Algebra, Geometry and Analysis of Commuting Ordinary Differential Operators

Alexander Zheglov

Textbook / Reference

This list of references are compiled for the students of the I International Undergraduate Mathematics Summer School, 2018. It is by no means an extensive list of reference on the subject covered by the class, but just a suggestion for a starting point (or for further reading) with most relevance for this class.

History: First articles about commuting differential operators

- G. Wallenberg, Über die Vertauschbarkeit homogener linearer Differentialausdrücke, Arch. der Math. u. Phys. (3) 4, 252–268 (1903).
- I. Schur, Über vertauschbare lineare Differentialausdrücke, Sitzungsber. Berl. Math. Ges. 4, 2–8 (1905).

Classification of commutative subalgebras of ODOs

- J. Burchnall, T. Chaundy, Commutative ordinary differential operators, Proc. London Math. Soc. 21 (1923) 420–440.
- J. Burchnall, T. Chaundy, Commutative ordinary differential operators, Proc. Royal Soc. Lon- don (A) 118, 557–583 (1928).
- J. Burchnall, T. Chaundy, Commutative ordinary differential operators. II: The identity $P^n = Q^m$, Proc. Royal Soc. London (A) 134, 471–485 (1931).
- I. Krichever, Methods of algebraic geometry in the theory of nonlinear equations, Uspehi Mat. Nauk 32 (1977), no. 6 (198), 183–208, 287.
- I. Krichever, Commutative rings of ordinary linear differential operators, Func. Anal. Appl. 12 no. 3 (1978), 175–185.
- V. Drinfeld, Commutative subrings of certain noncommutative rings, Funct. Anal. Appl. 11 (1977), no. 1, 11–14, 96.
- D. Mumford, An algebro–geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation, Proceedings of the International Symposium on Algebraic Geometry, 115–153, Kinokuniya Book Store, Tokyo (1978).
- J.-L. Verdier, Équations différentielles algébriques, Séminaire Bourbaki, 30e année (1977/78), Exp. no. 512, 101-122, Lecture Notes in Math. 71, Springer (1979).
- M. Mulase, Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians, Internat. J. Math. 1 (1990), no. 3, 293–342.

Explicit examples of commuting ODOs and explicit determination of the spectral data

- J. Dixmier, Sur les alg‘ebres de Weyl, Bull. Soc. Math. France 96 (1968) 209–242.
- I. Krichever, S. Novikov, Holomorphic bundles over algebraic curves and nonlinear equations, Russian Math. Surveys, 35:6 (1980), 47–68.
- P. Grinevich, Rational solutions of equations of commutation of differential operators, Func. Anal. Appl. 16 (1982), no. 1, 19–24, 96.
- O. Mokhov, Commuting differential operators of rank 3 and nonlinear equations, Math. USSR– Izv. 35 (1990), no. 3, 629–655.
- E. Previato, G. Wilson, Differential operators and rank 2 bundles over elliptic curves, Compositio Math. 81 (1992), 107–119.
- A. Mironov, Self–adjoint commuting ordinary differential operators, Invent. Math. 197 (2014), no. 2, 417–431.
- O. Mokhov, Commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients, Topology, geometry, integrable systems, and mathematical physics, 323–336, Amer. Math. Soc. Transl. Ser. 2, 234, Amer. Math. Soc. (2014).
- A. E. Mironov, A. B. Zheglov, Commuting ordinary differential operators with polynomial coefficients and automorphisms of the first Weyl algebra, Int. Math. Res. Not. IMRN, 10, 2974– 2993 (2016).
- I. Burban, A. Zheglov, Fourier-Mukai transform on Weierstrass cubics and commuting differential operators, Oberwolfach Preprints (OWP), 3, Mathematisches Forschungsinstitut Oberwolfach Oberwolfach, Germany, 1–32 (2016); https://arxiv.org/abs/1602.08694

Survey articles and books (contain not only the theory of commuting ODOs, but are mainly devoted to various aspects of nonlinear differential equations

- I. M. Krichever, Integration of nonlinear equations by the methods of algebraic geometry, Russian Math. Surveys, 32 (6), 183—208 (1977).
- Y. Manin, Algebraic aspects of nonlinear differential equations, Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. 11, 5—152 (1978).
- G. Segal, G. Wilson, Loop groups and equations of KdV type, Inst. Hautes E´tudes Sci. Publ. Math. no. 61 (1985), 5–65.
- M. Mulase, Algebraic theory of the KP equations, Perspectives in Mathematical Physics, R.Penner and S.Yau, Editors, 151—218 (1994).
- E. Previato, Seventy years of spectral curves: 1923–1993, Integrable systems and quantum groups 419–481, Lecture Notes in Math. 1620, Springer (1996).
- B.A. Dubrovin, I.M. Krichever, S.P. Novikov, Integrable systems. I., in Dynamical systems. IV. Symplectic geometry and its applications. Transl. from the Russian by G. Wasserman, Encycl. Math. Sci. 4, 173-280 (1990); translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 4, 179-248 (1985).
- D. Mumford, Tata lectures on Theta II, Birkh¨auser, Boston, 1984

Special lectures for students

- E. E. Demidov, The KadomtsevPetviashvili hierarchy and the Schottky problem, Fundam. Prikl. Mat., 4:1 (1998), 367460; available at: http://www.mathnet.ru

Lecture notes

Yes.

Grades

Take-home exam. Exercises will be given after each lecture.

Syllabus

Description

There are two classical problems related to integrable systems, appeared and studied already in the works of I. Schur, J. Burchnall, T. Chaundy in the beginning of 20th cen- tury: how to construct explicitly a pair of commuting differential operators and how to classify all commutative subalgebras of differential operators. Both problems have broad connections with many branches of modern mathematics, first of all with integrable systems, since explicit examples of commuting operators provide explicit solutions of many non-linear partial differen- tial equations. The theory of commuting differential operators is far to be complete, but it is well developed for commuting ordinary differential operators.

This course involves an explanation of basic ideas and constructions from the theory of commuting ordinary differential operators as well as an overview of related open problems from algebra, algebraic geometry and complex analysis.

Course topics

The course will cover the following topics:

1. Basic algebraic properties of the ring of ordinary differential operators. Pseudo-differential operators and Schur's theory.

2. Basic facts and constructions from Commutative Algebra and Algebraic Geometry (needed mostly for algebraic curves): Nullstellensatz, Krull dimension, localisation of rings and modules, discrete valuation rings, smooth points via regular local rings, torsion free modules over regular local rings.

3. Basic algebro-geometric properties of commutative subalgebras of ordinary differential op- erators. The Burchnall-Chaundy lemma, the notion of algebro-geometric spectral data (spectral curve, spectral bundle).

4. Analytic theory of commuting differential operators: the Baker-Akhieser function, the Jacobian of the spectral curve.

5. Definition and basic properties of the theta-functions. Krichever's explicit formulae of the Baker-Akhieser functions. Classification of commutative subalgebras of ordinary differential operators.

6. Efficiency of the classification: explicit examples of commuting ODOs and problems related with their construction. Algebro-geometric solutions of non-linear equations: an overview.

Basic Algebraic Geometry

Wei-Ping Li

Textbook / Reference

1. Griffiths and Harris: Principles of Algebraic Geometry;
2. Griffiths: Introduction to Algebraic Curves.

Lecture notes

Yes.

Grades

Take-home exam.

Syllabus

The content will include the following materials: Concepts and examples of complex manifolds, line bundles, divisors, sheaves and cohomology of sheaves, Riemann surfaces, Riemann-Hurwitz formula, Riemann-Roch formula,

Fiber bundles and Characteristic Classes

Xianzhe Dai

Textbook / Reference

1. *Characteristic classes* by Milnor and Stasheff
2. Witten deofrmation and Chern-Weil theory by Weiping Zhang
3. Fiber bundles and Chern-Weil theory by Johan Dupont
4. *Differential forms in algebraic topology* by Bott and Tu
5. Foundations of Differential Geometry, I by Kobayashi & Nomizu
6. Geometry, Topology and Physics by M. Nakahara

Lecture notes

Yes.

Grades

50% homework and 50% take-home exam

Syllabus

Characteristic classes are invariants of fiber bundles or principal bundles on manifolds which describes the 'twisting' of the bundle (think Möbius band). Besides their importance in the classification of manifolds, they play a crucial role in differential geometry, complex geometry, gauge theory and physics. The goal of this minicourse is to introduce the basic notions of vector bundles and principal bundles, and the theory of characteristic classes, with an emphasis on the geometric aspect and its geometric application.

Harmonic Analysis

Christopher Sogge

Textbook / Reference

Book: "Fourier integrals in classical analysis" by C. D. Sogge, Cambridge University Press.

Lecture notes

Yes

Grades

Take-home exam

Syllabus

1. A) Classical Harmonic Analysis

    - Fourier transform
    - Distributions
    - Interpolations and multiplier theorems
    - Hardy-Littlewood-Sobolev theorem and elliptic regularity
    - Wave front sets (microlocal analysis of singularities)
    - Oscillatory integral distributions

    If time permits we shall also cover

2. B) Method of stationary phase

3. C) Oscillatory integral estimates and related topics

Symplectic Geometry

Tianjun Li & Bo Dai

Textbook / Reference

1. Dusa McDuff and Dietemar Salamon, Introduction to Symplectic Topology.
2. Tian-Jun Li, Symplectic Calabi-Yau surfaces. Handbook of geometric analysis, No. 3, pp.231-356,2010

Lecture notes

Yes.

Grades

Take-home exam.

Syllabus

Symplectic geometry is the geometry of symplectic manifolds. Two centuries ago, symplectic geometry provided a language for classical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differential geometry and topology. Contact geometry is in many ways an odd-dimensional counterpart of symplectic geometry. Both symplectic geometry and contact geometry are motivated by the mathematical formalism of classical mechanics, where one can consider either the even-dimensional phase space of a mechanical system or constant energy hypersurface of odd dimension. The goal of this short course is to provide a fast introduction to symplectic geometry as well as contact geometry. The first week lectures contain basics of symplectic geometry; the second week lectures include basics of contact geometry, and advanced topics on closed symplectic 4-manifolds and symplectic 4-manifolds with contact boundary. The plan of lectures is as follows.

Week 1

1. Lecture 1—Linear symplectic geometry

   Symplectic vector spaces, symplectic linear groups, Lagrangian subspaces, complex structures, Hermitian vector spaces, 4-dimensional geometry

2. Lecture 2—Symplectic manifolds

   Symplectic vector bundles, almost complex structures, examples of symplectic manifolds, spaces of symplectic forms, Moser stability and Darboux theorem, symplectic and Lagrangian submanifolds, neighborhood theorems.

3. Lecture 3—Almost Kahler geometry

   Integrability of almost complex structures, differential calculus on almost Hermitian manifolds, spin^c structures and Dirac operators on 4-manifolds, identities on almost Kahler manifolds.

Week 2

1. Lecture 1—Contact structure

   Liouville vector field, hyper surfaces of contact type, contact structure, symplectic manifolds with contact boundary

2. Lecture 2—Closed symplectic 4-manifolds

   Examples including rational and ruled surfaces, Constructions, Geography,

   Kodaira dimension, Properties of Seiberg-Witten (little proof for this part)

3. Lecture 3—Symplectic fillings in dimension 4

   Stein domains, Symplectic fillings and caps of contact 3-manifolds, Geography of symplectic fillings,