcal{P}$ of seminorms on X is said to be `separating` if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References

  i [Rudin, 1991, Chapter 1].

Seminorms

- A `seminorm` on a vector space X is a real valued function p such that
  i
  $$p(x+y) \leq p(x) + p(y), \qquad \forall x, y \in X; \tag{29}$$
  ii
  $$p(\alpha x) = |\alpha| p(x), \qquad \forall \alpha \in \mathbf{K}, \forall x \in X; \tag{30}$$
- A family $\mathcal{P}$ of seminorms on X is said to be `separating` if to each $x \neq \theta$, there is at least on $p \in \mathcal{P}$ such that $p(x) \neq 0$.
- References
  i [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let
$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1 + p(x-y)}. \tag{31}$$
- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0, \forall i$.
- (X, d) is a topological vector space, `Frechét space`, i.e., `locally convex vector space` with a `complete tranlation-invariant metric`.
- A set $E \subset X$ is `bounded` if and only if every p is bounded on E.
- References
  - [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1+p(x-y)}. \tag{31}$$

- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0, \forall i$.
- (X, d) is a topological vector space, Frechét space, i.e., locally convex vector space with a complete tranlation-invariant metric.
- A set $E \subset X$ is bounded if and only if every p is bounded on E.
- References
  i [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1+p(x-y)}. \tag{31}$$

- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0, \forall i$.
- (X, d) is a topological vector space, `Frechét space`, i.e., `locally convex vector space` with a `complete tranlation-invariant metric`.
- A set $E \subset X$ is `bounded` if and only if every p is bounded on E.
- References
  i [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1 + p(x-y)}. \tag{31}$$

- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0$, $\forall i$.
- (X, d) is a topological vector space, Frechét space, i.e., locally convex vector space with a complete tranlation-invariant metric.
- A set $E \subset X$ is bounded if and only if every p is bounded on E.
- References
  - [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let
$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1+p(x-y)}. \tag{31}$$
- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0$, $\forall i$.
- (X, d) is a topological vector space, Frechét space, i.e., locally convex vector space with a complete tranlation-invariant metric.
- A set $E \subset X$ is bounded if and only if every p is bounded on E.
- References
  - [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1+p(x-y)}. \tag{31}$$

- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0$, $\forall i$.
- (X, d) is a topological vector space, `Frechét space`, i.e., `locally convex vector space` with a `complete tranlation-invariant metric`.
- A set $E \subset X$ is `bounded` if and only if every p is bounded on E.
- References
  - [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let
$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1 + p(x-y)}. \qquad (31)$$
- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0, \forall i$.
- (X, d) is a topological vector space, `Frechét space`, i.e., `locally convex vector space` with a `complete tranlation-invariant metric`.
- A set $E \subset X$ is `bounded` if and only if every p is bounded on E.
- References
  - [Rudin, 1991, Chapter 1].

Metric by Countable Separating Family of Seminorms

- If $\mathcal{P} = \{p_i : i = 1, 2, 3, \cdots\}$ is a countable separating family of seminorms on X.
- Let
$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p(x-y)}{1+p(x-y)}. \tag{31}$$
- d is a metric on X, and compatiable with the topology induced by $\mathcal{P}$.
- $\{x_n\}$ converges to x if and only if $p_i(x_n - x) \to 0, \forall i$.
- (X, d) is a topological vector space, `Frechét space`, i.e., `locally convex vector space` with a `complete tranlation-invariant metric`.
- A set $E \subset X$ is `bounded` if and only if every p is bounded on E.
- References
  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.
- Theorem

  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i. f is continuous;
  ii. f is bounded;
  iii. If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv. If $x_n \to \theta$, then $f(x_n) \to 0$.
- References

  i. [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- **Theorem**
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,
  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References
  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

  - References
    i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References

  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- **Theorem**
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i. f is continuous;
  ii. f is bounded;
  iii. If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv. If $x_n \to \theta$, then $f(x_n) \to 0$.

- References

  i. [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References
  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem

  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References

  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem
  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References

  i [Rudin, 1991, Chapter 1].

Dual Spaces

- The `dual space` of a topolological vector space is the space X^* whose elements are the continuous linear functionals on X.

- Theorem

  Let X be a metrizable topological vector space and $f \in X^*$. The following for properties are equivalent,

  i f is continuous;
  ii f is bounded;
  iii If $x_n \to \theta$, then $\{f(x_n)\}$ is bounded;
  iv If $x_n \to \theta$, then $f(x_n) \to 0$.

- References

  i [Rudin, 1991, Chapter 1].

Outline

Function Spaces
- Functional Analysis
- Function Spaces

Theory of the Fourier Transform
- The L^1 Theory of the Fourier Transform
- Inverse Fourier Transform
- The L^2 Theory of the Fourier Transform

Tempered Distributions
- Operators for Tempered Distributions
  - Convolution
  - Differentiation

$C(\Omega)$

- Let Ω be an open set of $\mathbf{R}^n$.
- Ω is the union of countable many compact sets K_n.
- $C(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \tag{32}$$

- References
  - [Rudin, 1991, Chapter 1].

$C(\Omega)$

- Let Ω be an open set of $\mathbf{R}^n$.
- Ω is the union of countable many compact sets K_n.
- $C(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \tag{32}$$

- References
  - [Rudin, 1991, Chapter 1].

$C(\Omega)$

- Let Ω be an open set of $\mathbf{R}^n$.
- Ω is the union of countable many compact sets K_n.
- $C(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \tag{32}$$

- References
  - [Rudin, 1991, Chapter 1].

$C(\Omega)$

- Let Ω be an open set of $\mathbf{R}^n$.
- Ω is the union of countable many compact sets K_n.
- $C(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \tag{32}$$

- References
  - [Rudin, 1991, Chapter 1].

$C(\Omega)$

- Let Ω be an open set of $\mathbf{R}^n$.
- Ω is the union of countable many compact sets K_n.
- $C(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in K_n\} \tag{32}$$

- References
  i [Rudin, 1991, Chapter 1].

Notations for Analsis on $\mathbf{R}^n$

- The term multi-index denotes an ordered n-tuple

$$\alpha = (\alpha_1, \cdots, \alpha_n), \tag{33}$$

of nonnegative integers.

- With each multi-index α is associated the differential operator

$$\partial^\alpha = D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \tag{34}$$

whose order is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{35}$$

- If $|\alpha| = 0$, $D^\alpha f = f$.
- References
  - [Rudin, 1991, Chapter 1].

Notations for Analsis on $\mathbf{R}^n$

- The term `multi-index` denotes an ordered n-tuple

$$\alpha = (\alpha_1, \cdots, \alpha_n), \tag{33}$$

of nonnegative integers.

- With each multi-index α is associated the `differential operator`

$$\partial^\alpha = D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \tag{34}$$

whose order is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{35}$$

- If $|\alpha| = 0$, $D^\alpha f = f$.
- References
  - [Rudin, 1991, Chapter 1].

Notations for Analsis on $\mathbf{R}^n$

- The term `multi-index` denotes an ordered n-tuple

$$\alpha = (\alpha_1, \cdots, \alpha_n), \tag{33}$$

of nonnegative integers.

- With each multi-index α is associated the `differential operator`

$$\partial^\alpha = D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \tag{34}$$

whose order is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{35}$$

- If $|\alpha| = 0$, $D^\alpha f = f$.
- References
  - i [Rudin, 1991, Chapter 1].

Notations for Analsis on $\mathbf{R}^n$

- The term `multi-index` denotes an ordered n-tuple

$$\alpha = (\alpha_1, \cdots, \alpha_n), \tag{33}$$

of nonnegative integers.

- With each multi-index α is associated the `differential operator`

$$\partial^\alpha = D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \tag{34}$$

whose order is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{35}$$

- If $|\alpha| = 0$, $D^\alpha f = f$.
- References
  - i [Rudin, 1991, Chapter 1].

Notations for Analsis on $\mathbf{R}^n$

- The term `multi-index` denotes an ordered n-tuple

$$\alpha = (\alpha_1, \cdots, \alpha_n), \tag{33}$$

  of nonnegative integers.

- With each multi-index α is associated the `differential operator`

$$\partial^\alpha = D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \tag{34}$$

  whose order is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{35}$$

- If $|\alpha| = 0$, $D^\alpha f = f$.
- References
  i [Rudin, 1991, Chapter 1].

$C^k(\Omega)$

- $C^k(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of order up to and including k, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq k, x \in K_n\}, \qquad (36)$$

for $n = 1, \cdots$.

- References
  - [Rudin, 1991, Chapter 1].

$C^k(\Omega)$

- $C^k(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of order up to and including k, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq k, x \in K_n\}, \tag{36}$$

for $n = 1, \cdots$.
- References
  - [Rudin, 1991, Chapter 1].

$C^k(\Omega)$

- $C^k(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of order up to and including k, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq k, x \in K_n\}, \qquad (36)$$

for $n = 1, \cdots$.

- References

  i [Rudin, 1991, Chapter 1].

$C^\infty(\Omega)$

- $C^\infty(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of any order, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq n, x \in K_n\} \qquad (37)$$

- References
  - [Rudin, 1991, Chapter 1].

$C^\infty(\Omega)$

- $C^\infty(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of any order, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq n, x \in K_n\} \tag{37}$$

- References
  - [Rudin, 1991, Chapter 1].

$C^\infty(\Omega)$

- $C^\infty(\Omega)$ is the vector space of all (complex) valued continuous functions on Ω which have continuous partial derivatives of any order, topologized by the separating family of seminorms

$$p_n(f) = \sup\{|D^\alpha f(x)| : |\alpha| \leq n, x \in K_n\} \qquad (37)$$

- References
  i [Rudin, 1991, Chapter 1].

Support

- If $\varphi \in C(\Omega)$, the support of φ, denoted by $\mathrm{supp}\ \varphi$, is the closure of the set

$$\{x \in \Omega : \varphi(x) \neq 0\}, \tag{38}$$

i.e., $\mathrm{supp}\ \varphi$ is the smallest closed subset of Ω such that $\varphi = 0$ in $\Omega \setminus \mathrm{supp}\ \varphi$.

- References
  - [Hömander, 1990].

Support

- If $\varphi \in C(\Omega)$, the $\mathtt{support}$ of φ, denoted by $\mathrm{supp}\ \varphi$, is the closure of the set

$$\{x \in \Omega : \varphi(x) \neq 0\}, \tag{38}$$

i.e., $\mathrm{supp}\ \varphi$ is the smallest closed subset of Ω such that $\varphi = 0$ in $\Omega \setminus \mathrm{supp}\ \varphi$.

- References
  - [Hömander, 1990].

Support

- If $\varphi \in C(\Omega)$, the $\texttt{support}$ of φ, denoted by $\texttt{supp}\ \varphi$, is the closure of the set

$$\{x \in \Omega : \varphi(x) \neq 0\}, \tag{38}$$

i.e., $\texttt{supp}\ \varphi$ is the smallest closed subset of Ω such that $\varphi = 0$ in $\Omega \setminus \texttt{supp}\ \varphi$.

- References
  - [Hömander, 1990].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \qquad (39)$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\operatorname{supp} f \Subset \Omega$.
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathrm{supp}\ f \Subset \Omega$.
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\operatorname{supp} f \Subset \Omega$.
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathrm{supp}\ f \Subset \Omega$.
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathrm{supp}\ f \Subset \Omega$.
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathtt{supp}\ f \Subset \Omega$.
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \tag{39}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathrm{supp}\ f \Subset \Omega$.
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I].

$C_K^k(\Omega)$ and $C_0^k(\Omega)$

- For a compact set $K \subset \Omega$, let $C_K^k(\Omega)$ denote the space of all $f \in C^k(\Omega)$ whose support lies in K.
- $C_K^k(\Omega)$ is a closed subspace of $C^k(\Omega)$.
- $$C_0^k(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^k(\Omega), \qquad (39)$$
  where the union is for all compact subsets $K \subset \Omega$.
- $C_0^k(\Omega)$ consists of functions in $C^k(\Omega)$ with compact supports contained in Ω.
- This condition on function supports is written as $\mathrm{supp}\ f \Subset \Omega$.
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I].

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \tag{40}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \tag{41}$$
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \tag{40}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \tag{41}$$
- References
  1. [Rudin, 1991, Chapter 1 and 6].
  2. [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \tag{40}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \tag{41}$$
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \tag{40}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \tag{41}$$
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \qquad (40)$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \qquad (41)$$
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K\subset\Omega} C_K^\infty(\Omega), \tag{40}$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \tag{41}$$
- References
  - i [Rudin, 1991, Chapter 1 and 6].
  - ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \qquad (40)$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \qquad (41)$$
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I]

$\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$

- For a compact set $K \subset \Omega$, let $\mathcal{D}_K(\Omega) = C_K^\infty(\Omega)$ denote the space of all $f \in C^\infty(\Omega)$ whose support lies in K.
- $\mathcal{D}_K(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.
- $$\mathcal{D}(\Omega) = C_0^\infty(\Omega) \triangleq \bigcup_{K \subset \Omega} C_K^\infty(\Omega), \qquad (40)$$
  where the union is for all compact subsets $K \subset \Omega$.
- $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact supports contained in Ω.
- $$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \Subset \Omega\}. \qquad (41)$$
- References
  i [Rudin, 1991, Chapter 1 and 6].
  ii [Yosida, 1980, Chapter I]

Test Functions $\mathcal{D}(\Omega)$

- The elements of $\mathcal{D}(\Omega)$ are called `test functions`.
- The heuristics for this term is clear from the following theorem.

Theorem

([Hömander, 1990, Theorem 1.2.4]) If f and g are locally integrable functions on Ω and

$$\int_\Omega f\phi\,dx = \int_\Omega g\phi\,dx, \quad \forall\varphi \in \mathcal{D}(\Omega), \tag{42}$$

then $f = g$ almost everywhere (a.e.) in Ω.

Test Functions $\mathcal{D}(\Omega)$

- The elements of $\mathcal{D}(\Omega)$ are called `test functions`.
- The heuristics for this term is clear from the following theorem.

Theorem

([Hömander, 1990, Theorem 1.2.4]) If f and g are locally integrable functions on Ω and

$$\int_\Omega f\phi \, dx = \int_\Omega g\phi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{42}$$

then $f = g$ almost everywhere (a.e.) in Ω.

Test Functions $\mathcal{D}(\Omega)$

- The elements of $\mathcal{D}(\Omega)$ are called `test functions`.
- The heuristics for this term is clear from the following theorem.

Theorem

([Hömander, 1990, Theorem 1.2.4]) If f and g are locally integrable functions on Ω and

$$\int_\Omega f\phi\, dx = \int_\Omega g\phi\, dx, \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{42}$$

then $f = g$ almost everywhere (a.e.) in Ω.

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p norm of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the essential supremum of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ ($1 \leq p < \infty$) is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \tag{43}$$

- $||f||_p$ is called the L^p norm of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the essential supremum of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p norm of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the essential supremum of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \le p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p norm of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the essential supremum of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \le p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p `norm` of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all `essentially bounded measurable functions` on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the `essential supremum` of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p `norm` of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all `essentially bounded measurable functions` on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the `essential supremum` of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \qquad (43)$$

- $||f||_p$ is called the L^p norm of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the essential supremum of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  - [Adams, 2003, Chapter 2],

$L^p(\Omega)$

- Another class of spaces is the $L^p(\Omega)$ space, $1 \leq p < \infty$, of all measurable functions f on Ω such that

$$||f||_p = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty. \tag{43}$$

- $||f||_p$ is called the L^p `norm` of f.
- For $p = \infty$, the space $L^\infty(\Omega)$ consists of all `essentially bounded measurable functions` on Ω.
- For $f \in L^\infty(\Omega)$, let $||f||_\infty$ be the `essential supremum` of $|f|(x)$.
- $L^p(\Omega)$ is a Banach space.
- The dual of $L^p(\Omega)$ $(1 \leq p < \infty)$ is $L^q(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$.
- References
  i [Adams, 2003, Chapter 2],

$C_0(\mathbf{R}^n)$

- $C_0(\Omega)$ is the subspace of $C(\mathbf{R}^n)$ of functions `vanishing at infinity`,

$$\lim_{x \to \infty} f = 0, \tag{44}$$

with the max-norm, or L^∞-norm,

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbf{R}^n\} \tag{45}$$

- It is a Banach space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

$C_0(\mathbf{R}^n)$

- $C_0(\Omega)$ is the subspace of $C(\mathbf{R}^n)$ of functions `vanishing at infinity`,

$$\lim_{x \to \infty} f = 0, \tag{44}$$

with the max-norm, or L^∞-norm,

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbf{R}^n\} \tag{45}$$

- It is a Banach space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

$C_0(\mathbf{R}^n)$

- $C_0(\Omega)$ is the subspace of $C(\mathbf{R}^n)$ of functions `vanishing at infinity`,

$$\lim_{x \to \infty} f = 0, \tag{44}$$

with the max-norm, or L^∞-norm,

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbf{R}^n\} \tag{45}$$

- It is a Banach space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

$C_0(\mathbf{R}^n)$

- $C_0(\Omega)$ is the subspace of $C(\mathbf{R}^n)$ of functions vanishing at infinity,

$$\lim_{x \to \infty} f = 0, \tag{44}$$

with the max-norm, or L^∞-norm,

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbf{R}^n\} \tag{45}$$

- It is a Banach space.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Outline

Fourier transform

- If $f \in L^1(\mathbf{R}^n)$, the Fourier transform $\mathcal{F}f$ of f is the function $\mathcal{F}f = \hat{f}$ defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-2\pi i \xi \cdot x}\, dx, \quad \forall \xi \in \mathbf{R}^n. \tag{46}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier transform

- If $f \in L^1(\mathbf{R}^n)$, the `Fourier transform` $\mathcal{F}f$ of f is the function $\mathcal{F}f = \hat{f}$ defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-2\pi i\xi\cdot x}\,dx, \quad \forall \xi \in \mathbf{R}^n. \tag{46}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier transform

- If $f \in L^1(\mathbf{R}^n)$, the `Fourier transform` $\mathcal{F}f$ of f is the function $\mathcal{F}f = \hat{f}$ defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-2\pi i \xi \cdot x}\, dx, \quad \forall \xi \in \mathbf{R}^n. \tag{46}$$

- References
  - i [Stein and Weiss, 1971, Chapter 1].

Fourier transform in L^1

Theorem

([Stein and Weiss, 1971, Theorem I.1.1–1.2])

(a) *The mapping $\mathcal{F} : f \to \hat{f}$ is a bounded linear transform from $L^1(\mathbf{R}^n)$ into $L^\infty(\mathbf{R}^n)$. In fact, $||\hat{f}||_\infty \leq ||f||_1$.*

(b) *If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}$ is uniformly continuous.*

(c) *(Riemann–Lebesgue) If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Fourier transform in L^1

Theorem
([Stein and Weiss, 1971, Theorem I.1.1–1.2])

(a) *The mapping $\mathcal{F} : f \to \hat{f}$ is a bounded linear transform from $L^1(\mathbf{R}^n)$ into $L^\infty(\mathbf{R}^n)$. In fact, $||\hat{f}||_\infty \leq ||f||_1$.*

(b) *If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}$ is uniformly continuous.*

(c) *(Riemann–Lebesgue) If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Fourier transform in L^1

Theorem
([Stein and Weiss, 1971, Theorem I.1.1–1.2])

(a) *The mapping $\mathcal{F} : f \to \hat{f}$ is a bounded linear transform from $L^1(\mathbf{R}^n)$ into $L^\infty(\mathbf{R}^n)$. In fact, $||\hat{f}||_\infty \leq ||f||_1$.*

(b) *If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}$ is uniformly continuous.*

(c) *(Riemann–Lebesgue) If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Fourier transform in L^1

Theorem
([Stein and Weiss, 1971, Theorem I.1.1–1.2])

(a) *The mapping* $\mathcal{F} : f \to \hat{f}$ *is a bounded linear transform from* $L^1(\mathbf{R}^n)$ *into* $L^\infty(\mathbf{R}^n)$. *In fact,* $||\hat{f}||_\infty \leq ||f||_1$.

(b) *If* $f \in L^1(\mathbf{R}^n)$, *then* $\hat{f}$ *is uniformly continuous.*

(c) *(*`Riemann-Lebesgue`*) If* $f \in L^1(\mathbf{R}^n)$, *then* $\hat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$.

Convolution in L^1

- In addition to the vector space operations, $L^1(\mathbf{R}^n)$ is endowed with a "multiplication" making it a `Banach algebra`.
- This operation, called `convolution`, is defined in the following way.
- If f and $g \in L^1(\mathbf{R}^n)$, their `convolution` $h = f * g$ is the function defined by

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy, \quad \forall x \in \mathbf{R}^n. \tag{47}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution in L^1

- In addition to the vector space operations, $L^1(\mathbf{R}^n)$ is endowed with a “multiplication” making it a Banach algebra.
- This operation, called convolution, is defined in the following way.
- If f and $g \in L^1(\mathbf{R}^n)$, their convolution $h = f * g$ is the function defined by

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy, \quad \forall x \in \mathbf{R}^n. \tag{47}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution in L^1

- In addition to the vector space operations, $L^1(\mathbf{R}^n)$ is endowed with a "multiplication" making it a `Banach algebra`.
- This operation, called `convolution`, is defined in the following way.
- If f and $g \in L^1(\mathbf{R}^n)$, their `convolution` $h = f * g$ is the function defined by

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy, \quad \forall x \in \mathbf{R}^n. \tag{47}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution in L^1

- In addition to the vector space operations, $L^1(\mathbf{R}^n)$ is endowed with a "multiplication" making it a Banach algebra.
- This operation, called convolution, is defined in the following way.
- If f and $g \in L^1(\mathbf{R}^n)$, their convolution $h = f * g$ is the function defined by

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy, \quad \forall x \in \mathbf{R}^n. \tag{47}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution in L^1

- In addition to the vector space operations, $L^1(\mathbf{R}^n)$ is endowed with a "multiplication" making it a `Banach algebra`.
- This operation, called `convolution`, is defined in the following way.
- If f and $g \in L^1(\mathbf{R}^n)$, their `convolution` $h = f * g$ is the function defined by

$$h(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy, \quad \forall x \in \mathbf{R}^n. \tag{47}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution in $L^p \times L^1$

More generally, $h = f * g$ is defined for $f \in L^p(\mathbf{R}^n)$ and $g \in L^1(\mathbf{R}^n)$. In fact, we have the following result.

Theorem

([Stein and Weiss, 1971, Theorem I.1.3]) If $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbf{R}^n)$, then $h = f * g$ is well-defined and belongs to $L^p(\mathbf{R}^n)$. Moreover,

$$||h||_p \leq ||f||_p ||g||_1. \tag{48}$$

Convolution Theorem

An essential feature is the fact that the Fourier transform of the `convolution of two functions` is the (point-wise) `product of their Fourier transforms`.

Theorem

([Stein and Weiss, 1971, Theorem I.1.4]) If f and $g \in L^1(\mathbf{R}^n)$, then

$$\widehat{(f * g)} = \hat{f}\hat{g}. \tag{49}$$

Some notations

- Many other important operations of analysis have particularly simple relations with the Fourier transform.
- Let τ_h denote the `translation operator` by $h \in \mathbf{R}^n$, defined by
$$(\tau_h f)(x) = f(x - h). \tag{50}$$
- If $a > 0$, let D_a denote the `dilation operator`, defined by
$$(D_a f)(x) = f(a \cdot x). \tag{51}$$
- References
  - [Stein and Weiss, 1971, Chapter 1].

Some notations

- Many other important operations of analysis have particularly simple relations with the Fourier transform.
- Let τ_h denote the `translation operator` by $h \in \mathbf{R}^n$, defined by

$$(\tau_h f)(x) = f(x - h). \tag{50}$$

- If $a > 0$, let D_a denote the `dilation operator`, defined by

$$(D_a f)(x) = f(a \cdot x). \tag{51}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Some notations

- Many other important operations of analysis have particularly simple relations with the Fourier transform.
- Let τ_h denote the `translation operator` by $h \in \mathbf{R}^n$, defined by

$$(\tau_h f)(x) = f(x - h). \tag{50}$$

- If $a > 0$, let D_a denote the `dilation operator`, defined by

$$(D_a f)(x) = f(a \cdot x). \tag{51}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Some notations

- Many other important operations of analysis have particularly simple relations with the Fourier transform.
- Let τ_h denote the `translation operator` by $h \in \mathbf{R}^n$, defined by

$$(\tau_h f)(x) = f(x - h). \tag{50}$$

- If $a > 0$, let D_a denote the `dilation operator`, defined by

$$(D_a f)(x) = f(a \cdot x). \tag{51}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Some notations

- Many other important operations of analysis have particularly simple relations with the Fourier transform.
- Let τ_h denote the `translation operator` by $h \in \mathbf{R}^n$, defined by

$$(\tau_h f)(x) = f(x - h). \tag{50}$$

- If $a > 0$, let D_a denote the `dilation operator`, defined by

$$(D_a f)(x) = f(a \cdot x). \tag{51}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e., $D_a f = a^k f$, for $a > 0$, then $\hat{f}$ is a homogenous function of order $-n-k$.*

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i \xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k) f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e., $D_a f = a^k f$, for $a > 0$, then $\hat{f}$ is a homogenous function of order $-n-k$.*

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i \xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k) f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e., $D_a f = a^k f$, for $a > 0$, then $\hat{f}$ is a homogenous function of order $-n-k$.*

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i\xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i\xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k)f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e.,* $D_a f = a^k f$, *for* $a > 0$, *then* $\hat{f}$ *is a homogenous function of order* $-n-k$.

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i\xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If* $P(x)$ *is a polynomial in the variables* $x_1, \cdots, x_n$ *and* $P(\partial)$ *is the associated differential operator, i.e., we replace* x^α *by* ∂^α, *then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i\xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k)f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e., $D_a f = a^k f$, for $a > 0$, then $\hat{f}$ is a homogenous function of order $-n-k$.*

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i \xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k)f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e.,* $D_a f = a^k f$, *for* $a > 0$, *then* $\hat{f}$ *is a homogenous function of order* $-n-k$.

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i\xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k)f(x))^\wedge(\xi). \tag{53}$$

Convolution Properties

Theorem

([Stein and Weiss, 1971, pp. 3 – 5]) Assume that f and functions involved belong to $L^1(\mathbf{R}^n)$ in the following. Then

(i) $\widehat{(\tau_h f)}(\xi) = \mathbf{e}^{-2\pi i h\cdot\xi}\hat{f}(\xi)$.

(ii) $(\mathbf{e}^{2\pi i h\cdot x} f(x))^\wedge(\xi) = (\tau_h \hat{f})(\xi)$.

(iii) $\widehat{D_a f}(\xi) = \frac{1}{a^n}\hat{f}(\frac{1}{a}\xi)$.
In particular, if f is a `homogeneous function` *of order k, i.e., $D_a f = a^k f$, for $a > 0$, then $\hat{f}$ is a homogenous function of order $-n-k$.*

(iv) $\widehat{\frac{\partial f}{\partial x_k}}(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

(v) $\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-2\pi i x_k f(x))^\wedge(\xi)$.

(vi) *If $P(x)$ is a polynomial in the variables $x_1, \cdots, x_n$ and $P(\partial)$ is the associated differential operator, i.e., we replace x^α by ∂^α, then*

$$\widehat{P(\partial)f}(\xi) = P(2\pi i\xi)\hat{f}(\xi) \tag{52}$$

$$P(\partial)\hat{f}(\xi) = (P(-2\pi i x_k)f(x))^\wedge(\xi). \tag{53}$$

Multiplication Formula

Another important property is the `multiplication formula`.

Theorem

([Stein and Weiss, 1971, Theorem I.1.15]) If f and $g \in L^1(\mathbf{R}^n)$, then

$$\int_{\mathbf{R}^n} \hat{f}(x)g(x)\,dx = \int_{\mathbf{R}^n} f(x)\hat{g}(x)\,dx \tag{54}$$

Outline

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the inverse Fourier transform of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi, \quad \forall x \in \mathbf{R}^n. \qquad (55)$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain summability methods for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the inverse Fourier transform of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi, \quad \forall x \in \mathbf{R}^n. \tag{55}$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain summability methods for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the `inverse Fourier transform` of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi)\mathbf{e}^{2\pi i \xi \cdot x}\, d\xi, \quad \forall x \in \mathbf{R}^n. \tag{55}$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain `summability methods` for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the `inverse Fourier transform` of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi, \quad \forall x \in \mathbf{R}^n. \tag{55}$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain `summability methods` for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the `inverse Fourier transform` of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi, \quad \forall x \in \mathbf{R}^n. \tag{55}$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain `summability methods` for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Inverse Fourier Transform

- If $g \in L^1(\mathbf{R}^n)$, the `inverse Fourier transform` of g is the function $\check{f}$ defined by

$$\check{g}(x) = \int_{\mathbf{R}^n} g(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi, \quad \forall x \in \mathbf{R}^n. \tag{55}$$

- However, the Fourier transform of $f \in L^1(\mathbf{R}^n)$ is not always integrable.
- Hence, the inverse Fourier transfrom may not be applied directly to obtain f from $\hat{f}$ by the inverse transform.
- In order to get around this difficulty, we shall use certain `summability methods` for integrals.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\,dx. \tag{56}$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon \to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\,dx. \tag{57}$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon \to 0} A_\epsilon(f) \tag{58}$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\,dx$ is `Abel summable` to this limit.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\,dx. \tag{56}$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon \to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\,dx. \tag{57}$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon \to 0} A_\epsilon(f) \tag{58}$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\,dx$ is `Abel summable` to this limit.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\,dx. \tag{56}$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon \to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\,dx. \tag{57}$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon \to 0} A_\epsilon(f) \tag{58}$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\,dx$ is `Abel summable` to this limit.
- References
  - i [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\,dx. \tag{56}$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon \to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\,dx. \tag{57}$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon \to 0} A_\epsilon(f) \tag{58}$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\,dx$ is `Abel summable` to this limit.
- References
  - i [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\,dx. \quad (56)$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon\to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\,dx. \quad (57)$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon\to 0} A_\epsilon(f) \quad (58)$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\,dx$ is `Abel summable` to this limit.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Abel method of summability

- For each $\epsilon > 0$, define the `Abel mean`,

$$A_\epsilon(f) = A_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||}\, dx. \quad (56)$$

- If $f \in L^1(\mathbf{R}^n)$,

$$\lim_{\epsilon\to 0} A_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\, dx. \quad (57)$$

- On the other hand, these Abel means are well-defined even when f is not integrable. Nevertheless, their limit

$$\lim_{\epsilon\to 0} A_\epsilon(f) \quad (58)$$

may exist.
- Whenever, the limit in Eq. (58) exists and is finite we say the $\int_{\mathbf{R}^n} f(x)\, dx$ is `Abel summable` to this limit.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Gauss summability

- Gauss summability is defined by the Gauss means.

$$G_\epsilon(f) = G_\epsilon = \int_{\mathbf{R}^n} f(x) \mathbf{e}^{-\epsilon ||x||^2} \, dx. \qquad (59)$$

- $\int_{\mathbf{R}^n} f(x)\, dx$ is Gauss summable to

$$\lim_{\epsilon \to 0} A_\epsilon(f), \qquad (60)$$

if this limit exists and is finite.

- References
  - [Stein and Weiss, 1971, Chapter 1].

Gauss summability

- `Gauss summability` is defined by the `Gauss means`.

$$G_\epsilon(f) = G_\epsilon = \int_{\mathbf{R}^n} f(x)\mathbf{e}^{-\epsilon||x||^2}\, dx. \tag{59}$$

- $\int_{\mathbf{R}^n} f(x)\, dx$ is `Gauss summable` to

$$\lim_{\epsilon \to 0} A_\epsilon(f), \tag{60}$$

if this limit exists and is finite.

- References
  - [Stein and Weiss, 1971, Chapter 1].

Gauss summability

- `Gauss summability` is defined by the `Gauss means`.

$$G_\epsilon(f) = G_\epsilon = \int_{\mathbf{R}^n} f(x) \mathbf{e}^{-\epsilon ||x||^2} \, dx. \tag{59}$$

- $\int_{\mathbf{R}^n} f(x) \, dx$ is `Gauss summable` to

$$\lim_{\epsilon \to 0} A_\epsilon(f), \tag{60}$$

if this limit exists and is finite.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Gauss summability

- `Gauss summability` is defined by the `Gauss means`.

$$G_\epsilon(f) = G_\epsilon = \int_{\mathbf{R}^n} f(x) \mathbf{e}^{-\epsilon ||x||^2} \, dx. \tag{59}$$

- $\int_{\mathbf{R}^n} f(x)\, dx$ is `Gauss summable` to

$$\lim_{\epsilon \to 0} A_\epsilon(f), \tag{60}$$

  if this limit exists and is finite.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\, dx, \qquad (61)$$

where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.

- Then $\int_{\mathbf{R}^n} f(x)\, dx$ is summable to I if $\lim_{\epsilon \to 0} M_\epsilon(f) = I$.
- We call $M_\epsilon(f)$ the Φ `means` of this integral.
- We shall need the Fourier transform of the functions $e^{-\epsilon||x||^2}$ and $e^{-\epsilon||x||}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\,dx, \qquad (61)$$

  where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.
- Then $\int_{\mathbf{R}^n} f(x)\,dx$ is summable to l if $\lim_{\epsilon\to 0} M_\epsilon(f) = l$.
- We call $M_\epsilon(f)$ the Φ `means` of this integral.
- We shall need the Fourier transform of the functions $e^{-\epsilon\|x\|^2}$ and $e^{-\epsilon\|x\|}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\, dx, \qquad (61)$$

where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.
- Then $\int_{\mathbf{R}^n} f(x)\, dx$ is summable to I if $\lim_{\epsilon\to 0} M_\epsilon(f) = I$.
- We call $M_\epsilon(f)$ the Φ means of this integral.
- We shall need the Fourier transform of the functions $\mathbf{e}^{-\epsilon||x||^2}$ and $\mathbf{e}^{-\epsilon||x||}$.
- References

[Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_{\epsilon}(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\,dx, \qquad (61)$$

  where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.
- Then $\int_{\mathbf{R}^n} f(x)\,dx$ is summable to l if $\lim_{\epsilon\to 0} M_{\epsilon}(f) = l$.
- We call $M_{\epsilon}(f)$ the Φ `means` of this integral.
- We shall need the Fourier transform of the functions $\mathbf{e}^{-\epsilon||x||^2}$ and $\mathbf{e}^{-\epsilon||x||}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\,dx, \qquad (61)$$

  where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.
- Then $\int_{\mathbf{R}^n} f(x)\,dx$ is summable to I if $\lim_{\epsilon\to 0} M_\epsilon(f) = I$.
- We call $M_\epsilon(f)$ the Φ `means` of this integral.
- We shall need the Fourier transform of the functions $\mathbf{e}^{-\epsilon||x||^2}$ and $\mathbf{e}^{-\epsilon||x||}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Methods of summability

- Both the methods can be put in the form

$$M_{\epsilon,\Phi}(f) = M_\epsilon(f) = \int_{\mathbf{R}^n} f(x)\Phi(\epsilon x)\,dx, \qquad (61)$$

where $\Phi \in C_0$ (cf. slide 11) and $\Phi(0) = 1$.
- Then $\int_{\mathbf{R}^n} f(x)\,dx$ is summable to l if $\lim_{\epsilon\to 0} M_\epsilon(f) = l$.
- We call $M_\epsilon(f)$ the Φ `means` of this integral.
- We shall need the Fourier transform of the functions $\mathbf{e}^{-\epsilon||x||^2}$ and $\mathbf{e}^{-\epsilon||x||}$.
- References
  - i [Stein and Weiss, 1971, Chapter 1].

Theorem on summability

Theorem

([Stein and Weiss, 1971, Theorem I.1.13 and 1.14])

(a) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} e^{-2\pi i \xi \cdot x} e^{-\pi\alpha ||x||^2} \, dx = \alpha^{-n/2} e^{-\frac{\pi ||x||^2}{\alpha}}. \tag{62}$$

(b) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} e^{-2\pi i \xi \cdot x} e^{-2\pi\alpha ||x||} \, dx = c_n \frac{\alpha}{(\alpha^2 + ||\xi||^2)^{\frac{n+1}{2}}}. \tag{63}$$

where $c_n = \frac{\Gamma[\frac{n+1}{2}]}{\pi^{\frac{n+1}{2}}}$.

Theorem on summability

Theorem ([Stein and Weiss, 1971, Theorem I.1.13 and 1.14])

(a) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} e^{-2\pi i \xi \cdot x} e^{-\pi\alpha ||x||^2} \, dx = \alpha^{-n/2} e^{-\frac{\pi ||x||^2}{\alpha}}. \tag{62}$$

(b) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} e^{-2\pi i \xi \cdot x} e^{-2\pi\alpha ||x||} \, dx = c_n \frac{\alpha}{(\alpha^2 + ||\xi||^2)^{\frac{n+1}{2}}}. \tag{63}$$

where $c_n = \frac{\Gamma[\frac{n+1}{2}]}{\pi^{\frac{n+1}{2}}}$.

Theorem on summability

Theorem

([Stein and Weiss, 1971, Theorem I.1.13 and 1.14])

(a) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} \mathbf{e}^{-2\pi i \xi \cdot x} \mathbf{e}^{-\pi\alpha ||x||^2}\, dx = \alpha^{-n/2} \mathbf{e}^{-\frac{\pi ||x||^2}{\alpha}}. \tag{62}$$

(b) *For all* $\alpha > 0$,

$$\int_{\mathbf{R}^n} \mathbf{e}^{-2\pi i \xi \cdot x} \mathbf{e}^{-2\pi\alpha ||x||}\, dx = c_n \frac{\alpha}{(\alpha^2 + ||\xi||^2)^{\frac{n+1}{2}}}. \tag{63}$$

where $c_n = \frac{\Gamma[\frac{n+1}{2}]}{\pi^{\frac{n+1}{2}}}$.

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

-

$$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

- $$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  - i [Stein and Weiss, 1971, Chapter 1].

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

-

$$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

-

$$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  - i [Stein and Weiss, 1971, Chapter 1].

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

-

$$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Weierstrass-Gauss kernel and Poisson kernel

- In the following, let W and P be the Fourier transforms of

$$e^{-4\pi^2\alpha||x||^2} \tag{64}$$

$$e^{-2\pi\alpha||x||} \tag{65}$$

for $\alpha > 0$, respectively,

-

$$W(t,\alpha) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{-\frac{||t||^2}{4\alpha}}, \tag{66}$$

$$P(t,\alpha) = c_n \frac{\alpha}{(\alpha^2 + ||t||^2)^{\frac{n+1}{2}}}. \tag{67}$$

- W is called the `Weierstrass kernel` (or `Weierstrass-Gauss kernel`).
- P is called the `Poisson kernel`.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Characterization of the Φ Means

Theorem

([Stein and Weiss, 1971, Theorem I.1.16]) If f and $\Phi \in L^1(\mathbf{R}^n)$ and $\varphi = \hat{\Phi}$, then

$$\int_{\mathbf{R}^n} \hat{f}(\xi)\mathbf{e}^{2\pi i\xi\cdot x}\Phi(\epsilon\xi)\,d\xi = \int_{\mathbf{R}^n} f(y)\varphi_\epsilon(x-y)\,dy, \qquad (68)$$

where

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon}) = \widehat{(D_\epsilon\Phi)}(x). \qquad (69)$$

In particular,

$$\int_{\mathbf{R}^n} \hat{f}(\xi)\mathbf{e}^{2\pi i\xi\cdot x}\mathbf{e}^{-2\pi\epsilon||x||}\,d\xi = \int_{\mathbf{R}^n} f(x)P(x-y,\epsilon)\,dy, \qquad (70)$$

and

$$\int_{\mathbf{R}^n} \hat{f}(\xi)\mathbf{e}^{2\pi i\xi\cdot x}\mathbf{e}^{-4\pi^2\epsilon||x||}\,d\xi = \int_{\mathbf{R}^n} f(x)W(x-y,\epsilon)\,dy. \qquad (71)$$

Approximation Properties of the Φ Means

Theorem

([Stein and Weiss, 1971, Theorem I.1.18]) Let $\varphi \in L^1(\mathbf{R}^n)$, with $\int_{\mathbf{R}^n} \varphi(x)\,dx = 1$, and for $\epsilon > 0$, let

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon}). \tag{72}$$

If $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$, or $f \in C_0 \subset L^\infty(\mathbf{R}^n)$, then $||f * \varphi_\epsilon - f||_p \to 0$ as $\epsilon \to 0$. In particular,

$$u(x,\epsilon) = \int_{\mathbf{R}^n} f(x)P(x-y,\epsilon)\,dy, \quad \texttt{Poisson integral}, \tag{73}$$

and

$$s(x,\epsilon) = \int_{\mathbf{R}^n} f(x)W(x-y,\epsilon)\,dy, \quad \texttt{Gauss-Weierstrass integral}, \tag{74}$$

converges to f in the L^p norm as $\epsilon \to 0$.
If $\int_{\mathbf{R}^n} \varphi(x)\,dx = 0$, then $||f * \varphi_\epsilon||_p \to 0$ as $\epsilon \to 0$.

L^1 Convergence of the Φ Means

Theorem

([Stein and Weiss, 1971, Theorem I.1.20]) If Φ and its Fourier transform $\varphi = \hat{\Phi}$ are integrable and $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$ (i.e., $\Phi(0) = 1$), then the Φ `means` of the integral

$$\int_{\mathbf{R}^n} \hat{f}(\xi) \mathbf{e}^{2\pi i \xi \cdot x}\, d\xi \tag{75}$$

converges to $f(x)$ in the L^1 norm.
In particular, the Abel and Gauss means of this integral converges to $f(x)$ in the L^1 norm, respectively.

Point-wise Convergence of the Φ Means

Theorem

([Stein and Weiss, 1971, Corollary 1.18]) If both f and $\hat{f} \in L^1(\mathbf{R}^n)$, then

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) \mathbf{e}^{2\pi i \xi \cdot x} \, d\xi \tag{76}$$

for almost every x.

A Corollary for Inverse Fourier Transform in $L^1(\mathbf{R}^n)$

Corollary

If both f and $\hat{f}$ are integrable, then

$$(\hat{f})^\wedge = f(-x) \tag{77}$$

for almost every x.

Outline

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- **Theorem**
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

$$||\mathcal{F}f||_2 = ||f||_2. \tag{78}$$

  - It follows that
    - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
    - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
  - $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

$$||\mathcal{F}f||_2 = ||f||_2. \tag{78}$$

- It follows that
  - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
  - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
- $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

$$||\mathcal{F}f||_2 = ||f||_2. \tag{78}$$

  - It follows that
    - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
    - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
  - $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

  $$||\mathcal{F}f||_2 = ||f||_2. \tag{78}$$

  - It follows that
    - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
    - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
  - $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

$$||\mathcal{F} f||_2 = ||f||_2. \tag{78}$$

  - It follows that
    - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
    - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
  - $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Extension of $\mathcal{F}$ from $L^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.1]) If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, then

$$||\mathcal{F}f||_2 = ||f||_2. \tag{78}$$

  - It follows that
    - $\mathcal{F}$ is a bounded linear operator defined on the dense subset $\in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ of $L^2(\mathbf{R}^n)$;
    - there exists a unique bounded extension, $\mathcal{F}$ (using the same notation), to all of $L^2(\mathbf{R}^n)$.
  - $\mathcal{F}$ will be called the `Fourier transform on` $L^2(\mathbf{R}^n)$.

Fourier Transform on L^2

- If $f \in L^2(\mathbf{R}^n)$, $\hat{f}$ is the L^2 limit of the sequence $\{\hat{h}_k\}$ where $\{h_k\}$ is any sequence in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ converging to f in the L^2 norm.
- Let

$$h_k(x) \begin{cases} f(x), & ||x|| \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{79}$$

- $\hat{f}$ is the L^2 limit of the sequence of functions defined by

$$\hat{h}_k(\xi) = \int_{||x|| \leq k} f(x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx. \tag{80}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier Transform on L^2

- If $f \in L^2(\mathbf{R}^n)$, $\hat{f}$ is the L^2 limit of the sequence $\{\hat{h}_k\}$ where $\{h_k\}$ is any sequence in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ converging to f in the L^2 norm.
- Let

$$h_k(x) \begin{cases} f(x), & \|x\| \leq k, \\ 0, & \text{otherwise}. \end{cases} \tag{79}$$

- $\hat{f}$ is the L^2 limit of the sequence of functions defined by

$$\hat{h}_k(\xi) = \int_{\|x\| \leq k} f(x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx. \tag{80}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier Transform on L^2

- If $f \in L^2(\mathbf{R}^n)$, $\hat{f}$ is the L^2 limit of the sequence $\{\hat{h}_k\}$ where $\{h_k\}$ is any sequence in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ converging to f in the L^2 norm.
- Let

$$h_k(x) \begin{cases} f(x), & \|x\| \le k, \\ 0, & \text{otherwise}. \end{cases} \tag{79}$$

- $\hat{f}$ is the L^2 limit of the sequence of functions defined by

$$\hat{h}_k(\xi) = \int_{\|x\| \le k} f(x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx. \tag{80}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier Transform on L^2

- If $f \in L^2(\mathbf{R}^n)$, $\hat{f}$ is the L^2 limit of the sequence $\{\hat{h}_k\}$ where $\{h_k\}$ is any sequence in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ converging to f in the L^2 norm.
- Let

$$h_k(x) \begin{cases} f(x), & ||x|| \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{79}$$

- $\hat{f}$ is the L^2 limit of the sequence of functions defined by

$$\hat{h}_k(\xi) = \int_{||x|| \leq k} f(x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx. \tag{80}$$

- References
  - [Stein and Weiss, 1971, Chapter 1].

Fourier Transform on L^2

- If $f \in L^2(\mathbf{R}^n)$, $\hat{f}$ is the L^2 limit of the sequence $\{\hat{h}_k\}$ where $\{h_k\}$ is any sequence in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ converging to f in the L^2 norm.
- Let

$$h_k(x) \begin{cases} f(x), & ||x|| \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{79}$$

- $\hat{f}$ is the L^2 limit of the sequence of functions defined by

$$\hat{h}_k(\xi) = \int_{||x|| \leq k} f(x) \mathbf{e}^{-2\pi i \xi \cdot x} \, dx. \tag{80}$$

- References
  - i [Stein and Weiss, 1971, Chapter 1].

Plancherel Theorem I

- A linear operator on $L^2(\mathbf{R}^n)$ that is an isometry and maps onto $L^2(\mathbf{R}^n)$ is called a unitary operator.
- $\mathcal{F}$ is an isometry and onto $L^2(\mathbf{R}^n)$.
- The first of Plancherel theorems in the L^2 theory of the Fourier transform is the following theorem.

Theorem

([Stein and Weiss, 1971, Theorem I.2.3]) The Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

Plancherel Theorem I

- A linear operator on $L^2(\mathbf{R}^n)$ that is an `isometry` and maps onto $L^2(\mathbf{R}^n)$ is called a `unitary operator`.
- $\mathcal{F}$ is an `isometry` and onto $L^2(\mathbf{R}^n)$.
- The first of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.
- Theorem

([Stein and Weiss, 1971, Theorem I.2.3]) The Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

Plancherel Theorem I

- A linear operator on $L^2(\mathbf{R}^n)$ that is an `isometry` and maps onto $L^2(\mathbf{R}^n)$ is called a `unitary operator`.
- $\mathcal{F}$ is an `isometry` and onto $L^2(\mathbf{R}^n)$.
- The first of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.
- Theorem

([Stein and Weiss, 1971, Theorem I.2.3]) The Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

Plancherel Theorem I

- A linear operator on $L^2(\mathbf{R}^n)$ that is an `isometry` and maps onto $L^2(\mathbf{R}^n)$ is called a `unitary operator`.
- $\mathcal{F}$ is an `isometry` and onto $L^2(\mathbf{R}^n)$.
- The first of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.

- Theorem

([Stein and Weiss, 1971, Theorem I.2.3]) The Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

Plancherel Theorem I

- A linear operator on $L^2(\mathbf{R}^n)$ that is an `isometry` and maps onto $L^2(\mathbf{R}^n)$ is called a `unitary operator`.
- $\mathcal{F}$ is an `isometry` and onto $L^2(\mathbf{R}^n)$.
- The first of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.

- **Theorem**
  ([Stein and Weiss, 1971, Theorem I.2.3]) The Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$.

Plancherel Theorem II

- The second of Plancherel theorems in the L^2 theory of the Fourier transform is the following theorem.

- Theorem

  ([Stein and Weiss, 1971, Theorem I.2.4]) The inverse of the Fourier transform, $\mathcal{F}^{-1}$, can be obtained by letting $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ for all $g \in L^2(\mathbf{R}^n)$.

Plancherel Theorem II

- The second of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.

- **Theorem**
  ([Stein and Weiss, 1971, Theorem I.2.4]) The inverse of the Fourier transform, $\mathcal{F}^{-1}$, can be obtained by letting $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ for all $g \in L^2(\mathbf{R}^n)$.

Plancherel Theorem II

- The second of `Plancherel theorems` in the L^2 theory of the Fourier transform is the following theorem.

- Theorem
  ([Stein and Weiss, 1971, Theorem I.2.4]) The inverse of the Fourier transform, $\mathcal{F}^{-1}$, can be obtained by letting $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ for all $g \in L^2(\mathbf{R}^n)$.

Distributions are Linear Functionals

- The basic idea in the theory of `distributions` is to consider them as linear functionals on some space of “regular” functions — the so-called “testing functions”.
- The `space of testing functions` is assumed to be well-behaved with respect to the operations (differentiation, Fourier transform, convolution, translation, etc).
- We are naturally led to the definition of such a space of testing functions by the following considerations.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Distributions are Linear Functionals

- The basic idea in the theory of `distributions` is to consider them as linear functionals on some space of “regular” functions — the so-called “testing functions”.
- The `space of testing functions` is assumed to be well-behaved with respect to the operations (differentiation, Fourier transform, convolution, translation, etc).
- We are naturally led to the definition of such a space of testing functions by the following considerations.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Distributions are Linear Functionals

- The basic idea in the theory of `distributions` is to consider them as linear functionals on some space of "regular" functions — the so-called "testing functions".
- The `space of testing functions` is assumed to be well-behaved with respect to the operations (differentiation, Fourier transform, convolution, translation, etc).
- We are naturally led to the definition of such a space of testing functions by the following considerations.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Distributions are Linear Functionals

- The basic idea in the theory of `distributions` is to consider them as linear functionals on some space of “regular” functions — the so-called “testing functions”.
- The `space of testing functions` is assumed to be well-behaved with respect to the operations (differentiation, Fourier transform, convolution, translation, etc).
- We are naturally led to the definition of such a space of testing functions by the following considerations.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Distributions are Linear Functionals

- The basic idea in the theory of `distributions` is to consider them as linear functionals on some space of "regular" functions — the so-called "testing functions".
- The `space of testing functions` is assumed to be well-behaved with respect to the operations (differentiation, Fourier transform, convolution, translation, etc).
- We are naturally led to the definition of such a space of testing functions by the following considerations.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Space of Testing Functions

- Suppose we want these operations to be defined on a function space, $\mathcal{S}$, and to preserve it.
- Then it would certainly have to consist of functions that are indefinitely differentiable.
- This, in view of Property (vi) in Theorem 2.4, indicates that each function of $\mathcal{S}$, after being multiplied by a polynomial, must still be in $\mathcal{S}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Space of Testing Functions

- Suppose we want these operations to be defined on a function space, $\mathcal{S}$, and to preserve it.
- Then it would certainly have to consist of functions that are indefinitely differentiable.
- This, in view of Property (vi) in Theorem 2.4, indicates that each function of $\mathcal{S}$, after being multiplied by a polynomial, must still be in $\mathcal{S}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Space of Testing Functions

- Suppose we want these operations to be defined on a function space, $\mathcal{S}$, and to preserve it.
- Then it would certainly have to consist of functions that are indefinitely differentiable.
- This, in view of Property (vi) in Theorem 2.4, indicates that each function of $\mathcal{S}$, after being multiplied by a polynomial, must still be in $\mathcal{S}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Space of Testing Functions

- Suppose we want these operations to be defined on a function space, $\mathcal{S}$, and to preserve it.
- Then it would certainly have to consist of functions that are indefinitely differentiable.
- This, in view of Property (vi) in Theorem 2.4, indicates that each function of $\mathcal{S}$, after being multiplied by a polynomial, must still be in $\mathcal{S}$.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Space of Testing Functions

- Suppose we want these operations to be defined on a function space, $\mathcal{S}$, and to preserve it.
- Then it would certainly have to consist of functions that are indefinitely differentiable.
- This, in view of Property (vi) in Theorem 2.4, indicates that each function of $\mathcal{S}$, after being multiplied by a polynomial, must still be in $\mathcal{S}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \tag{81}$$

for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  i [Stein and Weiss, 1971, Chapter 1];
  ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \tag{81}$$

for all n-tuples α and β of nonnegative integers.

- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  - i [Stein and Weiss, 1971, Chapter 1];
  - ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \tag{81}$$

  for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  i [Stein and Weiss, 1971, Chapter 1];
  ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \tag{81}$$

  for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  - i [Stein and Weiss, 1971, Chapter 1];
  - ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x\in\mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \tag{81}$$

  for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  i [Stein and Weiss, 1971, Chapter 1];
  ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \qquad (81)$$

  for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  - i [Stein and Weiss, 1971, Chapter 1];
  - ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ I

- The space $\mathcal{S}$ of `rapidly decreasing functions` is defined to be the class of all those C^∞ functions φ on $\mathbf{R}^n$ such that

$$\sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)| < \infty \qquad (81)$$

  for all n-tuples α and β of nonnegative integers.
- The space $\mathcal{S}$ is called the Schwartz space.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||^2} \in \mathcal{S}$, $\lambda > 0$.
  - $\varphi(x) = \mathbf{e}^{-\lambda ||x||}$ fails to be differentiable at the origin and, therefore, does not belong to $\mathcal{S}$.
- References
  i [Stein and Weiss, 1971, Chapter 1];
  ii [Rudin, 1991, Chapter 6];

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its "parent" space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its "parent" space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its “parent” space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its "parent" space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its "parent" space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

The Function Space $\mathcal{S}$ II

- $\mathcal{S}$ contains the space $\mathcal{D}$ of all C^∞ functions with compact support.
- If P is a polynomial in n variables and $\varphi \in \mathcal{S}$, then $P(x)\varphi(x)$ and $P(\partial)\varphi(x)$ are again in $\mathcal{S}$.
- The space C_0 and $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, contains $\mathcal{S}$.
- Each subspace is a `dense subspace` of its "parent" space.
- References
  - [Stein and Weiss, 1971, Chapter 1].

$\mathcal{S}$ under Fourier Transform

By Property (vi) of Theorem 2.4,

Theorem
([Stein and Weiss, 1971, Theorem I.3.2]) If $\varphi \in \mathcal{S}$, then $\hat{\varphi} \in \mathcal{S}$.

$\mathcal{S}$ under Convolution

- If φ and $\psi \in \mathcal{S}$, the above theorem implies that $\hat{\varphi}$ and $\hat{\psi} \in \mathcal{S}$.
- Therefore, $\hat{\varphi}\hat{\psi} \in \mathcal{S}$.
- Since $(\varphi * \psi)^\wedge = \hat{\varphi}\hat{\psi}$, applying the inverse Fourier transform shows that

Theorem

([Stein and Weiss, 1971, Theorem I.3.3]) If φ and $\psi \in \mathcal{S}$, so is $\varphi * \psi$.

$\mathcal{S}$ under Convolution

- If φ and $\psi \in \mathcal{S}$, the above theorem implies that $\hat{\varphi}$ and $\hat{\psi} \in \mathcal{S}$.
- Therefore, $\hat{\varphi}\hat{\psi} \in \mathcal{S}$.
- Since $(\varphi * \psi)^\wedge = \hat{\varphi}\hat{\psi}$, applying the inverse Fourier transform shows that

Theorem

([Stein and Weiss, 1971, Theorem I.3.3]) If φ and $\psi \in \mathcal{S}$, so is $\varphi * \psi$.

$\mathcal{S}$ under Convolution

- If φ and $\psi \in \mathcal{S}$, the above theorem implies that $\hat{\varphi}$ and $\hat{\psi} \in \mathcal{S}$.
- Therefore, $\hat{\varphi}\hat{\psi} \in \mathcal{S}$.
- Since $(\varphi * \psi)^{\wedge} = \hat{\varphi}\hat{\psi}$, applying the inverse Fourier transform shows that

Theorem

([Stein and Weiss, 1971, Theorem I.3.3]) If φ and $\psi \in \mathcal{S}$, so is $\varphi * \psi$.

$\mathcal{S}$ under Convolution

- If φ and $\psi \in \mathcal{S}$, the above theorem implies that $\hat{\varphi}$ and $\hat{\psi} \in \mathcal{S}$.
- Therefore, $\hat{\varphi}\hat{\psi} \in \mathcal{S}$.
- Since $(\varphi * \psi)^\wedge = \hat{\varphi}\hat{\psi}$, applying the inverse Fourier transform shows that

Theorem

([Stein and Weiss, 1971, Theorem I.3.3]) If φ and $\psi \in \mathcal{S}$, so is $\varphi * \psi$.

Metric on $\mathcal{S}$

- For each ordered pair of n-tuples nonnegative integer indices (α, β), define, in view of Eq. (81),

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \forall \varphi \in \mathcal{S}. \tag{82}$$

- $\{\rho_{\alpha\beta}\}$ is a countable family of separting seminorms.
- Define

$$\rho(\varphi) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} d_{\alpha\beta}(\varphi), \tag{83}$$

$$d(\varphi, \psi) = \rho(\varphi - \psi), \tag{84}$$

for φ and $\psi \in \mathcal{S}$.

- References
  - [Stein and Weiss, 1971, Chapter 1].

Metric on $\mathcal{S}$

- For each ordered pair of n-tuples nonnegative integer indices (α, β), define, in view of Eq. (81),

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \forall \varphi \in \mathcal{S}. \tag{82}$$

- $\{\rho_{\alpha\beta}\}$ is a countable family of separting seminorms.
- Define

$$\rho(\varphi) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} d_{\alpha\beta}(\varphi), \tag{83}$$

$$d(\varphi, \psi) = \rho(\varphi - \psi), \tag{84}$$

for φ and $\psi \in \mathcal{S}$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Metric on $\mathcal{S}$

- For each ordered pair of n-tuples nonnegative integer indices (α, β), define, in view of Eq. (81),

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \forall \varphi \in \mathcal{S}. \tag{82}$$

- $\{\rho_{\alpha\beta}\}$ is a `countable family of separting seminorms.`
- Define

$$\rho(\varphi) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} d_{\alpha\beta}(\varphi), \tag{83}$$

$$d(\varphi, \psi) = \rho(\varphi - \psi), \tag{84}$$

for φ and $\psi \in \mathcal{S}$.

- References
  - [Stein and Weiss, 1971, Chapter 1].

Metric on $\mathcal{S}$

- For each ordered pair of *n*-tuples nonnegative integer indices (α, β), define, in view of Eq. (81),

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \forall \varphi \in \mathcal{S}. \tag{82}$$

- $\{\rho_{\alpha\beta}\}$ is a `countable family of separting seminorms.`
- Define

$$\rho(\varphi) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} d_{\alpha\beta}(\varphi), \tag{83}$$

$$d(\varphi, \psi) = \rho(\varphi - \psi), \tag{84}$$

for φ and $\psi \in \mathcal{S}$.

- References
  - [Stein and Weiss, 1971, Chapter 1].

Metric on $\mathcal{S}$

- For each ordered pair of n-tuples nonnegative integer indices (α, β), define, in view of Eq. (81),

$$\rho_{\alpha\beta}(\varphi) = \sup_{x \in \mathbf{R}^n} |x^\alpha (D^\beta \varphi)(x)|, \quad \forall \varphi \in \mathcal{S}. \tag{82}$$

- $\{\rho_{\alpha\beta}\}$ is a `countable family of separting seminorms.`
- Define

$$\rho(\varphi) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} d_{\alpha\beta}(\varphi), \tag{83}$$

$$d(\varphi, \psi) = \rho(\varphi - \psi), \tag{84}$$

for φ and $\psi \in \mathcal{S}$.

- References
  i [Stein and Weiss, 1971, Chapter 1].

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping $\varphi(x) \to x^\alpha(D^\beta\varphi)(x)$ is continuous.*

(b) *If $\varphi \in \mathcal{S}$, then $\lim_{h\to 0} \tau_h\varphi = \varphi$.*

(c) *$(\mathcal{S}, d)$ is a complete metric space (F-space).*

(d) *The Fourier transform is a homeomorphism of $\mathcal{S}$ onto itself.*

(e) *$\mathcal{D}$ is a dense subset of $\mathcal{S}$.*

(f) *$\mathcal{S}$ is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping* $\varphi(x) \to x^\alpha (D^\beta \varphi)(x)$ *is continuous.*

(b) *If* $\varphi \in \mathcal{S}$, *then* $\lim_{h \to 0} \tau_h \varphi = \varphi$.

(c) $(\mathcal{S}, d)$ *is a complete metric space (F-space).*

(d) *The Fourier transform is a* `homeomorphism` *of* $\mathcal{S}$ *onto itself.*

(e) $\mathcal{D}$ *is a dense subset of* $\mathcal{S}$.

(f) $\mathcal{S}$ *is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping* $\varphi(x) \to x^\alpha(D^\beta\varphi)(x)$ *is continuous.*

(b) *If* $\varphi \in \mathcal{S}$, *then* $\lim_{h\to 0} \tau_h\varphi = \varphi$.

(c) $(\mathcal{S}, d)$ *is a complete metric space (F-space).*

(d) *The Fourier transform is a* `homeomorphism` *of* $\mathcal{S}$ *onto itself.*

(e) $\mathcal{D}$ *is a dense subset of* $\mathcal{S}$.

(f) $\mathcal{S}$ *is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping* $\varphi(x) \to x^\alpha (D^\beta \varphi)(x)$ *is continuous.*

(b) *If* $\varphi \in \mathcal{S}$, *then* $\lim_{h\to 0} \tau_h \varphi = \varphi$.

(c) $(\mathcal{S}, d)$ *is a complete metric space (F-space).*

(d) *The Fourier transform is a* `homeomorphism` *of* $\mathcal{S}$ *onto itself.*

(e) $\mathcal{D}$ *is a dense subset of* $\mathcal{S}$.

(f) $\mathcal{S}$ *is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem
([Stein and Weiss, 1971, p. 21])

(a) *The mapping* $\varphi(x) \to x^\alpha (D^\beta \varphi)(x)$ *is continuous.*

(b) *If* $\varphi \in \mathcal{S}$, *then* $\lim_{h \to 0} \tau_h \varphi = \varphi$.

(c) $(\mathcal{S}, d)$ *is a complete metric space (F-space).*

(d) *The Fourier transform is a* `homeomorphism` *of* $\mathcal{S}$ *onto itself.*

(e) $\mathcal{D}$ *is a dense subset of* $\mathcal{S}$.

(f) $\mathcal{S}$ *is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping $\varphi(x) \to x^\alpha(D^\beta\varphi)(x)$ is continuous.*

(b) *If $\varphi \in \mathcal{S}$, then $\lim_{h\to 0} \tau_h\varphi = \varphi$.*

(c) *$(\mathcal{S}, d)$ is a complete metric space (F-space).*

(d) *The Fourier transform is a homeomorphism of $\mathcal{S}$ onto itself.*

(e) *$\mathcal{D}$ is a dense subset of $\mathcal{S}$.*

(f) *$\mathcal{S}$ is separable.*

$(\mathcal{S}, d)$ as a Topological Vector Space

Theorem

([Stein and Weiss, 1971, p. 21])

(a) *The mapping* $\varphi(x) \to x^\alpha(D^\beta\varphi)(x)$ *is continuous.*

(b) *If* $\varphi \in \mathcal{S}$*, then* $\lim_{h\to 0} \tau_h\varphi = \varphi$.

(c) $(\mathcal{S}, \mathrm{d})$ *is a complete metric space (F-space).*

(d) *The Fourier transform is a* $homeomorphism$ *of* $\mathcal{S}$ *onto itself.*

(e) $\mathcal{D}$ *is a dense subset of* $\mathcal{S}$.

(f) $\mathcal{S}$ *is separable.*

Tempered Distributions

- The collection $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ is called the space of tempered distributions.
- It is also called generalized functions.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Tempered Distributions

- The collection $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ is called the space of `tempered distributions`.
- It is also called `generalized functions`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Tempered Distributions

- The collection $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ is called the space of `tempered distributions`.
- It is also called `generalized functions`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Tempered Distributions

- The collection $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ is called the space of `tempered distributions`.
- It is also called `generalized functions`.
- References
  - [Stein and Weiss, 1971, Chapter 1].

$L^p(\mathbf{R}^n)$ Functions as Tempered Distributions

Example

([Stein and Weiss, 1971, p. 21]) For $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, the linear functional $L = L_f$ defined by

$$L(\varphi) = L_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx, \tag{85}$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

Measures as Tempered Distributions

Example

([Stein and Weiss, 1971, p. 21]) If μ is a finite Borel measure, the linear functional $L = L_\mu$ defined by

$$L(\varphi) = L_\mu(\varphi) = \int_{\mathbf{R}^n} \varphi \, d\mu, \tag{86}$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

Tempered L^p Functions

Example

([Stein and Weiss, 1971, pp. 21 – 22])

- *A measurable function f such that*

$$\frac{f(x)}{(1+||x||^2)^k} \in L^p(\mathbf{R}^n), \tag{87}$$

for some $1 \le p \le \infty$ and some positive integer k, is called a tempered L^p function.

- *The linear functional $L = L_f$ defined by*

$$L(\varphi) = L_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx, \tag{88}$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

- *When $p = \infty$, such a function is often called a `slowly increasing function`.*

Tempered L^p Functions

Example

([Stein and Weiss, 1971, pp. 21 – 22])

- *A measurable function f such that*

$$\frac{f(x)}{(1+||x||^2)^k} \in L^p(\mathbf{R}^n), \tag{87}$$

for some $1 \leq p \leq \infty$ and some positive integer k, is called a tempered L^p function.

- *The linear functional $L = L_f$ defined by*

$$L(\varphi) = L_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx, \tag{88}$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

- *When $p = \infty$, such a function is often called a* `slowly increasing function`.

Tempered L^p Functions

Example

([Stein and Weiss, 1971, pp. 21 – 22])

- *A measurable function f such that*

$$\frac{f(x)}{(1+||x||^2)^k} \in L^p(\mathbf{R}^n), \tag{87}$$

for some $1 \le p \le \infty$ and some positive integer k, is called a tempered L^p function.

- *The linear functional $L = L_f$ defined by*

$$L(\varphi) = L_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx, \tag{88}$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

- *When $p = \infty$, such a function is often called a* `slowly increasing function.`

Tempered L^p Functions

Example

([Stein and Weiss, 1971, pp. 21 – 22])

- *A measurable function f such that*

$$\frac{f(x)}{(1+||x||^2)^k} \in L^p(\mathbf{R}^n), \qquad (87)$$

for some $1 \leq p \leq \infty$ and some positive integer k, is called a tempered L^p function.

- *The linear functional $L = L_f$ defined by*

$$L(\varphi) = L_f(\varphi) = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx, \qquad (88)$$

for $\varphi \in \mathcal{S}$, is a tempered distribution.

- *When $p = \infty$, such a function is often called a* `slowly increasing function`.

Tempered Measures

Example

([Stein and Weiss, 1971, p. 22])

- *A tempered measure is a Borel measure μ such that $\int_{\mathbf{R}^n} \frac{1}{(1+||x||^2)^k} d\mu < \infty$, for some integer k.*
- *As in Example 3.5, L_μ defines a tempered distribution.*

Tempered Measures

Example

([Stein and Weiss, 1971, p. 22])

- *A tempered measure is a Borel measure μ such that $\int_{\mathbf{R}^n} \frac{1}{(1+||x||^2)^k} d\mu < \infty$, for some integer k.*
- *As in Example 3.5, L_μ defines a tempered distribution.*

Tempered Measures

Example

([Stein and Weiss, 1971, p. 22])

- *A tempered measure is a Borel measure* μ *such that* $\int_{\mathbf{R}^n} \frac{1}{(1+||x||^2)^k} d\mu < \infty$, *for some integer* k.
- *As in Example 3.5,* L_μ *defines a tempered distribution.*

Dirac δ-function

Example

([Stein and Weiss, 1971, p. 22])

- *For $x_0 \in \mathbf{R}^n$ and an n tuple β,*

$$L(\varphi) = (D^\beta \varphi)(x_0), \tag{89}$$

for $\varphi \in \mathcal{S}$, defines a tempered distribution.

- *The* `Dirac δ-function` *at x_0*

$$L(\varphi) = \varphi(x_0), \tag{90}$$

This is a special case of the measures having mass 1 concentrated at z_0.

- *When $D^\beta = \frac{\partial}{\partial x_i}$, i.e., $L(\varphi) = \frac{\partial \varphi}{\partial x_i}(0)$, it is a tempered distribution that is not within the previous four types of distributions.*

Dirac δ-function

Example

([Stein and Weiss, 1971, p. 22])

- *For $x_0 \in \mathbf{R}^n$ and an n tuple β,*

$$L(\varphi) = (D^\beta \varphi)(x_0), \tag{89}$$

for $\varphi \in \mathcal{S}$, defines a tempered distribution.

- *The* `Dirac δ-function` *at x_0*

$$L(\varphi) = \varphi(x_0), \tag{90}$$

This is a special case of the measures having mass 1 concentrated at z_0.

- *When $D^\beta = \frac{\partial}{\partial x_i}$, i.e., $L(\varphi) = \frac{\partial \varphi}{\partial x_i}(0)$, it is a tempered distribution that is not within the previous four types of distributions.*

Dirac δ-function

Example

([Stein and Weiss, 1971, p. 22])

- *For $x_0 \in \mathbf{R}^n$ and an n tuple β,*

$$L(\varphi) = (D^{\beta}\varphi)(x_0), \tag{89}$$

for $\varphi \in \mathcal{S}$, defines a tempered distribution.

- *The* `Dirac δ-function` *at x_0*

$$L(\varphi) = \varphi(x_0), \tag{90}$$

This is a special case of the measures having mass 1 concentrated at z_0.

- *When $D^{\beta} = \frac{\partial}{\partial x_i}$, i.e., $L(\varphi) = \frac{\partial \varphi}{\partial x_i}(0)$, it is a tempered distribution that is not within the previous four types of distributions.*

Dirac δ-function

Example

([Stein and Weiss, 1971, p. 22])

- *For $x_0 \in \mathbf{R}^n$ and an n tuple β,*

$$L(\varphi) = (D^\beta \varphi)(x_0), \tag{89}$$

for $\varphi \in \mathcal{S}$, defines a tempered distribution.

- *The* `Dirac δ-function` *at x_0*

$$L(\varphi) = \varphi(x_0), \tag{90}$$

This is a special case of the measures having mass 1 concentrated at z_0.

- *When $D^\beta = \frac{\partial}{\partial x_i}$, i.e., $L(\varphi) = \frac{\partial \varphi}{\partial x_i}(0)$, it is a tempered distribution that is not within the previous four types of distributions.*

Functions and Tempered Distributions

- The tempered distributions of L^p functions in Example 3.4, or more generally, tempered L^p functions Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called measures.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Functions and Tempered Distributions

- The tempered distributions of L^p functions in Example 3.4, or more generally, tempered L^p functions Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called measures.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Functions and Tempered Distributions

- The tempered distributions of L^p `functions` in Example 3.4, or more generally, `tempered` L^p `functions` Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called `measures`.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Functions and Tempered Distributions

- The tempered distributions of L^p `functions` in Example 3.4, or more generally, `tempered` L^p `functions` Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called `measures`.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Functions and Tempered Distributions

- The tempered distributions of L^p `functions` in Example 3.4, or more generally, `tempered` L^p `functions` Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called `measures`.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  - [Stein and Weiss, 1971, Chapter 1].

Functions and Tempered Distributions

- The tempered distributions of L^p `functions` in Example 3.4, or more generally, `tempered` L^p `functions` Example 3.6, are called functions.
- Example 3.5 and Example 3.7 define the distributions that are called `measures`.
- We shall write, in these cases, f and μ, instead of L_f and L_μ.
- These functions and measures may be considered as embedded in $\mathcal{S}'$.
- References
  i [Stein and Weiss, 1971, Chapter 1].

Characterization of Tempered Distributions

Theorem

([Stein and Weiss, 1971, Theorem I.3.11]) A linear functional L on $\mathcal{S}$ is a tempered distribution if and only if there exists a constant C and integers m and k such that

$$|L(\varphi)| \leq C \sum_{|\alpha| \leq m, |\beta| \leq k} \rho_{\alpha\beta}(\varphi) \tag{91}$$

for all $\varphi \in \mathcal{S}$.

Outline

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^* \psi \rangle \tag{92}$$

and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi\rangle = \langle \varphi, T^*\psi\rangle \tag{92}$$

and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi\rangle = \langle \varphi, T^{*}\psi\rangle \tag{92}$$

and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^* \psi \rangle \tag{92}$$

and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^* \psi \rangle \tag{92}$$

  and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^* \psi \rangle \tag{92}$$

and then use the result to define extended operations.

Operations for Tempered Distributions

- Operations on functions
  - convolution,
  - differentiation,
  - translastion,
  - dilation,
  - Fourier transform,

  can be extended to tempered distributions in $\mathcal{S}'$.
- The basid approach is to use the `adjoint operator` for testing functions

$$\langle T\varphi, \psi\rangle = \langle \varphi, T^*\psi\rangle \tag{92}$$

and then use the result to define extended operations.

Convolution for Tempered Distributions I

- If g is any function on $\mathbf{R}^n$, its reflection, $\tilde{g}$, is defined as

$$\tilde{g}(x) = g(-x). \quad (93)$$

- Fubini's theorem implies, if u, φ and ψ are all in $\mathcal{S}$,

$$\int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\,dx = \int_{\mathbf{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\,dx. \quad (94)$$

- The mappings

$$\psi \xrightarrow{u*\varphi} \int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\,dx, \quad (95)$$

$$\theta \xrightarrow{u} \int_{\mathbf{R}^n} u(x)\theta(x)\,dx, \quad (96)$$

are continuous linear functionals on $\mathcal{S}$, respectively.

Convolution for Tempered Distributions I

- If g is any function on $\mathbf{R}^n$, its `reflection`, $\tilde{g}$, is defined as

$$\tilde{g}(x) = g(-x). \tag{93}$$

- Fubini's theorem implies, if u, φ and ψ are all in $\mathcal{S}$,

$$\int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\,dx = \int_{\mathbf{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\,dx. \tag{94}$$

- The mappings

$$\psi \xrightarrow{u*\varphi} \int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\,dx, \tag{95}$$

$$\theta \xrightarrow{u} \int_{\mathbf{R}^n} u(x)\theta(x)\,dx, \tag{96}$$

are continuous linear functionals on $\mathcal{S}$, respectively.

Convolution for Tempered Distributions I

- If g is any function on $\mathbf{R}^n$, its `reflection`, $\tilde{g}$, is defined as
$$\tilde{g}(x) = g(-x). \tag{93}$$
- Fubini's theorem implies, if u, φ and ψ are all in $\mathcal{S}$,
$$\int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\, dx = \int_{\mathbf{R}^n} u(x)(\tilde{\varphi} * \psi)(x)\, dx. \tag{94}$$
- The mappings
$$\psi \xrightarrow{u*\varphi} \int_{\mathbf{R}^n} (u * \varphi)(x)\psi(x)\, dx, \tag{95}$$
$$\theta \xrightarrow{u} \int_{\mathbf{R}^n} u(x)\theta(x)\, dx, \tag{96}$$

are continuous linear functionals on $\mathcal{S}$, respectively.

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \stackrel{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.
- Thus, the `convolution` of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \stackrel{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.
- Thus, the `convolution` of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \stackrel{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.

- Thus, the convolution of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \overset{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.

- Thus, the convolution of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \stackrel{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.

- Thus, the `convolution` of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions II

- Denoting them by $u * \varphi$ and u, Eq. (94) implies

$$(u * \varphi)(\psi) = u(\tilde{\varphi} * \psi). \tag{97}$$

- If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$, the right-hand side of Eq. (97) is well-defined since

$$\tilde{\varphi} * \psi \in \mathcal{S}. \tag{98}$$

- Furthermore, the mapping

$$\psi \longrightarrow \tilde{\varphi} * \psi \overset{u}{\longrightarrow} u(\tilde{\varphi} * \psi), \tag{99}$$

being the composition of two continuous functions, is continuous.

- Thus, the `convolution` of the distribution u with the testing function φ, $u * \varphi$, is defined by Eq. (97).
- References
  i [Stein and Weiss, 1971, Chapter 1].

Convolution for Tempered Distributions III

- This convolution is associative in the sense that

$$(u * \varphi) * \psi = u * (\varphi * \psi), \tag{100}$$

for $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$.

- Theorem

([Stein and Weiss, 1971, Theorem I.3.13]) If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, then the convolution $u * \varphi$ is the function f, whose value at $x \in \mathbf{R}^n$, is

$$f(x) = u(\tau_x \tilde{\varphi}). \tag{101}$$

Moreover, f belongs to C^∞, and it, as well as all its derivatives, are `slowly increasing`.

Convolution for Tempered Distributions III

- This convolution is associative in the sense that

$$(u * \varphi) * \psi = u * (\varphi * \psi), \tag{100}$$

for $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$.

- Theorem

([Stein and Weiss, 1971, Theorem I.3.13]) If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, then the convolution $u * \varphi$ is the function f, whose value at $x \in \mathbf{R}^n$, is

$$f(x) = u(\tau_x \tilde{\varphi}). \tag{101}$$

Moreover, f belongs to C^∞, and it, as well as all its derivatives, are `slowly increasing`.

Convolution for Tempered Distributions III

- This convolution is associative in the sense that

$$(u * \varphi) * \psi = u * (\varphi * \psi), \tag{100}$$

for $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$.

- Theorem
([Stein and Weiss, 1971, Theorem I.3.13]) If $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, then the convolution $u * \varphi$ is the function f, whose value at $x \in \mathbf{R}^n$, is

$$f(x) = u(\tau_x \tilde{\varphi}). \tag{101}$$

Moreover, f belongs to C^∞, and it, as well as all its derivatives, are `slowly increasing`.

Differentiation for Tempered Distributions I

- Integration by parts implies

$$\int_{\mathbf{R}^n} (D^\beta u)(x)\varphi(x)\,dx = (-1)^{|\beta|} \int_{\mathbf{R}^n} u(x)(D^\beta \varphi)(x)\,dx \tag{102}$$

for $u, \varphi \in \mathcal{S}$.

- The mapping

$$\varphi \xrightarrow{D^\beta u} \int_{\mathbf{R}^n} (D^\beta u)(x)\varphi(x)\,dx, \tag{103}$$

$$\theta \xrightarrow{u} \int_{\mathbf{R}^n} u(x)\theta(x)\,dx, \tag{104}$$

are continuous linear functionals on $\mathcal{S}$, repectively.

Differentiation for Tempered Distributions I

- Integration by parts implies

$$\int_{\mathbf{R}^n} (D^\beta u)(x)\varphi(x)\,dx = (-1)^{|\beta|}\int_{\mathbf{R}^n} u(x)(D^\beta\varphi)(x)\,dx \tag{102}$$

  for $u, \varphi \in \mathcal{S}$.
- The mapping

$$\varphi \xrightarrow{D^\beta u} \int_{\mathbf{R}^n} (D^\beta u)(x)\varphi(x)\,dx, \tag{103}$$

$$\theta \xrightarrow{u} \int_{\mathbf{R}^n} u(x)\theta(x)\,dx, \tag{104}$$

  are continuous linear functionals on $\mathcal{S}$, repectively.

Differentiation for Tempered Distributions II

- Denoting them by $D^{\beta}u$ and u, Eq. (102) implies

$$(D^{\beta}u)(\varphi) = (-1)^{|\beta|}u(D^{\beta}\varphi). \tag{105}$$

- The right-hand side is well-defined since $D^{\beta}\varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^{\beta}\varphi \xrightarrow{u} u(D^{\beta}\varphi), \tag{106}$$

being the composition of two continuous functions, is continuous.
- Thus, the `the partial derivative` $D^{\beta}u$ of the distribution u, is defined by Eq. (105).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Differentiation for Tempered Distributions II

- Denoting them by $D^\beta u$ and u, Eq. (102) implies

$$(D^\beta u)(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi). \tag{105}$$

- The right-hand side is well-defined since $D^\beta \varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^\beta \varphi \overset{u}{\longrightarrow} u(D^\beta \varphi), \tag{106}$$

being the composition of two continuous functions, is continuous.
- Thus, the `the partial derivative` $D^\beta u$ of the distribution u, is defined by Eq. (105).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Differentiation for Tempered Distributions II

- Denoting them by $D^\beta u$ and u, Eq. (102) implies

$$(D^\beta u)(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi). \tag{105}$$

- The right-hand side is well-defined since $D^\beta \varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^\beta \varphi \xrightarrow{u} u(D^\beta \varphi), \tag{106}$$

being the composition of two continuous functions, is continuous.

- Thus, the `the partial derivative` $D^\beta u$ of the distribution u, is defined by Eq. (105).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Differentiation for Tempered Distributions II

- Denoting them by $D^\beta u$ and u, Eq. (102) implies

$$(D^\beta u)(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi). \tag{105}$$

- The right-hand side is well-defined since $D^\beta \varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^\beta \varphi \stackrel{u}{\longrightarrow} u(D^\beta \varphi), \tag{106}$$

being the composition of two continuous functions, is continuous.
- Thus, the the partial derivative $D^\beta u$ of the distribution u, is defined by Eq. (105).
- References
  - [Stein and Weiss, 1971, Chapter 1].

Differentiation for Tempered Distributions II

- Denoting them by $D^\beta u$ and u, Eq. (102) implies

$$(D^\beta u)(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi). \qquad (105)$$

- The right-hand side is well-defined since $D^\beta \varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^\beta \varphi \stackrel{u}{\longrightarrow} u(D^\beta \varphi), \qquad (106)$$

being the composition of two continuous functions, is continuous.

- Thus, the `the partial derivative` $D^\beta u$ of the distribution u, is defined by Eq. (105).
- References

i [Stein and Weiss, 1971, Chapter 1].

Differentiation for Tempered Distributions II

- Denoting them by $D^\beta u$ and u, Eq. (102) implies

$$(D^\beta u)(\varphi) = (-1)^{|\beta|} u(D^\beta \varphi). \tag{105}$$

- The right-hand side is well-defined since $D^\beta \varphi \in \mathcal{S}$.
- Furthermore, , the mapping

$$\varphi \longrightarrow D^\beta \varphi \stackrel{u}{\longrightarrow} u(D^\beta \varphi), \tag{106}$$

being the composition of two continuous functions, is continuous.

- Thus, the `the partial derivative` $D^\beta u$ of the distribution u, is defined by Eq. (105).
- References
  - i [Stein and Weiss, 1971, Chapter 1].

References I

- Adams, R. A. (2003).
  Sobolev Spaces.
  Academic Press, New York.
- Dugundji, J. (1966).
  Topology.
  Series in advanced mathematics. Allyn and Bacon.
- Hömander, L. (1990).
  The Analysis of Linear Partial Differential Operators, volume I.
  Springer–Verlag, Berlin, 2nd edition.
- Natterer, F. (2001).
  The Mathematics of Computerized Tomography.
  SIAM, Philadelphia, PA.

References II

- Natterer, F. and Wübbeling, F. (2001).
  Mathematical Methods in Image Reconstruction.
  SIAM, Philadelphia, PA.
- Rudin, W. (1991).
  Functional analysis.
  International Series in Pure and Applied Mathematics. McGraw-Hill, New York, 2nd edition.
- Stein, E. M. (1970).
  Singular Integrals and Differentiability Properties of Functions.
  Princeton University Press, Princeton, New Jersey.
- Stein, E. M. and Weiss, D. (1971).
  Introduction to Fourier Analysis on Euclidean Spaces.
  Princeton University Press, Princeton, New Jersey.

References III

Yosida, K. (1980).
Functional Analysis.
Springer–Verlag, Berlin.