The Reversible Nearest Particle System
on a Finite Set *

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Abstract: In this paper we study the one-parameter family of attractive reversible nearest particle systems on \{1, 2, \cdots, N\}. Denote by \(\sigma_N\) the time that the system first hits the empty set. Then, \(\sigma_N\) has a logarithmic increasing rate as the parameter \(\lambda\) is small enough, but an exponential increasing rate as \(\lambda\) is large enough. Especially, it has a polynomial increasing rate in the critical case, i.e. \(\lambda = 1\).

Keywords: nearest particle system, first hitting time.

1 Introduction

A nearest particle system (or NPS for short) on \(S \subset \mathbb{Z}\) is a spin system taking values in subsets of \(S\), and with flip rates for any finite \(A \subset S\)

\[
q(A, A \setminus \{x\}) = 1 \quad \text{if } x \in A, \\
q(A, A \cup \{x\}) = \beta(l_x(A), r_x(A)) \quad \text{if } x \in S \setminus A, \\
q(A, B) = 0 \quad \text{otherwise},
\]

where \(l_x(A)\) and \(r_x(A)\) are the distances from \(x\) to the nearest points in \(A\) to the left and right respectively. See Chapter 7 of [4] for more details.

Suppose the system is reversible, equivalently, by Theorem VI.1.2 of [4]

\[
\beta(l, r) = \frac{\beta(l)\beta(r)}{\beta(l + r)}, \quad \beta(l, \infty) = \beta(\infty, l) = \beta(l),
\]

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where $\sum_{l=1}^{\infty} \beta(l) < \infty$. In this paper we consider the one-parameter family $\beta_\lambda(l) = \lambda \psi(l)$, where $\psi(\cdot)$ is strictly positive and satisfies $\sum_{n=1}^{\infty} \psi(n) = 1$. Suppose that $\psi(n)/\psi(n + 1)$ decreases to 1 as $n \to \infty$, which ensures that the process is Feller and attractive.

Denote by $\{\xi^N_t : t \geq 0\}$ the NPS on $\{1, 2, \cdots, N\}$ starting from all sites occupied, and by $\sigma_N$ the first time it hits the empty set. We estimate $\sigma_N$, and the results read as follows.

**Theorem 1.1** Suppose

$$M \Delta \sup_n \sum_{l+r=n} \frac{\psi(l)\psi(r)}{\psi(n)} < \infty. \quad (1)$$

Then, for any $C_N$ such that $\lim_{N \to \infty} C_N = \infty$,

$$\lim_{N \to \infty} P_{\lambda}(\sigma_N \leq C_N f_{\lambda}(N)) = 1,$$

where

$$f_{\lambda}(N) = \begin{cases} 
\log N, & \text{if } \lambda M < 1, \\
N \log N, & \text{if } \lambda M = 1, \\
(\lambda M)^N, & \text{if } \lambda M > 1.
\end{cases}$$

**Theorem 1.2** Suppose there exists $n_0$ such that

$$\frac{\lambda}{\lambda'_c} > \max \left\{ \frac{2\psi(3n_0)}{\sum_{l=n_0}^{2n_0} \psi(l)\psi(3n_0-l)}, \frac{1}{\sum_{n_0}^{2n_0} \psi(l)} \right\}, \quad (2)$$

where $\lambda'_c$ is the critical value for the contact process on $\mathbb{Z}$. Then there is a constant $\gamma > 0$ such that

$$\lim_{N \to \infty} P(\sigma_N \geq e^{\gamma N}) = 1.$$

**Theorem 1.3** Suppose $\lambda = 1$, and the initial distribution of the NPS follows the renewal measure $\text{Ren}(\beta)$ whenever the initial state is not $\emptyset$. Suppose $C_N$ and $C'_N$ are two sequences of constants such that $\lim_{N \to \infty} C_N = \infty$ and $\lim_{N \to \infty} C'_N = 0$. Then

$$\lim_{N \to +\infty} P\left( C'_N N \leq \sigma_N \leq C_N N^2 \right) = 1.$$
The initial distribution $\text{Ren}(\beta)$ of the previous theorem will be further elaborated in the beginning part of Section 4. The inequality (1) is not very restrictive. For example, let $\psi(n) = cn^{-\alpha}$, where $\alpha > 1$. A standard coupling shows that

$$
\lim_{N \to \infty} P(\sigma_N > (1 - \varepsilon) \log N) = 1, \quad \forall \varepsilon > 0.
$$

(3)

Theorem 1.1 and (3) imply that $\sigma_N$ has a logarithmic increasing rate as $\lambda$ is small enough. By Theorem 1.1 and Theorem 1.2, $\sigma_N$ has an exponential increasing rate as $\lambda$ is large enough. Theorem 1.3 tells us that $\sigma$ has a polynomial increasing rate as $\lambda = 1$, the critical point of the NPS on $\mathbb{Z}$.

This study is inspired by a series of papers by R. Durrett et al [1, 2, 3], in which the contact process is concerned. Namely, let $\{\zeta^N_t : t \geq 0\}$ be the contact process on $\{1, 2, \cdots, N\}$ with the parameter $\lambda'$ starting from all sites occupied, and $\tau_N$ be the first time it hits the empty set. Denote by $\lambda'_c$ the critical value of the contact process on $\mathbb{Z}$.

**Theorem 1.4** (i) If $\lambda' > \lambda'_c$, then there is a constant $\gamma_1(\lambda') > 0$ so that as $N \to \infty$, $\tau_N / (\log N) \to 1/\gamma_1(\lambda')$ in probability ([1], Theorem 1).

(ii) If $\lambda' > \lambda'_c$, then there is a constant $\gamma_2(\lambda') > 0$ so that as $N \to \infty$, $\log \tau_N / N \to \gamma_2(\lambda')$ in probability ([2], Theorem 2).

(iii) If $\lambda' = \lambda'_c$ and $a, b \in (0, \infty)$, then $P(aN \leq \tau_N \leq bN^4) \to 1$ as $N \to \infty$ ([3], Theorem 1.6).

The contact process is a non-reversible NPS. We believe the same conclusion is also true for the reversible NPS. However, we are only able to show it for very small $\lambda$ and very large $\lambda$. Moreover, the parameters in the lower estimate and the upper estimate should be amended to the same.

Theorems 1.1, 1.2 and 1.3 are proved in Sections 2, 3 and 4 in turn. In Section 3, we give a proof of (3) for the completeness.

## 2 Upper Estimate of $\sigma_N$

To get an upper bound of $\sigma_N$, we compare the evolution of $\{||\xi^N_t|| : t \geq 0\}$ with a birth and death process on $\{0, 1, \cdots, N\}$. On one hand, for any configuration
$|\xi| = i$, there are at most $i + 1$ intervals of consecutive vacant sites, which do not intersect mutually; in each interval, the rate that a new particle is born is no more than $\lambda M$. Hence the rate that $|\xi_t^N|$ increases 1 is not more than $(i + 1)\lambda M$. On the other hand, when $|\xi_t^N| = i$, the rate that $|\xi_t^N|$ decreases 1 equals $i$, the total rate that there is a particle dying.

Let $\{X_t : t \geq 0\}$ be the birth and death process on $\{0, 1, \cdots, N\}$ with death rate $a_i = i$, for any $i = 1, \cdots, N$; and birth rate $b_i = (i + 1)\lambda M$, for any $i = 0, \cdots, N - 1$. If initially $X_0 = x \geq |\xi_t^N|$, there is a coupling of $\{X_t : t \geq 0\}$ and $\{\xi_t^N : t \geq 0\}$ such that

$$P^{x, \xi_t^N} (X_t \geq |\xi_t^N|, \forall t \geq 0) = 1,$$

where $P^{x, \xi_t^N}$ is the conditional distribution of the initial state $(x, \xi_t^N)$.

**Proof of Theorem 1.1.** Let $\tau = \inf\{t > 0 : X_t = 0\}$ be the first time that $\{X_t : t \geq 0\}$ hits 0. Let $P^i$ be the conditional probability distribution on the initial state $i$, and $E^i$ be the expectation with respect to $P^i$, where $i = 0, \cdots, N$. By (4), $\sigma_N$ is stochastically dominated by $\tau$ if $X_0 \geq |\xi_t^N|$. Therefore, for any $t \geq 0$,

$$P(\sigma_N \geq t) \leq P^N(\tau \geq t).$$

By the Chebyshev inequality, for any $c > 0$,

$$P^N(\tau \geq cE^N\tau) \leq \frac{E^N\tau^2}{(cE^N\tau)^2}.$$  
(6)

It is shown in [6] that

$$E^N\tau = \sum_{i=1}^{N} e_i, \quad E^N\tau^2 = \sum_{i=1}^{N} \varepsilon_i,$$

where

$$e_i = \frac{1}{a_i} + \sum_{k=0}^{N-2-i} \frac{b_i b_{i+1} \cdots b_{i+k}}{a_i a_{i+1} \cdots a_{i+k} a_{i+k+1}} + \frac{b_i b_{i+1} \cdots b_{N-1}}{a_i a_{i+1} \cdots a_{N-1} a_N},$$

$$\varepsilon_i = \frac{2m_i}{a_i} + \sum_{k=0}^{N-2-i} \frac{2b_i b_{i+1} \cdots b_{i+k} m_{i+k+1}}{a_i a_{i+1} \cdots a_{i+k} a_{i+k+1}} + \frac{2b_i b_{i+1} \cdots b_{N-1} m_N}{a_i a_{i+1} \cdots a_{N-1} a_N},$$

and $m_i = E^i\tau$ for $i = 1, \cdots, N$. Notice that $m_i \leq m_N$ for any $i \leq N$. It follows that $\varepsilon_i \leq 2m_Ne_i$. Therefore,

$$E^N\tau^2 = \sum_{i=1}^{N} \varepsilon_i \leq 2m_N \sum_{i=1}^{N} e_i \leq 2m_NE^N\tau = 2 \left( E^N\tau \right)^2.$$
This together with (5) and (6) yields that
\[ P(\sigma_N \geq c_N E^N \tau) \leq 2c_N^{-2}. \] 
(8)

Therefore, an upper estimate of \( \sigma_N \) can be taken as \( c_N E^N \tau \). Suppose \( C_N \to \infty \) as \( N \to \infty \). Let \( c_N = C_N/C \), where \( C \) is the constant given in Lemma 2.1. Then the result holds by (8) and the next lemma.

\[ \square \]

**Lemma 2.1** There is a constant \( C \) such that for large \( N \),
\[
E^N \tau \leq \begin{cases} 
C \log N & \text{if } \lambda M < 1, \\
CN \log N & \text{if } \lambda M = 1, \\
C(\lambda M)^N & \text{if } \lambda M > 1.
\end{cases}
\]

**Proof.** By (7), for \( i = 1, \cdots, N \),
\[
e_i \leq \begin{cases} 
(1 - (\lambda M)^{N-i+1}) / ((1 - \lambda M) i), & \text{if } \lambda M \neq 1; \\
(N - i + 1)/i, & \text{if } \lambda M = 1.
\end{cases}
\]
Hence, if \( \lambda M < 1 \), there is a constant \( C \) so that
\[
E^N \tau = \sum_{i=1}^{N} e_i \leq (1 - \lambda M)^{-1} \sum_{i=1}^{N} i^{-1} \leq C \log N;
\]
if \( \lambda M > 1 \), there is a constant \( C \) so that
\[
E^N \tau = \sum_{i=1}^{N} e_i \leq (\lambda M - 1)^{-1} \sum_{i=1}^{N} (\lambda M)^{N-i+1} \leq C(\lambda M)^N;
\]
and if \( \lambda M = 1 \), there is a constant \( C \) so that
\[
E^N \tau = \sum_{i=1}^{N} e_i \leq (N + 1) \sum_{i=1}^{N} i^{-1} - N \leq CN \log N.
\]
\[ \square \]

### 3 Lower Estimate of \( \sigma_N \)

We begin this section with recalling the monotone property of spin systems. Let \( \{\xi_t : t \geq 0\} \) and \( \{\zeta_t : t \geq 0\} \) be two spin systems with the same state space. Suppose that whenever \( \xi \leq \zeta \),
\[
c_1(x, \xi) \leq c_2(x, \zeta) \quad \text{if } \xi(x) = \zeta(x) = 0,
\]
and
\[ c_1(x, \xi) \geq c_2(x, \zeta) \quad \text{if} \quad \xi(x) = \zeta(x) = 1. \]

Then by Theorem III.1.5 of [4] there is a coupling such that \( P^\xi \zeta (\xi_t \leq \zeta_t) = 1 \) for all \( \xi \leq \zeta \) and all \( t \geq 0 \). This together with Theorem 1.4 enlightens us to compare a NPS with a contact process by the renormalization argument. In other words, we divide \( \{1, \cdots, N\} \) into some subintervals and consider the existence of particles in each interval rather than at each site.

**Proof of Theorem 1.2.** Given \( n_0 \), let \( L = \lfloor N/n_0 \rfloor \) be the integer part of \( N/n_0 \), and divide \( \{1, 2, \cdots, Ln_0\} \) into subintervals
\[
I_k = \{(k-1)n_0 + 1, (k-1)n_0 + 2, \cdots, kn_0\}, \quad k = 1, 2, \cdots, L.
\]

We compare \( \{\xi^N_t : t \geq 0\} \) with a contact process \( \{\varsigma^L_t : t \geq 0\} \) on \( \{1, \cdots, L\} \), whose initial state is
\[
\varsigma^L_0(k) = \begin{cases} 
1 & \text{if} \quad \sum_{x \in I_k} \xi^N_0(x) \geq 1; \\
0 & \text{otherwise}.
\end{cases}
\]

We claim that, by choosing carefully the infection parameter of \( \varsigma^L_t \),
\[
\sum_{x \in I_k} \xi^N_t(x) \geq \varsigma^L_t(k), \quad \forall t \geq 0, k = 1, \cdots, L. \tag{9}
\]

This can be violated only when \( \sum_{x \in I_k} \xi^N_t(x) = \varsigma^L_t(k) \). So \( \xi^N_t \) and \( \varsigma^L_t \) evolve independently until \( \sum_{x \in I_k} \xi^N_t(x) = \varsigma^L_t(k) \) for some \( k \) and \( t > 0 \). A coupling is then needed to preserve the inequality 9. There are two cases.

**Case 1.** \( \sum_{x \in I_k} \xi^N_t(x) = \varsigma^L_t(k) = 1 \). Equivalently, there is only one particle in the \( k \)-th subinterval in the configuration \( \xi \) of the NPS, and the individual at site \( k \) is infected in the configuration \( \varsigma \) of the contact process. Because both death rates are 1, we let both particles die at the same time.

**Case 2.** \( \sum_{x \in I_k} \xi^N_t(x) = \varsigma^L_t(k) = 0 \). Equivalently, there are no particles in \( I_k \) and the individual at site \( k \) of \( \varsigma \) is healthy. Consider birth rates of both processes.

If \( k = 1 \), by attractiveness of the NPS, the total birth rate in \( I_1 \) is at least \( \lambda \sum_{l=0}^{2n_0} \psi(l) \) if there are particles in \( I_2 \). The case \( k = L \) is similar. If \( 1 < k < L \), the total birth rate in \( I_k \) is at least \( \lambda \sum_{l=0}^{2n_0} \psi(l) \) if there is at least one particle in \( I_{k-1} \) and no particle in \( I_{k+1} \), or vice versa. If there are particles in both \( I_{k-1} \) and \( I_{k+1} \), then
the total birth rate in $I_k$ is at least $\lambda \sum_{l=n_0}^{2n_0} \psi(l) \psi(3n_0 - l)/\psi(3n_0)$. Assumption (2) implies that we can choose the infection rate $\lambda'$ of the contact process $\varsigma_t^L$ to satisfy the following inequality.

$$\lambda' < \lambda' \leq \min \left\{ \lambda \sum_{l=n_0}^{2n_0} \frac{\psi(l) \psi(3n_0 - l)}{2\psi(3n_0)}, \sum_{l=n_0}^{2n_0} \lambda \psi(l) \right\}.$$  

Then there is $P_N$, a coupling of $\{\xi_t^N : t \geq 0\}$ and $\{\varsigma_t^{[N/n_0]} : t \geq 0\}$, such that for any $t \geq 0$,

$$P_N \left( \sum_{x \in I_k} \xi_t^N(x) \geq \varsigma_t^{[N/n_0]}(k), \forall k = 1, \ldots, [N/n_0] \right) = 1. \quad (10)$$

For any $t \geq 0$,

$$P(\sigma_N \geq t) = P \left( \sum_{x \in I_k} \xi_t^N(x) \neq \emptyset \right) \geq P \left( \varsigma_t^L \neq \emptyset \right) \geq P(\tau_L \geq t),$$

where $\tau_L = \inf \{t : \varsigma_t^L = \emptyset\}$. This together with part (ii) of Theorem 1.4 implies that

$$\liminf_{N \to \infty} P(\sigma_N \geq e^{\gamma L/2}) \geq \lim_{N \to \infty} P(\tau_L \geq e^{\gamma L/2}) = 1.$$  

Let $\gamma = \gamma(\lambda')/4n_0$, then the result follows. □

**Proof of (3).** To be self-contained, we give a proof of (3). Let $\{\gamma_t^N : t \geq 0\}$ be a spin system on $\{1, 2, \ldots, N\}$ starting from all sites occupied, in which particles die independently with rate 1 and no new particles are born. Then there is a coupling such that $P(\gamma_t^N \leq \xi_t^N, \forall t > 0) = 1$. This implies that

$$P(\xi_t^N \neq \emptyset) \geq P(\gamma_t^N \neq \emptyset), \forall t \geq 0.$$  

Notice that $P(\gamma_t^N(x) = 1) \geq e^{-t}$ for any $x = 1, \ldots, N$, and $\gamma_t^N(x)$ are mutually independent. So

$$P(\sigma_N \geq \alpha(N)) \geq 1 - \left(1 - e^{-\alpha(N)}\right)^N, \forall \alpha(N) \geq 0.$$  

Choose $\alpha(N)$ such that $\left(1 - e^{-\alpha(N)}\right)^N$ converges to zero as $N \to \infty$. This gives the lower estimate of $\sigma_N$. Especially, let $\alpha(N) = (1 - \varepsilon) \log N$, where $\varepsilon > 0$. Then (3) follows. □
4 The Critical Case

Theorem 1.3 can be divided into two separate statements:

\[
\lim_{N \to \infty} P \left( \sigma_N \leq C_N N^2 \right) = 1; \tag{11}
\]

and

\[
\lim_{N \to \infty} P \left( C_N' N \leq \sigma_N \right) = 1. \tag{12}
\]

The two statements will be proved by two distinct approaches. We shall compare the critical NPS \( \{\xi_t^N : t \geq 0\} \) with a critical NPS on \( \mathbb{Z} \) to show (11), and compare it with a modified process to prove (12).

For any \( A = \{x_0, x_1, \cdots, x_k\} \subset \{1, 2, \cdots, N\} \), define

\[
\nu_\beta(A) = \beta(x_1 - x_0) \beta(x_2 - x_1) \cdots \beta(x_k - x_{k-1}) \sum_{l=x_0}^{\infty} \beta(l) \sum_{r=N+1-x_k}^{\infty} \beta(r).
\]

Let \( K_N = \sum_{A \in \mathcal{S}_N \setminus \{\emptyset\}} \nu_\beta(A) \) and \( \pi(A) = \nu_\beta(A)/K_N \). Then \( \pi \) is the induced probability measure of the renewal measure \( \text{Ren}(\beta) \) restricted on \( \{1, 2, \cdots, N\} \).

The critical NPS \( \{\xi_t^N\} \) is a Markov process taking values in \( \mathcal{S}_N \) with jump rate

\[
q(A, B) = \begin{cases} 
1 & \text{if } x \in A, B = A \setminus \{x\}, \\
\beta(l) \beta(r)/\beta(l + r) & \text{if } x \notin A, B = A \cup \{x\}; \\
0 & \text{otherwise},
\end{cases}
\]

and reversible with respect to \( \pi \) in the sense that \( \pi(A) q(A, B) = \pi(B) q(B, A) \) for \( A, B \in \mathcal{S}_N \ A \neq \emptyset, B \neq \emptyset \). Throughout this section we take \( \pi \) to be the initial distribution of \( \{\xi_t^N\} \).

Let \( \{\tilde{\xi}_t^N : t \geq 0\} \) be a Markov process on \( \mathcal{S}_N \), which has the same transition rates as \( \{\xi_t^N : t \geq 0\} \) except that particles can be born from the empty set. Namely, denote by \( \tilde{q} \) and \( q \) respectively the transition rates of \( \{\tilde{\xi}_t^N : t \geq 0\} \) and \( \{\xi_t^N : t \geq 0\} \), then

\[
\tilde{q}(A, B) = \begin{cases} 
q(A, B) & \text{if } A \neq \emptyset, \\
q & \text{if } A = \emptyset \text{ and } |B| = 1, \\
0 & \text{otherwise},
\end{cases}
\]

where \( q > 0 \) is a constant. Let

\[
\nu_\beta(\emptyset) = q^{-1}, \quad \nu_\beta|_{\emptyset} = \text{Ren}(\beta)|_{\emptyset}.
\]
Then $\tilde{\pi} = \nu_\beta / (K_N + q^{-1})$ is a reversible distribution of $\{\tilde{\xi}^N_t : t \geq 0\}$.

**Proof of Equation (12).** Let

$$\tau = \inf\{t \geq 0, \tilde{\xi}^N_t = \emptyset\},$$

$\tilde{P}$ be the distribution of $\{\tilde{\xi}^N_t : t \geq 0\}$ with initial distribution $\pi$, and $\tilde{E}$ be the expectation with respect to $\tilde{P}$. Notice that $\{\tilde{\xi}^N_t : t \geq 0\}$ is stationary under $\tilde{P}$. For any $t > 0$,

$$2t\pi(\emptyset) = \tilde{E} \int_0^{2t} 1_{\{\xi^N_s = \emptyset\}} ds.$$

By the Strong Markovian Property, the right side above equals

$$\tilde{E} \tilde{E} \left( \int_0^{2t} 1_{\{\xi^N_s = \emptyset\}} ds \bigg| \mathcal{F}_\tau \right) \geq \tilde{E} \tilde{E} \left( 1_{\{\tau < t\}} \int_0^{2t} 1_{\{\xi^N_s = \emptyset\}} ds \bigg| \mathcal{F}_\tau \right) \geq \tilde{E} \tilde{E} \left( 1_{\{\tau < t\}} \int_0^{\tau+t} 1_{\{\xi^N_s = \emptyset\}} ds \bigg| \mathcal{F}_\tau \right) = \tilde{P}(\tau < t) \tilde{E} \left( \int_0^{t} 1_{\{\xi^N_s = \emptyset\}} ds \bigg| \xi^N_0 = \emptyset \right).$$

Denote by $\sigma$ the first time $\{\tilde{\xi}^N_t : t \geq 0\}$ jumps. Then

$$\tilde{E} \left( \int_0^{\tau} 1_{\{\xi^N_s = \emptyset\}} ds \bigg| \xi^N_0 = \emptyset \right) \geq \tilde{E} \left( \sigma_{1_{\{\tau \leq \sigma\}}} \xi^N_0 = \emptyset \right) = \int_0^{\tau} \tilde{q}_0 e^{-\tilde{q}_0 s} ds,$$

where $\tilde{q}_0 = \sum_\xi \tilde{q}(\emptyset, \xi) = Nq$. Hence

$$\tilde{P}(\tau < t) \leq 2t\pi(\emptyset) \left( \int_0^{\tau} \tilde{q}_0 e^{-\tilde{q}_0 s} ds \right)^{-1} = \frac{2tq^{-1}}{K_N + q^{-1}} \left( \int_0^{\tau} Nqse^{-Nqs} ds \right)^{-1}.$$

Notice that

$$\tilde{P}(\tau < t) \geq \tilde{P}(\tau < t, \tilde{\xi}^N_0 \neq \emptyset) = \tilde{P}(\tilde{\xi}_0^N \neq \emptyset) \tilde{P}(\tau < t)_{\tilde{\xi}_0^N \neq \emptyset} = P(\sigma_N < t)K_N / (K_N + q^{-1}).$$

This together with (13) yields that

$$P(\sigma_N < t) \leq 2tK_N^{-1} \left( q \int_0^{\tau} Nqse^{-Nqs} ds \right)^{-1} = 2NtK_N^{-1} \left( \int_0^{Nqt} se^{-s} ds \right)^{-1}.$$

Notice that the right side does not depend on $q$, which is arbitrary. Let $q \to \infty$, then it follows that

$$P(\sigma_N < t) \leq 2NtK_N^{-1}, \quad \forall \ t > 0.$$

By (14), the lower estimate of $\sigma_N$ is such $t_N$ that $t_N/N$ converges to zero, which implies the result. □
Lemma 4.1 Let
\[ S_N(x, y) = \{ \xi^N : \xi^N(x) = \xi^N(y) = 1, \xi^N(z) = 0, \forall z < x, \text{ or } z > y \} . \]
Then \( \nu_\beta(S_N(x, y)) \leq 1 \) and there is a constant \( C > 0 \) such that \( \nu_\beta(S_N(x, y)) \geq C \) whenever \( y - x \) is large enough.

On the other hand, when \( N \) is large enough,
\[
K_N = \sum_{\xi \in S_N \setminus \{\emptyset\}} \nu_\beta(\xi) \geq \sum_{x=0}^{[N/3]} \sum_{y=[2N/3]}^N \nu_\beta(S_N(x, y)) \geq CN^2. \tag{14}
\]
where \( S_N = \{0, 1\}^{\{1,2,\ldots,N\}} \).

Proof of Lemma 4.1. Let \( X_n \) be the time until the first renewal \( \geq n \). Then \( \{X_n : n \geq 0\} \) is a Markov chain with transition probability \( p(0, n) = \beta(n + 1), p(n + 1, n) = 1 \) for all \( n \geq 0 \). Since \( \mu := \sum_{n=1}^{\infty} n \beta(n) < \infty \), \( \{X_n : n \geq 0\} \) has an invariant distribution \( \pi \), and \( \pi(0) = 1/\mu \). Thus \( P(X_n = 0) \) converges to \( 1/\mu \) as \( n \to +\infty \). Notice that \( X_n = 0 \) if and only if \( n \) is a renewal time. Hence, there exists \( n_0 > 0 \) such that \( P(X_n = 0) > (2 \mu)^{-1} \) for any \( n \geq n_0 \). Then
\[
\nu_\beta(S_N(x, y)) = P(X_{y-x} = 0) > (2 \mu)^{-1}, \quad \text{if } y - x > n_0.
\]
It is not difficult to check \( \nu_\beta(S_N(x, y)) = P(X_{y-x} = 0) \leq 1 \). \( \square \)

Let \( \{\eta_t : t \geq 0\} \) be a reversible nearest particle system on \( \mathbb{Z} \), and \( r_t \) the rightmost particle in \( \{\eta_t : t \geq 0\} \), i.e.
\[
r_t := \sup\{x : \eta_t(x) = 1\}.
\]

The properties of \( r_t \) of the critical NPS are studied in [5].

Lemma 4.2 ([5], Theorem 1) Let \( \{\eta_t : t \geq 0\} \) be the critical reversible nearest particle system on \( \mathbb{Z} \). Suppose the initial configurations have a particle at the origin and no particle on the left of the origin, and follows the renewal measure \( \text{Ren}(\beta) \) with density \( \beta(\cdot) \). Then, as \( a \to \infty \), \( \frac{r_n}{a} \) converges in distribution to a Brownian motion with diffusion constant \( D > 0 \) in the Skorohod space.

Proof of Equation (11). Since the transition mechanism of \( \{\eta_t : t \geq 0\} \) is translation invariant, we can regard the NPS on \([0, N]\) as the NPS on \([n, N + n]\) for any \( n \). So we do not distinguish them in symbols.
To use Lemma 4.2, we partition the configurations of \( \{0,1\}^N \) by the position of the rightmost particle. Namely, let \( A_x \) be the set of configurations whose rightmost particle is at \( x \), i.e.

\[
A_x = \{ \xi \in \{0,1\}^{\{0,\ldots,N\}} : \xi(x) = 1, \xi(y) = 0, \forall \ y > x. \}
\]

Denote by \( P \) be the distribution of \( \{\xi^N_t : t \geq 0\} \) with initial distribution in Theorem 1.3, and by \( P_{N,x} \) the distribution of the NPS on \([-N, \cdots, 0]\) whose initial configurations have a particle at \( x \), no particle to the right of \( x \), and follows the renewal measure \( \text{Ren}(\beta) \). Then

\[
P = \sum_{x=0}^{N} P(A_x) P_{N,x}.
\]

(15)

Denote by \( P \) the distribution of the NPS on \( \mathbb{Z} \) with the initial distribution in Lemma 4.2. Now regard \( P_{N,x} \) the distribution of the NPS on \([-N, \cdots, 0]\). Thanks to the attractive property, there is a coupling of \( P \) and \( \tilde{P}_{N,x} \) such that for all \( t > 0 \),

\[
\xi^N_t(i) \leq \eta_t(i), \quad -x \leq i \leq N - x
\]

almost surely if \( \eta_0([-x,0]) = \xi^N_0 \). By (16), \( \xi^N_t \equiv \emptyset \) once \( r_t < -x \), hence \( \sigma_N \leq \inf\{t : r_t < -x\} \) almost surely.

Suppose \( \lim_{N \to \infty} C_N = \infty \). For any \( C > 0 \),

\[
P_{N,x} (\sigma_N \leq C_N N^2) \geq P_{N,x} (\sigma_N \leq C_N x^2) \geq P (\exists t \leq C_N x^2 \text{ s.t. } r_t < -x)
\]

\[
\geq P (\exists t \leq C \text{ s.t. } r_{x^2 t} < -x) \geq P (\exists t \leq C \text{ s.t. } r_{x^2 t}/x < -1),
\]

whenever \( N \) is large. This together with Lemma 4.2 implies that

\[
\liminf_{N,x \to +\infty} P_{N,x} (\sigma_N \leq C_N N^2) \geq P (\exists t \leq C \text{ s.t. } B_t < -1), \quad \forall \ C > 0,
\]

where \( \{B_t : t \geq 0\} \) is a Brownian motion with diffusion constant \( D \). Let \( C \to +\infty \), the right side of the above equation converges to 1. Hence

\[
\lim_{N,x \to +\infty} P_{N,x} (\sigma_N \leq C_N N^2) = 1.
\]

Consequently, for any \( \varepsilon > 0 \), there exists \( N_0 > 0 \) such that

\[
P_{N,x} (\sigma_N \leq C_N N^2) > 1 - \varepsilon.
\]
for any $N \geq x \geq N_0$. This together with (15) implies that

$$P(\sigma_N \leq C_N N^2) = \sum_{x=0}^{N} P(A_x)P_{N,x}(\sigma_N \leq C_N N^2) \geq (1 - \varepsilon) \sum_{x=N_0}^{N} P(A_x). \quad (17)$$

By Lemma 4.1, on one hand,

$$\sum_{x=0}^{N_0-1} \nu_\beta(A_x) \leq \sum_{x=0}^{N_0-1} \sum_{y=0}^{x} \nu_\beta(S_N(y, x)) \leq N_0^2.$$

Therefore, as $N \to \infty$,

$$\sum_{x=N_0}^{N} P(A_x) \geq 1 - N_0^2/(CN^2) \to 1.$$

This together with (17) implies that

$$\liminf_{N \to \infty} P(\sigma_N \leq C_N N^2) \geq (1 - \varepsilon).$$

Let $\varepsilon \to 0$ and the result follows. \qed

**References**


