Dynamics of $\mathrm{SL}(2,\mathbb{R})$-Cocycles and Applications to Spectral Theory

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Goals of this Minicourse

- against the backdrop of Barry Simon's 21st century problems, describe the state of affairs in the spectral theory of Schrödinger operators around the turn of the century
- state some of the major results obtained in this century
- explain why the technical core of the proofs is solely of dynamical nature
- more generally, extract the dynamical aspects of recent advances in spectral theory and indicate how further progress can be obtained
- describe recent joint work with Artur Avila and Jairo Bochi on the denseness of uniform hyperbolicity in a context relevant to Schrödinger operators

Lecture 1

Barry Simon's 21st Century Problems

Goals of this Lecture

- state three of Barry Simon's fifteen Schrödinger operators problems for the 21st century
- explain why these problems were central issues in spectral theory at the time
- describe the results that led to complete solutions of these three problems

The Almost Mathieu Operator

Consider the Hilbert space

$$\ell^2(\mathbb{Z}) = \left\{ \psi : \mathbb{Z} \to \mathbb{C} : \sum_{n \in \mathbb{Z}} |\psi(n)|^2 < \infty \right\}$$

and, for $\lambda, \alpha, \omega \in \mathbb{R}$, the linear operator

$$H_{\lambda,\alpha,\omega} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$$

given by

$$[H_{\lambda,\alpha,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda \cos(2\pi(\omega + n\alpha))\psi(n)$$

and its spectrum

$$\sigma(H_{\lambda,\alpha,\omega}) = \{E \in \mathbb{R} : (H_{\lambda,\alpha,\omega} - E)^{-1} \text{ does not exist}\}$$

Barry Simon's 21st Century Problems

Here are the three problems problems that concern

$$[H_{\lambda,\alpha,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))\psi(n)$$

Problem 4. (Ten Martini problem) Prove for all $\lambda \neq 0$ and all irrational α that $\Sigma_{\lambda,\alpha} = \sigma(H_{\lambda,\alpha,\omega})$ (this set is ω-independent) is a Cantor set, that is, it is nowhere dense.

Problem 5. Prove for all irrational α and $|\lambda| = 1$ that $\Sigma_{\lambda,\alpha}$ has measure zero.

Problem 6. Prove for all irrational α and $|\lambda| < 1$ that the spectrum is purely absolutely continuous.

Remark. Periodic potentials are well understood, so one may restrict attention to $\lambda \neq 0$ and α irrational. One may further restrict to $\alpha, \omega \in [0, 1)$.

Barry Simon's 21st Century Problems

All three problems are completely solved by now. Here are the key contributing papers:

Problem 4. Prove for all $\lambda \neq 0$ and all irrational α that $\Sigma_{\lambda,\alpha}$ is a Cantor set.

Puig 2003, Avila-Jitomirskaya 2009+

Problem 5. Prove for all irrational α and $|\lambda| = 1$ that $\Sigma_{\lambda,\alpha}$ has measure zero.

Avila-Krikorian 2006

Problem 6. Prove for all irrational α and $|\lambda| < 1$ that the spectrum is purely absolutely continuous.

Avila-Jitomirskaya 2009+, Avila-D. 2008, Avila 2009+

The Relation to Physics

We will first address the following two preliminary questions that come to mind naturally:

- Why consider Schrödinger operators?
- Why consider the cosine potential in the almost Mathieu operator?

A Quantum Particle in a 1D Discrete World

The state of the quantum system is described by a normalized element ψ of

$$\ell^2(\mathbb{Z}) = \left\{ \psi : \mathbb{Z} \to \mathbb{C} : \sum_{n \in \mathbb{Z}} |\psi(n)|^2 < \infty \right\}$$

The interpretation is as follows:

$$\mathrm{Prob}\,(\text{ particle is in } A \text{ }) = \sum_{n \in A} |\psi(n)|^2$$

A Quantum Particle in a 1D Discrete World

The state changes with time according to the Schrödinger equation:

$$i\partial_t \psi = H\psi$$

Here, H is the Schrödinger operator

$$[H\psi](n) = \psi(n+1)\psi(n-1) + V(n)\psi(n)$$

where the potential $V : \mathbb{Z} \to \mathbb{R}$ models the environment the quantum particle is exposed to.

Formally, the solution is given by

$$\psi_t = e^{-itH}\psi_0$$

A Quantum Particle in a 1D Discrete World

The Schrödinger operator is self-adjoint:

$$\langle \phi, H\psi \rangle = \langle H\phi, \psi \rangle$$

and hence the spectral theorem allows one to rigorously define a unitary operator e^{-itH}.

The "allowed energies" are given by the spectrum of H:

$$\sigma(H) = \{E \in \mathbb{R} : (H - E)^{-1} \text{ does not exist}\}$$

Moreover, for every $\psi \in \ell^2(\mathbb{Z})$, there is a so-called spectral measure $d\mu_\psi$ so that

$$\langle \psi, g(H)\psi \rangle = \int_{\sigma(H)} g(E)\, d\mu_\psi(E)$$

A Quantum Particle in a 1D Discrete World

Spectral measures are important because they are related to the long time behavior of the solutions to the Schrödinger equation.

Indeed, if $\psi(t)$ solves $i\partial_t\psi = H\psi$ and $d\mu$ is the spectral measure of $\psi(0)$, then

- the particle "travels freely" if $d\mu$ is absolutely continuous
- the particle "travels somewhat" if $d\mu$ is singular continuous
- the particle "does not travel" if $d\mu$ is pure point

We say that H has purely absolutely continuous spectrum (resp., purely singular continuous spectrum, pure point spectrum) if all spectral measures are purely absolutely continuous (resp., purely singular continuous, pure point).

The Almost Mathieu Operator

The potential $V : \mathbb{Z} \to \mathbb{R}$ models the environment the quantum particle is exposed to. One may regard $V(n)$ as the (relative) height of an obstacle at site n.

Since quasi-periodic structures exist in nature, quasi-periodic potentials

$$V(n) = f(\omega + n\alpha)$$

with $f : \mathbb{T} \to \mathbb{R}$ are physically relevant. The special case of $f(\omega) = 2\lambda\cos(2\pi\omega)$ arises in this context as the simplest non-constant example.

Historically, however, $f(\omega) = 2\lambda\cos(2\pi\omega)$ arose for a slightly different reason. It is obtained by separation of variables of a $2D$ model with constant magnetic field.

The Almost Mathieu Operator

Since the early investigations of the almost Mathieu operator, three main issues have attracted attention.

- the shape of the spectrum
- the Lebesgue measure of the spectrum
- the type of the spectral measures

The Shape of the Spectrum

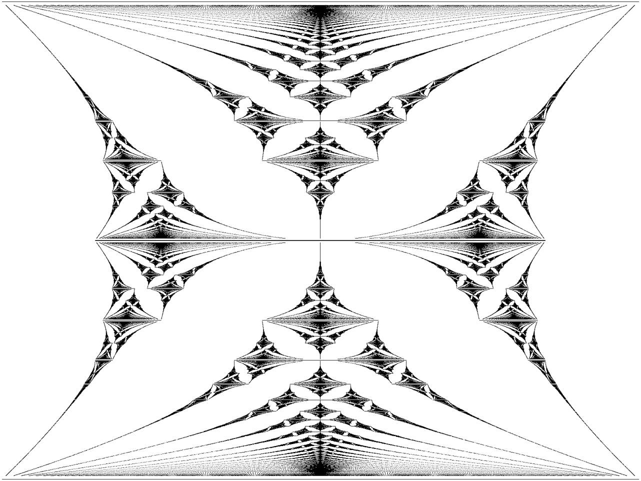
Based on numerics, the shape of the spectrum was conjectured to be a Cantor set.

In 1981, Mark Kac offered ten Martinis for a proof of Cantor spectrum for all non-periodic cases, that is, for every $\lambda \neq 0$ and every irrational α.

In 1982, Barry Simon coined the term *Ten Martini Problem*.

Here is the so-called Hofstadter butterfly, which shows the spectrum for $\lambda = 1$ and α ranging through $[0, 1]$:

The Hofstadter Butterfly



The Aubry-André Conjectures and the Role of λ

Aubry and André conjectured the following picture as the coupling constant runs from zero to infinity.

The measure of the spectrum is piecewise affine and runs from the value 4 down to zero and then back up. More precisely, it obeys the formula

$$\mathrm{Leb}(\sigma(H_{\lambda,\alpha,\omega})) = 4|1 - |\lambda||$$

Thus, the case $|\lambda| = 1$ is critical and, indeed, a phase transition occurs at that point:

The spectral measures are absolutely continuous for $|\lambda| < 1$, singular continuous for $|\lambda| = 1$, and pure point for $|\lambda| > 1$.

The Almost Mathieu Operator

A great many papers were devoted to the ten Martini problem and the Aubry-André conjectures in the 1980's and 1990's. All three issues were partially resolved by 1999.

It is probably fair to say that spectral theorists had exhausted their toolboxes and at the turn of the century, it was clear that entirely new ideas and approaches were needed to handle the remaining cases.

The three problems from Simon's list addressed this situation.

Looking back now, it turned out that tools from modern dynamics (especially the dynamics of $\mathrm{SL}(2, \mathbb{R})$-cocycles over irrational rotations) were exactly what was needed and, once this was realized, the three problems were solved quite quickly.

Generalized Eigenfunctions

Consider the difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

We say that $E \in \mathbb{R}$ is a generalized eigenvalue if this equation has a non-trivial solution u_E, called the corresponding generalized eigenfunction, satisfying

$$|u_E(n)| \leq C(1+|n|)^{\delta}$$

Theorem

(a) *Every generalized eigenvalue of H belongs to $\sigma(H)$.*
(b) *For almost every $E \in \mathbb{R}$ with respect to any spectral measure, there exists a generalized eigenfunction with $\delta = \frac{1}{2} + \varepsilon$.*
(c) *The spectrum of H is given by the closure of the set of generalized eigenvalues of H.*

Subordinate Solutions

A solution of $u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$ is called subordinate at ∞ if for every linearly independent solution $\tilde{u}$, we have

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} |u(n)|^2}{\sum_{n=1}^{N} |\tilde{u}(n)|^2} = 0$$

Subordinacy at $-\infty$ is defined analogously.

Theorem

Consider any spectral measure of H. Then μ_{ac} is supported by

$$\{E \in \mathbb{R} : \ \textit{at } +\infty \textit{ or } -\infty \textit{ there are no subordinate solutions}\}$$

and μ_{sing} is supported by

$$\{E \in \mathbb{R} : \ \textit{there is a solution that is subordinate at both } \pm\infty\}$$

Transfer Matrices and Cocycles

Now that we have seen that spectral analysis essentially reduces to a solution analysis, let us rewrite

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

as follows:

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = T_E(n) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$$

where

$$T_E(n) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M_E(n) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

where

$$M_E(n) = T_E(n) \cdots T_E(1)$$

Transfer Matrices and Cocycles

In the case where the potential V is dynamically defined

$$V(n) = f(T^n\omega)$$

(with $f : \Omega \to \mathbb{R}$, $T : \Omega \to \Omega$, and $\omega \in \Omega$), the transfer matrix

$$\begin{aligned} M_E(n) &= T_E(n) \cdots T_E(1) \\ &= \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V(1) & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} E - f(T^n\omega) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - f(T\omega) & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

takes the form of a linear cocycle.

Lecture 2

The Connection Between Dynamics and the Spectral Theory of Schrödinger Operators

Goals of this Lecture

- present the general framework: the Schrödinger operators with dynamically defined potentials and the associated family of Schrödinger cocycles
- state, motivate, and prove Johnson's theorem relating the spectrum of the operators and uniform hyperbolicity of the cocycles
- use Lyapunov exponents to subdivide the complement of uniform hyperbolicity further and discuss the connection of this subdivision to the spectral measures of the operators

Dynamically Defined Potentials

Now that we have seen that the spectrum and the spectral measures are of central importance in the study of $H_{\lambda,\alpha,\omega}$, let us relate them to dynamical quantities.

We consider a potential of the form

$$V(n) = f(T^n\omega)$$

where $T : \Omega \to \Omega$ is an invertible ergodic transformation (of a compact metric space), $\omega \in \Omega$, and $f : \Omega \to \mathbb{R}$ is bounded (continuous).

The almost Mathieu case corresponds to T being the rotation of the circle by α and $f(\omega) = 2\lambda \cos(2\pi\omega)$.

The (Almost Sure) Spectrum

Ergodicity of $T : \Omega \to \Omega$ with respect to μ, say, implies that there is a compact set $\Sigma \subset \mathbb{R}$ such that the spectrum of

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + f(T^n\omega)\psi(n)$$

is equal to Σ for μ-almost every $\omega \in \Omega$.

If Ω is compact, T is minimal, and f is continuous, then the spectrum is equal to Σ for *every* $\omega \in \Omega$ (choose any ergodic measure, apply the result above, and use strong operator convergence to go from almost everywhere to everywhere).

This implies the ω-independence of $\sigma(H_{\lambda,\alpha,\omega})$ mentioned in the first lecture.

Schrödinger Cocycles

It is clearly of interest to determine Σ. Let us discuss a dynamical characterization of this set. For simplicity, we will restrict our attention to the case where Ω is compact, T is minimal, and f is continuous. This covers the almost Mathieu operator and many other cases of interest.

For a given energy $E \in \mathbb{R}$, we consider the following skew-product:

$$(T, A_E) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2, \ (\omega, v) \mapsto (T\omega, A_E(\omega)v)$$

where

$$A_E(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

Schrödinger Cocycles

The maps

$$(T, A_E) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2, \ (\omega, v) \mapsto (T\omega, A_E(\omega)v)$$

form the canonical family (indexed by the energy E) of $\mathrm{SL}(2,\mathbb{R})$-cocycles associated with the Schrödinger operators H. For this reason, we will call them Schrödinger cocycles in these lectures.

Define the matrices $A_E^n(\omega)$ by $(T, A_E)^n(\omega, v) = (T^n\omega, A_E^n(\omega)v)$.

We say that (T, A_E) (or, abusing notation, A_E) is *uniformly hyperbolic* if there are constants $C_1, C_2 > 0$ such that for every $\omega \in \Omega$ and $n \in \mathbb{Z}_+$, we have

$$\|A_E^n(\omega)\| \geq C_1 e^{C_2|n|}$$

Spectrum and Uniform Hyperbolicity

Theorem (Johnson 1986)

$$\mathbb{R} \setminus \Sigma = \{E \in \mathbb{R} : A_E \textit{ is uniformly hyperbolic}\}$$

Before giving the proof of this result, we recall some basic principles.

The first is that the cocycle generates the transfer matrices which map solution data from the origin to some other site.

Concretely, consider the difference equation

$$u(n+1) + u(n-1) + f(T^n\omega)u(n) = Eu(n)$$

Spectrum and Uniform Hyperbolicity

Rewriting $u(n+1) + u(n-1) + f(T^n\omega)u(n) = Eu(n)$ as

$$\begin{aligned}\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} &= \begin{pmatrix} E - f(T^{n-1}\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \end{pmatrix} \\ &= A_E(T^{n-1}\omega) \begin{pmatrix} u(n-1) \\ u(n-2) \end{pmatrix}\end{aligned}$$

and iterating this, we see that u solves the difference equation if and only if

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A_E^n(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$

In particular, uniform hyperbolicity of the cocycle ensures that there are pairs of solutions $u_\pm(\omega)$ such that $u_\pm(\omega)$ solves the difference equation above, decays exponentially at $\pm\infty$ and grows exponentially at $\mp\infty$.

Spectrum and Uniform Hyperbolicity

The second basic principle is related to generalized eigenvalues and generalized eigenfunctions.

Suppose that the difference equation

$$u(n+1) + u(n-1) + f(T^n\omega)u(n) = Eu(n)$$

has a solution u that grows sub-exponentially at both $\pm\infty$.

Then, we can truncate u at $\pm N$ and divide by its ℓ^2 norm. This results in a normalized element u_N of $\ell^2(\mathbb{Z})$.

An easy calculation then shows that

$$\lim_{N\to\infty} \|(H-E)u_N\| = 0$$

which implies that $E \in \sigma(H)$. (Note: The choice of the interval $[-N, N]$ is not essential.)

Spectrum and Uniform Hyperbolicity

Theorem (Johnson 1986)

$$\mathbb{R} \setminus \Sigma = \{E \in \mathbb{R} : A_E \textit{ is uniformly hyperbolic}\}$$

Proof of "$\supseteq$."

Suppose A_E is uniformly hyperbolic. Apply the first basic principle and deduce the existence of the special solutions $u_\pm(\omega)$.

It is then a straightforward calculation that

$$\langle \delta_n, (H-E)^{-1}\delta_m \rangle = \frac{u_-(\omega, \min\{n,m\}) u_+(\omega, \max\{n,m\})}{u_-(\omega,0)u_+(\omega,1) - u_-(\omega,1)u_+(\omega,0)}$$

indeed defines an inverse of $(H-E)$ and hence $E \in \mathbb{R} \setminus \Sigma$. $\square$

Spectrum and Uniform Hyperbolicity

Theorem (Johnson 1986)

$$\mathbb{R} \setminus \Sigma = \{E \in \mathbb{R} : A_E \textit{ is uniformly hyperbolic}\}$$

Proof of "$\subseteq$."

Assume that A_E is not uniformly hyperbolic. Then, for suitable $\varepsilon_k \to 0$ we can find ω_k and $n_k \to \infty$ such that

$$\|A_E^{n_k}(\omega_k)\| \leq e^{\varepsilon_k n_k}$$

Shifting the potential (or adjusting the origin), we can in this way generate pieces of solutions that are of subexponential size.

The second basic principle (in combination with the first) then allows us to show that E is a generalized eigenvalue of H for a suitable ω and this in turn implies that $E \in \Sigma$. $\square$

Spectrum and Uniform Hyperbolicity

Johnson's theorem shows that, since everything that spectral theory and quantum dynamics care about happens *inside the spectrum*, we have to go

beyond uniform hyperbolicity

and analyze the dynamics of the cocycles A_E there.

Naturally, we use Lyapunov exponents to subdivide further:

$$\Sigma = \mathcal{Z} \cup \mathcal{NUH}$$

Lyapunov Exponents and Spectral Type

The decomposition

$$\Sigma = \mathcal{Z} \cup \mathcal{NUH}$$

is obtained as follows.

Consider the Lyapunov exponent

$$L(E) = \inf_{n \geq 1} \frac{1}{n} \int_{\Omega} \log \|A_E^n(\omega)\| \, d\mu(\omega) = \lim_{n \to \infty} \frac{1}{n} \log \|A_E^n(\omega)\|$$

(for μ-almost every $\omega \in \Omega$) and set

$$\mathcal{Z} = \{E \in \mathbb{R} : L(E) = 0\}$$
$$\mathcal{NUH} = \{E \in \mathbb{R} : A_E \text{ is not uniformly hyperbolic and } L(E) > 0\}$$

Lyapunov Exponents and Spectral Type

The decomposition

$$\Sigma = \mathcal{Z} \cup \mathcal{NUH}$$

is closely related to the decomposition of spectral measures into a.c, s.c., and p.p. parts.

Unfortunately, this connection is not as clean and as general as Johnson's result for the spectrum. The rule of thumb is the following:

Inside $\mathcal{Z}$, spectral measures are absolutely continuous.

Inside $\mathcal{NUH}$, spectral measures are pure point.

In exceptional cases, singular continuous spectral measures can appear in both $\mathcal{Z}$ and $\mathcal{NUH}$. (Note: Avila's recent work!)

Lyapunov Exponents and Spectral Type

There is one result connecting Lyapunov exponents and the spectral type that holds in complete generality.

Denote by Σ_{ac} the (almost sure) absolutely continuous spectrum of H. In other words, Σ_{ac} is the smallest closed set that supports all purely absolutely continuous spectral measures of H (for almost every $\omega \in \Omega$).

Theorem (Ishii 1973, Pastur 1980, Kotani 1984)

$$\Sigma_{\mathrm{ac}} = \overline{\mathcal{Z}}^{\mathrm{ess}} = \{E \in \mathbb{R} : \mathrm{Leb}\,(\mathcal{Z} \cap (E - \varepsilon, E + \varepsilon)) > 0 \; \forall \varepsilon > 0\}$$

Lyapunov Exponents and Spectral Type

Theorem (Ishii 1973, Pastur 1980, Kotani 1984)

$$\Sigma_{\mathrm{ac}} = \overline{\mathcal{Z}}^{\mathrm{ess}} = \{E \in \mathbb{R} : \mathrm{Leb}\,(\mathcal{Z} \cap (E-\varepsilon, E+\varepsilon)) > 0 \ \forall \varepsilon > 0\}$$

Instead of giving a proof of the Ishii-Pastur-Kotani theorem (which would take another minicourse), we try to elucidate the result with some remarks and some weaker, and yet still interesting, statements.

Lyapunov Exponents and Spectral Type

Let us first comment on the easier half of the theorem: $\Sigma_{\mathrm{ac}} \subseteq \overline{\mathcal{Z}}^{\mathrm{ess}}$.

Consider $E \in \mathcal{NUH}$, that is, an energy in the spectrum with $L(E) > 0$. The Osceledec theorem shows that, for μ-almost every $\omega \in \Omega$, there are solutions $u_\pm(\omega)$ that decay exponentially at $\pm\infty$.

If they are linearly independent, we can construct $(H - E)^{-1}$ as explained earlier and deduce that $E \in \mathbb{R} \setminus \Sigma$; contradiction.

Thus, the solutions must be linearly dependent (and hence be multiples of each other), which implies that E is in fact an eigenvalue of H with an exponentially decaying eigenfunction.

Problem: The energy-dependence of the exceptional sets of measure zero.

Lyapunov Exponents and Spectral Type

Consider now the harder half of the theorem: $\Sigma_{\mathrm{ac}} \supseteq \overline{\mathcal{Z}}^{\mathrm{ess}}$.

It is much easier to show that $\Sigma_{\mathrm{ac}} \supseteq \overline{\mathcal{B}}^{\mathrm{ess}}$, where

$$\mathcal{B} = \left\{ E \in \mathbb{R} : \sup_{\omega, n} \|A_E^n(\omega)\| < \infty \right\}$$

Boundedness of the cocycle can be shown, for example, by proving reducibility. In this way one gets (useful!) additional information about the dynamics of the cocycle, apart from the mere vanishing of the Lyapunov exponent.

The Avila-Krikorian result goes in this direction. We will say more about this in the next lecture.

Kotani's Little Known Gem

Here we describe a result of Kotani (hidden in a long survey paper) that was the key to my work with Avila on Simon's 6th problem.

Problem 6. Prove for all irrational α and $|\lambda| < 1$ that the spectrum is purely absolutely continuous.

We will see in the next lecture that by the turn of the century, this statement was known for Diophantine frequencies but the proof was clearly limited to such frequencies.

The key realization of Kotani is that averaging of spectral measures over the phase does not lose any information about absolute continuity in the regime of zero Lyapunov exponents!

Kotani's Little Known Gem

Consider the ω-dependent spectral measure associated with H and δ_0, ν_ω, which obeys

$$\int_\Sigma g(E)\, d\nu_\omega(E) = \langle \delta_0, g(H)\delta_0 \rangle$$

Now average with respect to the underlying ergodic measure to obtain the measure ν, so that

$$\int_\Sigma g(E)\, d\nu(E) = \int_\Omega \langle \delta_0, g(H)\delta_0 \rangle \, d\mu(\omega)$$

The measure ν is called the density of states measure. It is always continuous and often even more regular. In fact, it is absolutely continuous with a smooth density for sufficiently random potentials (where the operators have pure point spectrum).

Kotani's Little Known Gem

Thus, the regularity of ν is in general (much) better than that of the individual spectral measures.

However, Kotani has shown that this phenomenon can only occur in the regime of positive Lyapunov exponents:

Theorem (Kotani 1997)

Consider the restriction of all measures in question to $\mathcal{Z}$. Then the following are equivalent:
(a) *The measure ν is absolutely continuous.*
(b) *For μ-almost every $\omega \in \Omega$, all spectral measures of H are absolutely continuous.*

Lecture 3

Almost Mathieu Schrödinger Cocycles, Aubry Duality, and Localization

Goals of this Lecture

We will try to explain as much as possible about the three 21st century problems concerning the almost Mathieu operator and their solutions.

Recall that they concern the nowhere denseness of the spectrum, the Lebesgue measure of the spectrum, and the spectral type:

Problem 4. Prove for all $\lambda \neq 0$ and all irrational α that $\Sigma_{\lambda,\alpha}$ is a Cantor set.

Problem 5. Prove for all irrational α and $|\lambda| = 1$ that $\Sigma_{\lambda,\alpha}$ has measure zero.

Problem 6. Prove for all irrational α and $|\lambda| < 1$ that the spectrum is purely absolutely continuous.

Some Key Insights

- Herman's estimate for the Lyapunov exponent: The Lyapunov exponent is strictly positive for $|\lambda| > 1$.
- Jitomirskaya's localization result: For non-Liouville rotations, the positivity of the Lyapunov exponent implies pure point spectrum with exponentially decaying eigenfunctions.
- Aubry duality, which is essentially the Fourier transform, relates the coupling constants λ and λ^{-1}.
- Aubry duality and localization for λ imply purely absolutely continuous spectrum for λ^{-1}.
- Puig's approach to Cantor spectrum: Aubry duality and localization for λ imply Cantor spectrum for λ^{-1}.
- Avila-Krikorian's approach to zero measure spectrum at critical coupling: for almost every $E \in \mathcal{Z}$, the cocycle is reducible

Positivity of the LE: The Herman Estimate

Theorem (Herman 1983)

$$L(E) \geq \log|\lambda|$$

Recall that $V(n) = 2\lambda\cos(2\pi(\omega + n\alpha))$. Setting $z = e^{2\pi i\omega}$, we see that $V_\omega(n) = \lambda\left(e^{2\pi i\alpha n}z + e^{-2\pi i\alpha n}z^{-1}\right)$. Thus,

$$A_E(T^n\omega) = \begin{pmatrix} E - \lambda\left(e^{2\pi i\alpha n}z + e^{-2\pi i\alpha n}z^{-1}\right) & -1 \\ 1 & 0 \end{pmatrix}$$

If we define, initially on $|z| = 1$,

$$N_E^n(z) = z^n A_E^n(\omega) = (zA_E(T^{n-1}\omega))\cdots(zA_E(\omega)),$$

we see that N_E^n extends to an entire function and hence $z \mapsto \log\|N_n(z)\|$ is subharmonic. Note: $\log\|N_E^n(0)\| = \log|\lambda|$.

Positivity of the LE: The Herman Estimate

Proof of the Herman Estimate.

$$\begin{aligned} L(E) &= \inf_{n\geq 1} \frac{1}{n} \int_0^1 \log \|A_E^n(\omega)\| \, d\omega \\ &= \inf_{n\geq 1} \frac{1}{n} \int_0^1 \log \|e^{2\pi i n\omega} A_E^n(e^{2\pi i\omega})\| \, d\omega \\ &= \inf_{n\geq 1} \frac{1}{n} \int_0^1 \log \|N_E^n(e^{2\pi i\omega})\| \, d\omega \\ &\geq \log \|N_E^n(0)\| \\ &= \log |\lambda|. \end{aligned}$$

$\square$

Positivity of the LE: The Herman Estimate

In fact, the Herman estimate is optimal:

Theorem (Bourgain-Jitomirskaya 2002)

For every $E \in \Sigma_{\lambda,\alpha}$, we have

$$L_{\lambda,\alpha}(E) = \max\{\log|\lambda|, 0\}$$

The proof is based on continuity properties of the Lyapunov exponent and computations in the case of rational α due to Krasovsky.

The Central Localization Result

Theorem (Jitomirskaya 1999)

Suppose that α is Diophantine and $\Sigma = \mathcal{NUH}$. Then, for an explicit full measure set of ω's that contains zero, H has pure point spectrum with exponentially decaying eigenfunctions.

The Diophantine condition used in the 1999 paper reads as follows: There are $c > 0$ and $r > 1$ such that for every $m \in \mathbb{Z} \setminus \{0\}$,

$$|\sin(2\pi m\alpha)| \geq \frac{c}{|m|^r}$$

The assumption was weakened by Avila and Jitomirskaya (2009+). It suffices to assume that the continued fraction approximants p_k/q_k of α obey

$$\lim_{k\to\infty} q_k^{-1} \log q_{k+1} = 0$$

Aubry Duality

Consider the operator $H_{\lambda,\alpha} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$ given by

$$[H_{\lambda,\alpha}\varphi](\omega, n) = \varphi(\omega, n+1)+\varphi(\omega, n-1)+2\lambda \cos(2\pi(\omega+n\alpha))\varphi(\omega, n).$$

Introduce the duality transform $\mathcal{A} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$,

$$[\mathcal{A}\varphi](\omega, n) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi in\eta} \varphi(\eta, m)\, d\eta.$$

This definition assumes initially that φ is such that the sum in m converges, but note that in terms of the Fourier transform on $L^2(\mathbb{T} \times \mathbb{Z})$, we have $[\mathcal{A}\varphi](\omega, n) = \hat{\varphi}(n, \omega + n\alpha)$, which may be used to extend the definition to all of $L^2(\mathbb{T} \times \mathbb{Z})$ and shows that $\mathcal{A}$ is unitary.

Aubry Duality

Theorem

Suppose $\lambda \neq 0$ and $\alpha \in \mathbb{T}$ is irrational.
(a) $H_{\lambda,\alpha}\mathcal{A} = \lambda \mathcal{A} H_{\lambda^{-1},\alpha}$.
(b) $\Sigma_{\lambda,\alpha} = \lambda \Sigma_{\lambda^{-1},\alpha}$.
(c) $L_{\lambda,\alpha}(E) = \log|\lambda| + L_{\lambda^{-1},\alpha}(\frac{E}{\lambda})$.
(d) *If $H_{\lambda,\alpha,\omega}$ has pure point spectrum for almost every $\omega \in \mathbb{T}$, then $H_{\lambda^{-1},\alpha,\omega}$ has purely absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*

See Avron-Simon 1983 and Gordon-Jitomirskaya-Last-Simon 1997.

Localization and Duality Imply Cantor Spectrum

Theorem (Puig 2004)

If α is Diophantine and $|\lambda| \neq 0, 1$, then $\Sigma_{\lambda,\alpha}$ is a Cantor set.

Cantor spectrum was known for Liouville α (Choi-Elliott-Yui 1990). Note also that for $|\lambda| = 1$, zero measure spectrum (known for almost every α, Last 1994) implies Cantor spectrum.

Given this situation, Puig addressed the Cantor spectrum issue precisely in the regime where previous methods were inadequate. Moreover, it covers full measure sets of coupling constants and frequencies.

The "real" result of Puig, in itself an amazing discovery, is that localization and duality imply Cantor spectrum.

Localization and Duality Imply Cantor Spectrum

Lemma

(a) *Suppose u is an exponentially decaying solution of*

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi n\alpha)u(n) = Eu(n) \qquad (1)$$

Consider its Fourier series $\hat{u}(\omega) = \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\omega}$. Then, $\hat{u}$ is real-analytic on $\mathbb{T}$ and the sequence $\tilde{u}(n) = \hat{u}(\omega + n\alpha)$ is a solution of

$$u(n+1)+u(n-1)+2\lambda^{-1}\cos(2\pi(\omega+n\alpha))u(n) = (\lambda^{-1}E)u(n) \quad (2)$$

(b) *Conversely, suppose u is a solution of (2) with $\omega = 0$ of the form $u(n) = g(n\alpha)$ for some real-analytic function g on $\mathbb{T}$. Consider the Fourier series $g(\omega) = \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n\omega}$. Then, the sequence $\{\hat{g}(n)\}$ is an exponentially decaying solution of (1).*

Localization and Duality Imply Cantor Spectrum

Lemma

Suppose $\alpha \in \mathbb{T}$ *is Diophantine and* $A : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ *is analytic. Assume that there is a non-vanishing analytic map* $v : \mathbb{T} \to \mathbb{R}^2$ *such that*

$$v(\omega + \alpha) = A(\omega)v(\omega) \quad \text{for every } \omega \in \mathbb{T}$$

Then, there are $c \in \mathbb{R}$ *and an analytic map* $B : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ *such that*

$$B(\omega + \alpha)^{-1} A(\omega) B(\omega) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad \text{for every } \omega \in \mathbb{T}.$$

Localization and Duality Imply Cantor Spectrum

Proof.

Let $B_1(\omega) = \begin{pmatrix} v_1(\omega) & -\frac{v_2(\omega)}{d(\omega)} \\ v_2(\omega) & \frac{v_1(\omega)}{d(\omega)} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ where $v(\omega) = \begin{pmatrix} v_1(\omega) \\ v_2(\omega) \end{pmatrix}$ and $d(\omega) = v_1(\omega)^2 + v_2(\omega)^2 > 0$. We have

$$B_1(\omega+\alpha)^{-1} A(\omega) B_1(\omega) = \begin{pmatrix} 1 & \tilde{c}(\omega) \\ 0 & 1 \end{pmatrix}$$

Now let $c = \int_{\mathbb{T}} \tilde{c}(\omega)\, d\omega$ and use the Diophantine condition to find $b : \mathbb{T} \to \mathbb{R}$ analytic such that $b(\omega+\alpha) - b(\omega) = \tilde{c}(\omega) - c$ for every $\omega \in \mathbb{T}$. Conjugating again with $B_2(\omega) = \begin{pmatrix} 1 & b(\omega) \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, yields the desired matrix, so we set $B(\omega) = B_1(\omega) B_2(\omega)$. $\square$

Localization and Duality Imply Cantor Spectrum

Proof of Puig's theorem.

Consider a coupling constant $\lambda > 1$, an eigenvalue E of $H_{\lambda,\alpha,0}$ and a corresponding exponentially decaying eigenfunction.

Apply the lemmas and conjugate the λ^{-1}-cocycle to $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$.

The constant c cannot be zero, otherwise we would get two linearly independent ℓ^2 solutions by the first lemma.

Use $c \neq 0$ to show that a small perturbation of the energy can make the cocycle uniformly hyperbolic.

Since the eigenvalues of $H_{\lambda,\alpha,0}$ are dense in $\Sigma_{\lambda,\alpha}$, Aubry duality now implies that $\Sigma_{\lambda^{-1},\alpha}$ is nowhere dense.

Apply Aubry duality again to find that $\Sigma_{\lambda,\alpha}$ is nowhere dense. $\square$

Reducibility

Recall that the second lemma in the proof of Puig's theorem concerned the conjugation of the cocycle to a constant matrix. This is a central concept:

Definition

The cocycle A_E is called reducible if there exist an analytic map $B : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ and a matrix $C_E \in \mathrm{SL}(2, \mathbb{R})$ such that

$$B(\omega + \alpha)^{-1} A(\omega) B(\omega) = C_E$$

More precisely, in the case the cocycle is called analytically reducible modulo $\mathbb{Z}$.

Reducibility Almost Everywhere in $\mathcal{Z}$

Theorem (Avila-Krikorian 2006)

Suppose α satisfies a recurrent Diophantine condition and $|\lambda| = 1$. Then, $\Sigma_{\lambda,\alpha}$ has zero Lebesgue measure.

One says that α satisfies a recurrent Diophantine condition if infinitely many iterates of α under the Gauss map obey a uniform Diophantine condition. This is a full measure condition.

Moreover, for all other α's, the zero measure property was already known at the time and hence this result solved Simon's fifth problem completely.

The "real" result of Avila and Krikorian is that reducibility holds almost everywhere in $\mathcal{Z}$.

Reducibility Almost Everywhere in $\mathcal{Z}$

Theorem (Avila-Krikorian 2006)

Suppose α satisfies a recurrent Diophantine condition. Then, for almost every $E \in \mathcal{Z}$, the cocycle A_E is reducible.

A.E. Reducibility Implies Zero Measure Spectrum.

Consider the case $|\lambda| = 1$. Then, $\Sigma_{\lambda,\alpha} = \mathcal{Z}$ and $H_{\lambda,\alpha}$ is self-dual.

Suppose $\mathcal{Z}$ has positive measure. Then, there exists an energy $E \in \mathcal{Z}$ for which A_E is reducible.

Thus, by duality, there exists an ω so that E is an eigenvalue of the (dual operator) $H_{\lambda,\alpha,\omega}$ with an exponentially decaying eigenfunction.

This shows that $L_{\lambda,\alpha}(E) > 0$; contradiction since $E \in \mathcal{Z}$. $\square$

Absolute Continuity of the Density of States Measure

Theorem (Avila-D. 2008)

For $|\lambda| < 1$, α irrational, and almost every ω, $H_{\lambda,\alpha,\omega}$ has purely absolutely continuous spectrum.

Again, there is a "real result" behind this statement, and it is the following:

Theorem (Avila-D. 2008)

The density of states measure of the almost Mathieu operator is absolutely continuous if and only if $|\lambda| \neq 1$.

Kotani's gem and $\Sigma_{\lambda,\alpha} = \mathcal{Z}$ for $|\lambda| \le 1$ then imply the first statement.

Absolute Continuity of the Density of States Measure

Theorem (Avila-D. 2008)

The density of states measure of the almost Mathieu operator is absolutely continuous if and only if $|\lambda| \neq 1$.

Recall that if

$$\beta(\alpha) = \limsup_{k \to \infty} q_k^{-1} \log q_{k+1}$$

vanishes, the theorem follows from duality and localization.

Thus, one may assume $\beta(\alpha) > 0$. The good approximation of α by rational numbers then allows one to deduce absolute continuity by approximation from absolute continuity results for the rational AMO. (Note that $|\lambda| = 1$ is exceptional due to $\mathrm{Leb}(\Sigma_{\lambda,\alpha}) = 0$.)

Lecture 4

Denseness of Uniform Hyperbolicity and Genericity of Cantor Spectrum