Chapter 5. Differential Geometry of Surfaces

5.1 Surface in parametric form

In 3D, a surface can be represented by

1. Explicit form \( z = f(x,y) \)
2. Implicit form \( f(x,y,z) = 0 \)
3. Vector form \( \mathbf{r}(x,y) = (x,y,f(x,y))^T \), or more general \( \mathbf{r}(u,v) = (x(u,v),y(u,v),z(u,v))^T \) depending on two parameters.

Example 1. The sphere of radius \( a \) has the geographical form

\[
\mathbf{r}(\theta,\phi) = (a \cos \theta \cos \phi, a \cos \sin \phi, a \sin \theta)^T \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi
\]

Example 2. The cylinder built on the curve \( \mathbf{r}(t) = (x(t),y(t))^T \), \( a \leq t \leq b \) in the xy-plane has the form

\[
\mathbf{r}(u,v) = (x(u),y(u),v)^T \quad a \leq u \leq b, -\infty < v < \infty
\]

Example 3. Surface of revolution by rotating a curve \( \mathbf{r}(t) = (p(t),0,q(t))^T \) \( a \leq t \leq b \) about the z-axis

\[
\mathbf{r}(u,v) = \mathbf{R}_z \mathbf{r}(t) = \begin{bmatrix}
\cos v & -\sin v \\
\sin v & \cos v \\
0 & 1
\end{bmatrix} \begin{pmatrix} p(u) \\ 0 \\ q(u) \end{pmatrix} = (p(u) \cos v, p(u) \sin v, q(u))^T
\]

\[ a \leq u \leq b, 0 \leq v \leq 2\pi \]

Specical cases are: a torus with \( \mathbf{r}(t) = (R + a \cos t, 0, a \sin t)^T \), \( 0 \leq t \leq 2\pi \)
A cone with \( \mathbf{r}(t) = (t, 0, mt)^T \), \( -\infty \leq t \leq \infty \)

Tangent vectors on the surface are \( \mathbf{r}_u(u,v) \) and \( \mathbf{r}_v(u,v) \). Hence a unit normal \( \mathbf{n} \) at \( \mathbf{r}(u,v) \) is given by

\[
\mathbf{n} = \pm \frac{\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)}{|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)|}
\]
assuming \( \mathbf{r}_u \times \mathbf{r}_v \neq 0 \) at \( (u,v) \), (non-singular point).

If the surface is given implicitly \( f(x,y,z) = 0 \), then

\[
\mathbf{n} = \pm \frac{\nabla f}{|\nabla f|}
\]
5.2 Metric properties

Distance on the surface is measured by

\[
\begin{bmatrix}
\frac{dx}{du} \\
\frac{dy}{du} \\
\frac{dz}{du}
\end{bmatrix}
= \frac{d\vec{r}}{du} = \vec{r}_u du + \vec{r}_v dv
= \begin{bmatrix}
\frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\
\frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \\
\frac{\delta z}{\delta u} & \frac{\delta z}{\delta v}
\end{bmatrix}
\begin{bmatrix}
du \\
dv
\end{bmatrix}
= A\vec{u}
\]

\[
ds^2 = d\vec{r} \cdot d\vec{r} = (d\vec{r})^T d\vec{r}
= d\vec{u}^T A^T A \vec{u}
= d\vec{u}^T B d\vec{u}
\]

where \( B = A^T A = \begin{bmatrix}
\vec{r}_u \cdot \vec{r}_u & \vec{r}_u \cdot \vec{r}_v \\
\vec{r}_v \cdot \vec{r}_u & \vec{r}_v \cdot \vec{r}_v
\end{bmatrix}
= \begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
\]

in standard notation

This is the \( 1 \text{st fundamental form} \) of the surface:

\[
ds^2 = Edu^2 + 2Fdu dv + Gdv^2
\]

The unit tangent \( \hat{t} \) along the curve \( \vec{r}(t) = \vec{r}(u(t), v(t)) \) is

\[
\hat{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{A\dot{\vec{u}}}{(\dot{\vec{u}}^T B\dot{\vec{u}})^{1/2}}
\]

The length of the segment of the curve \( \vec{r}(t) \) from \( t = t_0 \) to \( t = t_1 \) is

\[
s = \int_{t_0}^{t_1} |\dot{\vec{r}}| dt
= \int_{t_0}^{t_1} (A\dot{\vec{u}}^T B\dot{\vec{u}})^{1/2} dt
\]

If two curves \( \vec{r}_i(t) = \vec{r}(u_i(t), v_i(t)) \) intersect at an angle \( \theta \) on the surface, then

\[
\cos \theta = \hat{t}_1 \cdot \hat{t}_2 = \frac{\dot{\vec{u}}_1 A^T \dot{\vec{u}}_2}{(\dot{\vec{u}}_1 B\dot{\vec{u}}_1)^{1/2}(\dot{\vec{u}}_2 B\dot{\vec{u}}_2)^{1/2}}
= \frac{\dot{\vec{u}}_1 B\dot{\vec{u}}_2}{ds_1 ds_2}
\]

\[
\vec{r}(u + \delta u, v + \delta v)
\]

\[
\vec{r}(u, v) 
\]

\[
\vec{r}(u, v + \delta v)
\]

\[
\vec{r}(u + \delta u, v)
\]

\[
dA
\]

\[
v
\]

\[
u
\]
An elementry area $dA$ will be given by

$$dA = \left| \left[ \vec{r}(u + du, v) - \vec{r}(u, v) \right] \times \left[ \vec{r}(u, v + dv) - \vec{r}(u, v) \right] \right|$$

$$\approx \left| \vec{r}_u du \times \vec{r}_v dv \right|$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| dudv$$

We have

$$\left| \vec{r}_u \times \vec{r}_v \right| = \left| \vec{r}_u \right|^2 \cdot \left| \vec{r}_v \right|^2 - \left( \vec{r}_u \cdot \vec{r}_v \right)^2 = EG - F^2 = \det(B) = |B|$$

Thus

$$A_R = \int_R \frac{1}{2} |B| dudv$$

for any Region $R$ on the surface.

5.3 Curvatures

Recalled that for a general space curve $\vec{y}(t)$,

$$\vec{y}'(t) = \dot{\vec{y}}(t) = \vec{s}'(t)$$

$$\vec{y}''(t) = \ddot{\vec{y}}(t) = \vec{s}''(t) + \dot{s}^2 \hat{N}$$

where $\hat{N}$ is the principal normal to the curve, not to confuse with the normal $\hat{n}$ to the surface. If $\vec{y}(t)$ lies on the surface, then $\vec{y}(t) = \vec{r}(u(t), v(t))$, so that

$$\vec{r} = \vec{r}_u \hat{u} + \vec{r}_v \hat{v}$$

Differentiating again

$$\vec{r}'' = \vec{r}_{uu} \hat{u}^2 + 2 \vec{r}_{uv} \hat{u} \hat{v} + \vec{r}_{vv} \hat{v}^2 + \vec{r}_u \ddot{\hat{u}} + \vec{r}_v \ddot{\hat{v}}$$

Notice that the surface normal $\hat{n}$ is perpendicular to $\hat{t}$, $\hat{u}$ and $\hat{v}$

$$\therefore \hat{v} \cdot \hat{n} = \hat{s}^2 k \hat{N} \cdot \hat{n} = \hat{n} \cdot \vec{r}_{uu} \hat{u}^2 + 2 \hat{n} \cdot \vec{r}_{uv} \hat{u} \hat{v} + \hat{n} \cdot \vec{r}_{vv} \hat{v}^2$$

$$= \hat{u}^T \begin{bmatrix} \vec{r}_{uu} & \vec{r}_{uv} \\ \vec{r}_{uv} & \vec{r}_{vv} \end{bmatrix} \hat{u} = \hat{u}^T \hat{D} \hat{u}$$
The right-hand side is the second fundamental form of the surface,
\[ D = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \] is standard notation.

The normal curvature \( \kappa_n \) of the curve \( \vec{y}(t) \) in the surface is defined to be
\[ \kappa_n = \frac{\vec{y} \cdot \hat{n}}{S^2} = \frac{\dot{u}^T D \ddot{u}}{\dot{u}^T B \dot{u}} \]

Note that \( \kappa_n = \kappa \hat{N} \cdot \hat{n} = \kappa \cos \theta \), i.e. it is the component of \( \kappa \) in the direction of \( \hat{n} \). The other component of \( \kappa \) in the tangent plane is known as the geodesic curvature \( \kappa_g \), because of orthogonality, it has the magnitude
\[ \kappa_g^2 = \kappa^2 - \kappa_n^2 = \kappa^2 \sin^2 \theta \]

A surface curve \( \vec{y}(t) \), for which \( \kappa_g = 0 \) at every point is called a geodesic, (as straight as possible on the surface).

Consider row \( \kappa_n \) as a function of the direction \( \dot{u} = \left( \frac{du}{dt}, \frac{dv}{dt} \right)^T \),

Let \( \lambda = \frac{dv}{du} \) then
\[ \kappa_n = \kappa_n(\lambda) = \frac{Lu^2 + 2Mu\dot{v} + Nu^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \]

As \( \lambda \) changes direction in the surface, \( \kappa_n \) will achieve a maximum and a minimum value unless \( L: M: N = E: F: G \), in that case \( \kappa_n \) is independent of \( \lambda \) (such locations are called umbilic points)

Setting \( \frac{d\kappa_n}{d\lambda} = 0 \), the principal directions \( \lambda \) and the corresponding principal curvatures \( k \) are governed by
\[ (FN - GM)\lambda^2 + (EN - GL)\lambda + (EM - FL) = 0 \]
and
\[ (EG - F^2)k^2 - (LG + NE - 2FM)k - (LN - M^2) = 0 \]
The solutions satisfy
\[ E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0 \]
\[ \frac{1}{2}(k_1 + k_2) = \frac{LG - 2FM + NE}{2(EG - F^2)} \]
\[ k_1k_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det(D)}{\det(B)} \]

\[ K = k_1k_2 \] is called the Gaussian Curvature.

\[ H = \frac{1}{2}(k_1 + k_2) \] is the mean Curvature (Germain Curvature).

If the two directions \( \lambda_1, \lambda_2 \) correspond to
\[ d\vec{r}_1 = d\vec{r}(\lambda_1) = \vec{r}_u du_1 + \vec{r}_v dv_1 \]
\[ d\vec{r}_2 = d\vec{r}(\lambda_2) = \vec{r}_u du_2 + \vec{r}_v dv_2 \]

then
\[ \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2 = E \frac{du_1}{dt} \frac{du_2}{dt} + F \left( \frac{du_1}{dt} \frac{dv_2}{dt} + \frac{du_2}{dt} \frac{dv_1}{dt} \right) + G \frac{dv_1}{dt} \frac{dv_2}{dt} = 0 \]

Therefore the two principal directions are orthogonal.

Some geometrical meaning of the curvatures are the following.

1) A surface is called minimal if \( H = 0 \) everywhere. A minimal surface with boundary \( l \) has the smallest surface area among all surfaces with boundary \( l \).

2) If the principal directions are taken as the parametric curves, then \( F \equiv 0 \equiv M \) and

\[ k_1 = \frac{L}{E}, \quad k_2 = \frac{N}{G} \]

curvature in any other direction \( \lambda \) is then given by
\[ K(\lambda) = \frac{L + N\lambda^2}{E + G\lambda^2} = k_1 \frac{E}{E + G\lambda^2} + k_2 \frac{G\lambda^2}{E + G\lambda^2} \]
\[ = k_1 \cos^2 \psi + k_2 \sin^2 \psi \]

where \( \psi \) is the angle between \( \vec{r}_u \) and \( \dot{\vec{r}} = \vec{r}_u \dot{u} + \vec{r}_v \dot{v} \), \( \left( \lambda = \frac{\dot{\psi}}{\dot{u}} \right) \) which is known as the Euler's formula.

3) If \( K > 0 \) at a point \( P \) on the surface, then \( P \) is an elliptic point. As \( k_1, k_2 \) have the same sign, so all the surface is bending the same way in all directions.
4) If $K<0$ at a point $P$, then it is an hyperbolic point. Tangent directions at $P$ can bend away or towards the tangent plane.

5) If $K=0$ then either  
(a) One principal curvature only is zero. The point $P$ is a parabolic point, and one of the principal direction is straight near $P$.  
(b) Both principal curvature are zero. The point is a special type of umbilic point and is planar. 
Note: Isolated planar points can exist on surfaces which is far from planar. E.g. the monkey saddle surface $z = x(x + \sqrt{3}y)(x - \sqrt{3}y)$ at $P(0,0,0)$

6) Consider a point $P$ on the surface $z = \psi(x,y)$ By a change of coordinates, choose the origin at $P$, and $x, y$ axes along the principal directions at $P$, also $z$-axis in the direction of the surface normal at $P$, then the surface has equation $z = f(x,y)$ local to $(0,0,0)$ with $f(0,0)=0$

$$\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0 \quad \text{(} \because \text{x-y is the tangent plane)}$$

and

$$z = f(x, y) = \frac{1}{2} \left\{ \frac{\partial^2 f(0,0)}{\partial x^2} x^2 + 2 \frac{\partial^2 f(0,0)}{\partial x \partial y} xy + \frac{\partial^2 f(0,0)}{\partial y^2} y^2 \right\} + O(x^3, y^3)$$
Now, taking $\vec{r} = (x, y, f(x, y))$

$$\vec{r}_x = \left(1, 0, \frac{\partial f}{\partial x}\right), \quad \vec{r}_y = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

At $P = (0,0,0)$, $\vec{r}_x (p) = (1,0,0)$, $\vec{r}_y (p) = (0,1,0)$ and $\vec{n} = (0,0,1)$

$$E = \vec{r}_x (p) \cdot \vec{r}_x (p) = 1, \quad F = \vec{r}_x (p) \cdot \vec{r}_y (p) = 0, \quad G = \vec{r}_y (p) \cdot \vec{r}_y (p) = 1$$

the principal directions are orthogonal. Also

$$\vec{r}_{xx} (p) = \left(0,0, \frac{\partial^2 f}{\partial x^2}\right)_p, \quad \vec{r}_{xy} (p) = \left(0,0, \frac{\partial^2 f}{\partial x \partial y}\right)_p, \quad \vec{r}_{yy} (p) = \left(0,0, \frac{\partial^2 f}{\partial y^2}\right)_p$$

Hence

$$L = \vec{n} \cdot \vec{r}_{xx} (p) = \frac{\partial^2 f(0,0)}{\partial x^2}$$

$$M = \vec{n} \cdot \vec{r}_{xy} (p) = \frac{\partial^2 f(0,0)}{\partial x \partial y}$$

$$N = \vec{n} \cdot \vec{r}_{yy} (p) = \frac{\partial^2 f(0,0)}{\partial y^2}$$

$$\therefore z = \frac{1}{2} \left\{ Lx^2 + 2Mxy + Ny^2 \right\} + O(x^3, y^3)$$

$$= \frac{1}{2} \left\{ k_1 x^2 + k_2 y^2 \right\} + O(x^3, y^3)$$

These conics justify the terminology used. The simplest surface on which all three cases of Gaussian curvature occur is the torus.

$$K = \begin{cases} > & \text{if } K > 0 \\ < & \text{if } K < 0 \end{cases}$$

5.4 Special cases

5.4.1 Developable surfaces

Consider a surface in $\mathbb{R}^3$ which is constructed by a moving straight line, this so called ruled surface has the form

$$\vec{r}(u,v) = \vec{r}_0 (u) + v \vec{a}(u)$$
where \( \vec{r}_0(u) \) is the position vector of a point on a given line, \( \vec{a}(u) \) is the direction of the moving line (generators)

or if straight lines are used to join two given curves \( \vec{r}_0(u), \vec{r}_1(u) \) then

\[
\vec{r}(u, v) = (1 - v)\vec{r}_0(u) + v\vec{r}_1(u)
\]

Examples are cylinders and cones.

Now we look for conditions so that a ruled surface can be unrolled into a flat plane without distortion, (i.e. distances are preserved). If a ruled surface is developable, then all the generators eventually lie on a plane, therefore they are either parallel or intersect one another.

Now the intersection of two generators \( \vec{a} \) and \( \vec{a} + \hat{a} du \) is governed by

\[
\left( \vec{r}_0 + \dot{r} du - \vec{r}_0 \right) \left[ \vec{a} \times (\vec{a} + \hat{a} du) \right] = \dot{r}_0 \cdot (\vec{a} \times \hat{a}) = 0
\]
In case all the generators are parallel, the above condition is also satisfied. Therefore it is the condition for a ruled surface becomes developable. As

\[ \vec{r}_u = \dot{\vec{r}}_0 + v\hat{a}, \quad \vec{r}_v = \vec{a}, \quad \hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \]

\[ \vec{r}_{uu} = \ddot{\vec{r}}_0 + v\ddot{a}, \quad \vec{r}_{uv} = \ddot{\vec{a}}, \quad \vec{r}_{vv} = 0 \]

therefore, \[ N = \hat{n} \cdot \vec{r}_{vv} = 0 \]

\[ M = \hat{n} \cdot \vec{r}_{uv} = \hat{n} \cdot \vec{a} = \frac{1}{|\vec{r}_u \times \vec{r}_v|} (\dot{\vec{r}}_0 + v\hat{a}) \times \vec{a} \cdot \hat{a} = 0 \]

\[ \therefore K = \frac{\det(D)}{\det(B)} = \frac{LN - M^2}{\det(B)} = 0 \quad \text{for developable surfaces.} \]

In case the ruled surface is governed by two curves \( \vec{r}_0(u), \vec{r}_1(u) \), the condition becomes

\( (\vec{r}_1 - \vec{r}_0) \cdot (\dot{\vec{r}}_0 \times \dot{\vec{r}}_1) = 0 \), this fact is used in the tangent plane method of generating developable surface passing through two curves.
5.4.2 Envelope of space curves

Regarding \( \vec{r} = \vec{r}(u, v) \) as a family of curves \( \vec{r} = \vec{r}_v(u) \) depending on a parameter \( v \). There may exist a curve \( \vec{r} = \vec{r}_e(v) \) which is tangential to every curve \( \vec{r}_v(u) \) at the parametric value \( v \). Such a curve, if it exists, is called the envelope of the family of curves.

In terms of the original parameters \( \vec{r}(u, v) \) it implies \( \vec{r}_u \) is parallel to \( \vec{r}_v \) and the surface normal is not defined at these points:

\[
\vec{r}_u \times \vec{r}_v = 0
\]

In case of the developable surface, the generators will have an envelope if they are not parallel. An envelope satisfies

\[
\vec{r}_u \times \vec{r}_v = (\vec{r}_0 + v\vec{a}) \times \vec{a} = 0
\]

so long as the generators are not parallel, \( \vec{a} \times \vec{a} \neq 0 \) hence the location of common tangent occurs at

\[
v = \left( \frac{\vec{a} \times \dot{\vec{a}}}{|\vec{a} \times \dot{\vec{a}}|^2} \right) \left( \vec{r}_0 \times \vec{a} \right)
\]

and the equation of the envelope of the generators is

\[
\vec{r}_e = \vec{r}_0(u) + \frac{(\vec{a} \times \dot{\vec{a}}) \cdot (\vec{r}_0 \times \vec{a})}{|\vec{a} \times \dot{\vec{a}}|^2} \vec{a}(u)
\]