1. Introduction of large deviation theory.

In the field of large deviations, people concern about asymptotic computation of small probabilities on an exponential scale. Since the remarkable works by Donsker-Varadhan (and others) in seventies and eighties, the field has been developed into a relatively complete system. There have been several “general” tricks that become standard approaches in dealing with large deviation problems. Perhaps the most useful is Gätner-Ellis Theorem.

We have no intension to state this theorem in its full generality. Let \( \{Y_n\} \) be a sequence of non-negative random variables and let \( \{b_n\} \) be a positive sequence such that \( b_n \to \infty \). The basic assumption is existence of the limit

\[
\Lambda(\theta) = \lim_{n \to \infty} \frac{1}{b_n} \log E \exp \left\{ \theta b_n Y_n \right\} \quad \theta > 0
\]  

**Theorem 1.1.** (Gätner-Ellis). Under some regularity conditions on the function \( \Lambda(\cdot) \),

\[
\lim_{n \to \infty} \frac{1}{b_n} \log P \{Y_n \geq \lambda\} = \Lambda^*(\lambda) \quad \lambda > 0
\]

where

\[
\Lambda^*(\lambda) = \sup_{\theta > 0} \left\{ \theta \lambda - \Lambda(\theta) \right\}
\]

If the exponential moment generating function

\[
E \exp \left\{ \theta b_n Y_n \right\}
\]

does not exist or, (1.1) is not in the right scale to describe the large deviation behavior of \( \{Y_n\} \), we assume

\[
\Lambda(\theta) = \lim_{n \to \infty} \frac{1}{b_n} \log E \exp \left\{ \theta b_n Y_n^{1/p} \right\} \quad \theta > 0
\]  

where \( p > 0 \) is fixed.

Replacing \( Y_n \) by \( Y_n^{1/p} \) in Theorem 1.1, we have

**Theorem 1.2.** Under some regularity conditions on the function \( \Lambda(\cdot) \),

\[
\lim_{n \to \infty} \frac{1}{b_n} \log P \{Y_n \geq \lambda\} = \Lambda^*(\lambda^p) \quad \lambda > 0
\]

By Taylor expansion,

\[
E \exp \left\{ \theta b_n Y_n^{1/p} \right\} = \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m E Y_n^{m/p}
\]

When establishing (1.3) by “standard” approaches becomes techniquely impossible, one may attempt to estimate

\[
E Y_n^{m/p}
\]

When \( p \neq 1 \), there are some good reasons to feel unpleasent to face sometimes fractional power \( m/p \).
Lemma 1.1. The following two statements (1.5) and (1.6) are equivalent:

\[
\lim_{n \to \infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p} = \Psi(\theta) \quad \theta > 0
\]  

(1.5)

\[
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} = p \Psi \left( \frac{\theta}{p} \right) \quad \theta > 0
\]  

(1.6)

Proof. Due to similarity, we only show that (1.5) implies (1.6). Given \( \epsilon > 0 \),

\[
\frac{1}{b_{n}} \left[ \frac{p^{\frac{1}{2}}}{n^{\frac{1}{2}} + 1} \right] \leq \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1 + \epsilon}{\theta} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p}
\]

By Jensen inequality,

\[
\mathbb{E} Y_n^{m/p} \leq \left[ \mathbb{E} Y_n^{(p-1)m+1} \right]^{\frac{1}{p}}
\]

On the other hand, as

\[
b_n^{p-1m+1} \left( \mathbb{E} Y_n^{p-1m+1} \right)^{\frac{1}{p}} \geq 1
\]

we have

\[
\left( b_n^{p-1m+1} \mathbb{E} Y_n^{p-1m+1} \right)^{\frac{1}{p}} \leq b_n^{p(p-1m+1)} \mathbb{E} Y_n^{p-1m+1}
\]

Summerizing what we have,

\[
b_m^m \mathbb{E} Y_n^{m/p} \leq b_n^{p-1m+1} \mathbb{E} Y_n^{p-1m+1}
\]

Consequently,

\[
\frac{1}{b_{n}} \left[ \frac{p^{\frac{1}{2}}}{n^{\frac{1}{2}} + 1} \right] \leq \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1 + \epsilon}{\theta} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p}
\]

By Stirling formula, there are constants \( C > 0 \) and \( \delta > 0 \) such that

\[
\frac{\theta^m}{m!} b_n^m \mathbb{E} Y_n^{m/p} \leq C(1 + \delta)^{-m} \left( \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1 + \epsilon}{\theta} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p} \right)^p
\]

Thus,

\[
\mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \leq C \left( 1 + \frac{\delta}{\theta} \right) \left( \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1 + \epsilon}{\theta} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p} \right)^p
\]
Consequently,
\[
\limsup_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \leq p \Psi \left( \frac{1 + \epsilon}{p} \right)
\]
Letting \( \epsilon \to 0^+ \) on the right gives
\[
\limsup_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \leq p \Psi \left( \frac{\theta}{p} \right)
\]
On the other hand,
\[
\mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \geq \frac{\theta^{pm}}{(pm)!} b_n^m \mathbb{E} Y_n^m
\]
By Stirling formula again, for any \( 0 < \delta < \epsilon \), there is \( C > 0 \) such that
\[
C(1 + \delta)^{-m} \left( \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \right)^{1/p} \geq \frac{1}{m!} \left( \frac{\theta}{(1 + \epsilon)p} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p}
\]
for all \( m \geq 0 \). Thus
\[
C \frac{1 + \delta}{\delta} \left( \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \right)^{1/p} \geq \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{\theta}{(1 + \epsilon)p} \right)^m b_n^m \left( \mathbb{E} Y_n^m \right)^{1/p}
\]
By (1.5),
\[
\liminf_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \geq p \Psi \left( \frac{\theta}{(1 + \epsilon)p} \right) \quad \theta > 0
\]
Letting \( \epsilon \to 0^+ \) on the right,
\[
\liminf_{n \to \infty} \frac{1}{b_n} \log \mathbb{E} \exp \left\{ \theta b_n Y_n^{1/p} \right\} \geq p \Psi \left( \frac{\theta}{p} \right) \quad \theta > 0
\]

By Lemma 1.1 and Theorem 1.3, immediately we obtain

**Theorem 1.4.** Under (1.5) and some regularity condition on \( \Psi(\cdot) \),
\[
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P} \{ Y_n \geq \lambda \} = -I(\lambda) \quad (\lambda > 0)
\]
where
\[
I(\lambda) = p \sup_{\theta > 0} \left\{ \lambda^{1/p} - \Psi(\theta) \right\}
\]
We now apply Theorem 4 to a more special case. Let \( Y \geq 0 \) be a random variable such that
\[
\lim_{m \to \infty} \frac{1}{m} \log \frac{1}{(m!)^{1/\gamma}} \mathbb{E} Y^m = -\kappa
\]
for some \( \gamma > 0 \) and \( \kappa \in \mathbb{R} \).
Theorem 1.5 (König and Morters (2002)). Under (1.8)

\[ \lim_{t \to \infty} t^{-1/\gamma} \log \mathbb{P}\{Y \geq t\} = -\gamma e^{\kappa/\gamma}. \] (1.9)

**Proof.** We only need to check the condition (1.5) with \( Y_t = Y/t \), \( b_t = t^{1/\gamma} \) and \( p = 2\gamma \). Indeed, for any \( \theta > 0 \),

\[
\begin{align*}
&\lim_{t \to \infty} \frac{1}{t^{1/\gamma}} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} t^{m/(2\gamma)} \left( \mathbb{E} Y^m \right)^{1/\gamma} \\
&= \lim_{t \to \infty} \frac{1}{t^{1/\gamma}} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} t^{m/(2\gamma)} \left( m! \right)^{1/\gamma} e^{-\gamma m} \\
&= \lim_{t \to \infty} \frac{1}{t^{1/\gamma}} \log \sum_{m=0}^{\infty} \frac{1}{\sqrt{2\pi m!}} \left( \theta t^{1/\gamma} e^{-\frac{\theta^2}{2\gamma^2}} \right)^m \\
&= \lim_{t \to \infty} \frac{1}{t^{1/\gamma}} \log \sum_{m=0}^{\infty} \frac{1}{2^m m!} \left( \theta t^{1/\gamma} e^{-\frac{\theta^2}{2\gamma^2}} \right)^{2m} \\
&= \frac{1}{2} \theta^2 e^{-\kappa/\gamma}
\end{align*}
\]

Hence,

\[ I(\lambda) = 2\gamma \sup_{\theta > 0} \left\{ \theta \lambda^{\frac{1}{\gamma}} - \frac{1}{2} \theta^2 e^{-\kappa/\gamma} \right\} = \lambda^{\frac{1}{\gamma}} \gamma e^{\kappa/\gamma} \]

\[ \square \]

2. Large deviation for Brownian intersection local times.

Recall a \( d \)-dimensional Brownian motion \( W(t) \) is a stochastic process in \( \mathbb{R}^d \) with the following properties:

(1). For any \( s < t \), the increment \( W(t) - W(s) \) is independent of the history (up to the time \( s \))

\[ \left\{ W(u); \ u \leq s \right\} \]

(2). For any \( t > 0 \), \( W(t) \) is a normal random variable with mean 0 and covariance matrix \( tI_d \) (where \( I \) is the \( d \times d \) identity matrix).

By convention, we usually assume that \( W(0) = 0 \). When the fact that \( W(t) \) is a Markov process is emphasized, however, we may allow \( W(t) \) to start at any point \( x \in \mathbb{R} \) (i.e., \( W(0) = x \)).
Let $W_1(t), \cdots, W_p(t)$ be independent $d$-dimensional Brownian motions. If we allow $W_j(\cdot)$ ran up to time $t_j$ ($j = 1, \cdots, p$), a natural question is to ask how much time is spent for the $p$ independent trajectories $W_1(t), \cdots, W_p(t)$ to intersect. In other words, we are interested in the time set

$$\{ (s_1, \cdots, s_p) \in [0, t_1] \times \cdots \times [0, t_p]; \ W_1(s_1) \approx \cdots \approx W_p(s_p) \}$$

If properly defined, the Lebesgue measure of this set is called the intersection local time of $W_1(t), \cdots, W_p(t)$ and is denoted by $\alpha([0, t_1] \times \cdots \times [0, t_p])$.

**Theorem 2.1.** (Dvoretzky-Erdős-Kakutani (1950, 1954))

$$W_1(0, \infty) \cap \cdots \cap W_p(0, \infty) \neq \emptyset$$

if and only if $p(d - 2) < d$.

In the rest of this section, we assume $p(d - 2) < d$.

There are two equivalent ways to construct Brownian intersection local time in the multi-dimensional case. The first approach (Geman, Horowitz and Rosen (1984)) corresponds to the notation

$$\alpha([0, t_1] \times \cdots \times [0, t_p])$$

$$= \int_0^{t_1} \cdots \int_0^{t_p} \delta_0(W_1(s_1) - W_2(s_2)) \cdots \delta_0(W_{p-1}(s_{p-1}) - W_p(s_p)) ds_1 \cdots ds_p \quad (2.1)$$

Geman, Horowitz and Rosen (1984) prove that $p(d - 2) < d$, the occupation measure on $\mathbb{R}^{d(p-1)}$ given by

$$\mu_A(B) = \int_A 1_B(W_1(s_1) - W_2(s_2), \cdots, W_{p-1}(s_{p-1}) - W_p(s_p)) ds_1 \cdots ds_p \quad B \subset \mathbb{R}^{d(p-1)}$$

is absolutely continuous, with probability 1, with respect to Lebesgue measure on $\mathbb{R}^{d(p-1)}$ for any Borel set $A \subset (\mathbb{R}^p)^+$ (in particular, for $A = [0, t_1] \times \cdots \times [0, t_p]$) and, the density $\alpha(x, A)$ of such measure can be chosen so that the function

$$(x, t_1, \cdots, t_p) \longrightarrow \alpha(x, [0, t_1] \times \cdots \times [0, t_p]) \quad x \in \mathbb{R}^{d(p-1)} \quad (t_1, \cdots, t_p) \in (\mathbb{R}^p)^+$$

is jointly continuous. The random measure $\alpha(\cdot)$ on $(\mathbb{R}^p)^+$ is defined as

$$\alpha(A) = \alpha(0, A) \quad \forall \text{ Borel set } A \subset (\mathbb{R}^p)^+. $$

Another approach (Le Gall (1990)) constitutes the notation

$$\alpha([0, t_1] \times \cdots \times [0, t_p]) = \int_{\mathbb{R}^d} \left[ \prod_{j=1}^{p} \int_0^{t_j} \delta_x(W(s)) ds \right] dx \quad (2.2)$$

5
Let \( f(x) \) be a nice probability density function on \( \mathbb{R}^d \). Given \( \epsilon > 0 \), write \( f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1} x) \) and define

\[
\alpha_\epsilon([0, t_1] \times \cdots \times [0, t_p]) = \int_{\mathbb{R}^d} \left[ \prod_{j=1}^{p} \int_{0}^{t_j} f_\epsilon(W(s) - x) ds \right] dx
\]

Under \( p(d - 2) < d \), Le Gall (1990) shows that there is a random variable \( \alpha([0, t_1] \times \cdots \times [0, t_p]) \) such that

\[
\lim_{\epsilon \to 0^+} \alpha_\epsilon([0, t_1] \times \cdots \times [0, t_p]) = \alpha([0, t_1] \times \cdots \times [0, t_p])
\]

holds in \( L^m \)-norm for any \( m \geq 1 \) and for any \( t_1, \cdots, t_p > 0 \).

In the special case \( d = 1 \), let \( L_1(t, x), \cdots, L_p(t, x) \) be the local times of \( W_1, \cdots, W_p \), respectively. By the second construction, one can see that

\[
\alpha([0, t_1] \times \cdots \times [0, t_p]) = \int_{-\infty}^{\infty} \prod_{j=1}^{p} L_j(t_j, x) dx
\]

By the scaling property of Brownian motions

\[
\alpha([0, t]^p) = t^{\frac{d}{2} - \frac{d(p-1)}{2}} \alpha([0, 1]^p). \tag{2.3}
\]

Our main theorem in this section is the following

**Theorem 2.2.** Under \( p(d - 2) < d \),

\[
\lim_{t \to \infty} t^{-\frac{d}{2} - \frac{d(p-1)}{d(p-1)}} \log \mathbb{P}\left\{ \alpha([0, 1]^p) \geq t \right\} = -\frac{p}{2} \kappa(d, p) - \frac{4p}{d(p-1)} \tag{2.4}
\]

where \( \kappa(d, p) \) is the best constant of the Gagliardo-Nirenberg inequality

\[
|f|_{2p} \leq C \left\| \nabla f \right\|_2^{\frac{d(p-1)}{2p}} \left\| f \right\|_2^{1 - \frac{d(p-1)}{2p}} \quad f \in W^{1,2} (\mathbb{R}^d)
\]

**Remark.** We point out some facts about \( \kappa(d, p) \) which will be used later. Let

\[
\mathcal{F} = \left\{ f \in W^{1,2}(\mathbb{R}^d); \int_{\mathbb{R}^d} |f(x)|^2 = 1 \right\}
\]

Then

\[
\sup_{f \in \mathcal{F}} \left\{ \left( \int_{\mathbb{R}^d} |f(x)|^{2p} dx \right)^{1/p} - \frac{1}{2} \int_{\mathbb{R}^d} \left\| \nabla f \right\|^2 dx \right\} = \frac{2p - d(p-1)}{2p} \left( \frac{d(p-1)}{p} \right)^{\frac{d(p-1)}{2p-d(p-1)}} \kappa(d, p)^{\frac{4p}{2p-d(p-1)}} \tag{2.5}
\]
The second fact is that
\[
\rho = \left( \frac{2p - d(p - 1)}{2p} \right)^{\frac{2p-d(p-1)}{2p}} \left( \frac{d(p-1)}{p} \right)^{\frac{d(p-1)}{2p}} \kappa(d, p)^2
\]  
(2.6)
where
\[
\rho = \sup_f \int_{\mathbb{R}^d \times \mathbb{R}^d} G(x - y) f(x) f(y)
\]  
(2.7)
where the supremum is taken for all \( f \) on \( \mathbb{R}^d \) satisfying
\[
\int_{\mathbb{R}^d} |f(x)|^{\frac{2p}{p-1}} dx = 1
\]
and where
\[
G(x) = \int_0^\infty e^{-t} \frac{1}{(2\pi t)^{d/2}} \exp \left\{ -\frac{|x|^2}{2t} \right\} dt \quad x \in \mathbb{R}^d
\]

We now discuss the proof of our theorem. By Theorem 1.5 and by the relation (2.6) between \( \kappa(d, p) \) and \( \rho \), we need only to establish
\[
\lim_{m \to \infty} \frac{1}{m} \log(m!) - m^{-\frac{d(p-1)}{2}} \mathbb{E} \left[ \alpha([0, 1]^m) \right]
= p \log \rho + \frac{2p - d(p - 1)}{2} \log \frac{2p}{2p - d(p - 1)}
\]  
(2.8)
To calculate the moment of \( \alpha([0, 1]^p) \), notice that by (2.2) for any \( t_1, \ldots, t_p > 0 \),
\[
\mathbb{E} \left[ \alpha([0, t_1] \times \cdots \times [0, t_p]) \right] = \mathbb{E} \left[ \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \int_{[0, t_j]^m} ds_1 \cdots ds_m \prod_{k=1}^m \delta_{x_1}(W_j(s_k)) \right]
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \int_{[0, t_j]^m} ds_1 \cdots ds_m \mathbb{E} \prod_{k=1}^m \delta_{x_k}(W(s_k))
\]
Let \( \Sigma_m \) be the permutation group on \( \{1, \ldots, m\} \). By time rearrangement,
\[
\int_{[0, t_j]^m} \prod_{k=1}^m \delta_{x_k}(W(s_k))
= \sum_{\sigma \in \Sigma_m} \int_{\{0 \leq s_1, \ldots, s_m \leq t_j\}} ds_1 \cdots ds_m \mathbb{E} \prod_{k=1}^m \delta_{x_{\sigma(k)}}(W(s_k))
= \sum_{\sigma \in \Sigma_m} \int_{\{0 \leq s_1, \ldots, s_m \leq t_j\}} ds_1 \cdots ds_m \mathbb{E} \prod_{k=1}^m \delta_{x_{\sigma(k)} - x_{\sigma(k-1)}}(W(s_k) - W(s_{k-1}))
= \sum_{\sigma \in \Sigma_m} \int_{\{0 \leq s_1, \ldots, s_m \leq t_j\}} ds_1 \cdots ds_m \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)})
\]
where
\[ p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{ -\frac{|x|^2}{2t} \right\} dt \quad x \in \mathbb{R}^d \]

is the density function of \( W(t) \) and, we follow the convention \( s_0 = 0, x_{\sigma(0)} = 0 \).

Therefore,
\[
\mathbb{E} \left[ \alpha([0,t_1] \times \cdots \times [0,t_p])^m \right] \\
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \sum_{\sigma \in \Sigma_m} \int_{\{0 \leq s_1 \leq \cdots \leq s_m \leq t_j\}} ds_1 \cdots ds_m \\
\times \prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \\
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m p_{s_k-s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p \\
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p
\]

Let \( \tau_1, \cdots, \tau_p \) be independent exponential times with parameter 1. We assume the independence between \( \{\tau_1, \cdots, \tau_p\} \) and \( \{W_1(\cdot), \cdots, W_p(\cdot)\} \). Replacing \( t_1, \cdots, t_p \) by \( \tau_1, \cdots, \tau_p \) gives
\[
\mathbb{E} \left[ \alpha([0,\tau_1] \times \cdots \times [0,\tau_p])^m \right] \\
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m \int_0^\infty dte^{-t} \int_{\{0 \leq s_1 \leq \cdots \leq s_m \leq t\}} ds_1 \cdots ds_m \\
\times d\tau_k(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p \\
= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p
\]

where the second step follows from the identity
\[
\int_0^\infty dte^{-t} \int_{\{0 \leq s_1 \leq \cdots \leq s_m \leq t\}} ds_1 \cdots ds_m \prod_{k=1}^m \varphi_k(s_k - s_{k-1}) \\
= \prod_{k=1}^m \int_0^\infty e^{-t} \varphi_k(t) dt
\]

In the next section, we shall establish that
\[
\lim_{m \to \infty} \frac{1}{m} \log \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p = p \log \rho
\]
Or
\[
\lim_{m \to \infty} \frac{1}{m} \log \frac{1}{(ml)^p} \mathbb{E} \left[ \alpha([0, \tau_1] \times \cdots \times [0, \tau_p])^m \right] = p \log \rho \quad (2.13)
\]

We now prove the upper bound of (2.8). First notice that \( \tau_{\text{min}} = \min\{\tau_1, \ldots, \tau_p\} \) is exponential with parameter \( p \). By (2.2),

\[
\begin{align*}
\mathbb{E} \left[ \alpha([0, \tau_1] \times \cdots \times [0, \tau_p])^m \right] \\
\geq \mathbb{E} \left[ \alpha([0, \tau_{\text{min}}]^p)^m \right] = \mathbb{E} \frac{2^p - d(p - 1)}{2m} \mathbb{E} \left[ \alpha([0, 1]^p)^m \right] \\
= p \frac{2^p - d(p - 1)}{2m} \Gamma \left( 1 + \frac{2p - d(p - 1)}{2m} \right) \mathbb{E} \left[ \alpha([0, 1]^p)^m \right].
\end{align*}
\]

Thus
\[
\mathbb{E} \alpha([0, 1]^p)^m \leq p \frac{2^p - d(p - 1)}{2m} \Gamma \left( 1 + \frac{2p - d(p - 1)}{2m} \right) \mathbb{E} \left[ \alpha([0, 1]^p)^m \right]
\]

By Stirling formula and (2.13),
\[
\limsup_{m \to \infty} \frac{1}{m} \log(m!) - \frac{d(p - 1)}{2} \mathbb{E} \left[ \alpha([0, 1]^p)^m \right] \\
\leq p \log \rho + \frac{2p - d(p - 1)}{2} \log \frac{2p}{2p - d(p - 1)} \quad (2.14)
\]

We now prove the lower bound of (2.8). Let \( t_1, \ldots, t_p > 0 \). By (2.9)
\[
\mathbb{E} \left[ \alpha([0, t_1] \times \cdots \times [0, t_p])^m \right] \\
\leq \prod_{j=1}^p \left\{ \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \left[ \sum_{\sigma \in S_m} \int_{\{0 \leq s_1 \leq \cdots \leq s_m \leq t_j\}} ds_1 \cdots ds_m \right] \right. \\
\times \left. \prod_{k=1}^m p_{s_k - s_{k-1}}(x_{\sigma(k)} - x_{\sigma(k-1)}) \right\}^{1/p} \\
= \prod_{j=1}^p \left( \mathbb{E} \left[ \alpha([0, t_j]^p)^m \right] \right)^{1/p}
\]
So we have

\[
\mathbb{E} \left[ \alpha \left( [0, \tau_1] \times \cdots \times [0, \tau_p] \right)^m \right] \\
= \int_0^\infty \cdots \int_0^\infty dt_1 \cdots dt_p \exp \left\{ - \sum_{j=1}^p t_j \right\} \mathbb{E} \left[ \alpha \left( [0, t_1] \times \cdots \times [0, t_p] \right)^m \right] \\
\leq \int_0^\infty \cdots \int_0^\infty dt_1 \cdots dt_p \exp \left\{ - \sum_{j=1}^p t_j \right\} \prod_{j=1}^p \left( \mathbb{E} \left[ \alpha \left( [0, t_j] \right)^m \right] \right)^{1/p} \\
= \left\{ \int_0^\infty e^{-t} \left( \mathbb{E} \left[ \alpha \left( [0, t] \right)^m \right] \right)^{1/p} dt \right\}^p \\
= \mathbb{E} \left[ \alpha \left( [0, 1] \right)^m \right] \left\{ \int_0^\infty t^{\frac{2p-d(p-1)}{2}} e^{-t} dt \right\}^p \\
= \mathbb{E} \left[ \alpha \left( [0, 1] \right)^m \right] \left[ \Gamma \left( \frac{2p-d(p-1)}{2} + 1 \right) \right]^p
\]

where the fourth step follows from (2.2). Consequently,

\[
\mathbb{E} \left[ \alpha \left( [0, 1] \right)^m \right] \geq \left[ \Gamma \left( \frac{2p-d(p-1)}{2} + 1 \right) \right]^{-p} \mathbb{E} \left[ \alpha \left( [0, \tau_1] \times \cdots \times [0, \tau_p] \right)^m \right]
\]

By Stirling formula and (2.13),

\[
\liminf_{m \to \infty} \frac{1}{m} \log(m!) - \frac{d(p-1)}{2} \mathbb{E} \left[ \alpha \left( [0, 1] \right)^m \right] \\
\geq p \log \rho + \frac{2p-d(p-1)}{2} \log \frac{2p}{2p-d(p-1)}
\]

Finally, (2.8) follows from (2.14) and (2.15).