Introduction to representation theory of braid groups

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Braid groups were studied by E. Artin in the 1920’s.

The isotopy classes of geometric braids as above form a group by composition. This is the braid group with $n$ strands denoted by $B_n$. 
Braid relations

$B_n$ is generated by $\sigma_i$, $1 \leq i \leq n - 1$ with relations

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1
\]
A braid and its closure (figure eight knot):
Quantum symmetry in representations of braid groups

Homological representations of braid groups

Monodromy of KZ connection

Hypergeometric integrals

Representations of braid groups via quantum groups

Drinfeld-K. Theorem
Monodromy representations of logarithmic connections
Plan

- Monodromy representations of logarithmic connections
- Knizhnik-Zamolodchikov (KZ) connection
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- Monodromy representations of logarithmic connections
- Knizhnik-Zamolodchikov (KZ) connection
- Homological representations and KZ connections
- Quantum symmetry in homological representations
$\mathcal{F}_n(X)$: configuration space of ordered distinct $n$ points in $X$.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n ; x_i \neq x_j \text{ if } i \neq j\},$$

$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$
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Suppose $X = \mathbb{C}$.

$\pi_1(\mathcal{F}_n(\mathbb{C})) = P_n, \quad \pi_1(\mathcal{C}_n(\mathbb{C})) = B_n$

We set $X_n = \mathcal{F}_n(\mathbb{C})$
We set
\[ \omega_{ij} = d \log(z_i - z_j), \quad 1 \leq i \neq j \leq n. \]

Consider a total differential equation of the form \( d\phi = \omega \phi \) for a logarithmic form
\[ \omega = \sum_{i < j} A_{ij} \omega_{ij} \]

with \( A_{ij} \in M_m(\mathbb{C}) \).
As the integrability condition we infinitesimal pure braid relations

\[ [A_{ik}, A_{ij} + A_{jk}] = 0, \quad (i, j, k \text{ distinct}), \]
\[ [A_{ij}, A_{k\ell}] = 0, \quad (i, j, k, \ell \text{ distinct}) \]

The following are generalized for the complement of the union of complex hyperplanes.
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Representation theory of braid groups
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- As the holonomy of the flat connection \( \omega \) we obtain linear representation of the pure braid group \( P_n \).
- The horizontal section of \( \omega \) is expressed as an infinite sum of iterated integrals of logarithmic forms (hyperlogarithms).
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- The horizontal section of \( \omega \) is expressed as an infinite sum of iterated integrals of logarithmic forms (hyperlogarithms).
- Infinitesimal pure braid relations describe the nilpotent completion of the pure braid group \( P_n \) over \( \mathbb{Q} \) (Malcev algebra).
\( g \) : complex semi-simple Lie algebra.
\( \{ I_\mu \} \) : orthonormal basis of \( g \) w.r.t. Killing form.
\[ \Omega = \sum_\mu I_\mu \otimes I_\mu \]
\( r_i : g \to \text{End}(V_i), \ 1 \leq i \leq n \) representations.
$g$: complex semi-simple Lie algebra.

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$\Omega_{ij}$: the action of $\Omega$ on the $i$-th and $j$-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i < j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbb{C} \setminus \{0\}$$

$\omega$ defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbb{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$
As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{\otimes n}).$$
Monodromy representations of braid groups

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We shall express the horizontal sections of the KZ connection : $d\varphi = \omega \varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$ : fiber of $\pi$, \quad $Y_{n,m} = X_{n,m}/\mathcal{S}_m$
Representations of $\mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

$\lambda \in \mathbb{C}$
$M_\lambda$: Verma module of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight vector $v$ such that

$$Hv = \lambda v, \quad Ev = 0$$

$M_\lambda$ is spanned by

$$v, Fv, F^2v, \cdots$$

For a non-negative integer $\lambda$ we obtain an irreducible representation $V_\lambda$ of dimension $\lambda + 1$ as a quotient of $M_\lambda$.  

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Representation theory of braid groups
Consider the case $\lambda = 1$. Put $V = V_\lambda$.

The monodromy representations of braid groups

$$\theta_\kappa : B_n \rightarrow GL(V \otimes n).$$

Set $q = e^{2\pi \sqrt{-1/\kappa}}$ and

$$g_i = q^{1/4}\theta_\kappa(\sigma_i)$$

Then we have

$$(g_i - q^{1/2})(g_i + q^{-1/2}) = 0.$$ 

The monodromy representations factor through the Iwahori-Hecke algebra $\mathcal{H}(q)$. The above quadratic relation leads to the skein relation of the Jones polynomial.
Fix $P = \{(1, 0), \cdots, (n, 0)\} \subset D$, where $D$ is a 2 dimensional disc.

$\Sigma = D \setminus P$

$$\mathcal{F}_{n,m}(D) = \mathcal{F}_m(\Sigma), \quad \mathcal{C}_{n,m}(D) = \mathcal{F}_m(\Sigma)/\mathfrak{S}_m$$
Homology of relative configuration spaces

\[ H_1(C_{n,m}(D); \mathbb{Z}) \cong \mathbb{Z}^n \oplus \mathbb{Z} \]
Consider the homomorphism

$$\alpha : H_1(C_{n,m}(D); \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. 
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Composing with the abelianization map

$$\pi_1(C_{n,m}(D), x_0) \rightarrow H_1(C_{n,m}(D); \mathbb{Z}),$$

we obtain the homomorphism

$$\beta : \pi_1(C_{n,m}(D), x_0) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$
$H_*(\tilde{C}_{n,m}(D); \mathbb{Z})$ considered to be a $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$-module by deck transformations.

Express $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$ as the ring of Laurent polynomials $R = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.

$H_{n,m} = H_m(\tilde{C}_{n,m}(D); \mathbb{Z})$
Homological representations

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\[ H_{n,m} = H_m(\tilde{C}_{n,m}(D); \mathbb{Z}) \]

\( H_{n,m} \) is a free \( R \)-module of rank

\[ d_{n,m} = \binom{m+n-2}{m}. \]

\( \rho_{n,m} : B_n \longrightarrow \text{Aut}_R H_{n,m} \) : homological representations \((m > 1)\) extensively studied by Bigelow and Krammer ; they are faithful representations.
The space of null vectors is defined by

\[ \Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n \]

Consider the tensor product \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \).
Space of null vectors

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Consider the tensor product \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \).

\( m \) : non-negative integer

\[ W[|\Lambda| - 2m] = \{ x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} ; \ Hx = (|\Lambda| - 2m)x \} \]
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\[ N[|\Lambda| - 2m] = \{ x \in W[|\Lambda| - 2m] ; \ Ex = 0 \}. \]
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The KZ connection \( \omega \) commutes with the diagonal action of \( g \) on \( M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \), hence it acts on the space of null vectors \( N[|\Lambda| - 2m] \).

The monodromy of KZ connection

\[
\theta_{k,\lambda} : P_n \longrightarrow \text{Aut} \ N[|\Lambda| - 2m]
\]
Comparison theorem

We fix a complex number $\lambda$ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$. We have

$$\theta_{\kappa,\lambda} : B_n \longrightarrow \text{Aut} \ N[n\lambda - 2m].$$

Theorem

There exists an open dense subset $U$ in $(\mathbb{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi \sqrt{-1} \lambda/\kappa}, \quad t = e^{2\pi \sqrt{-1} / \kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M^\otimes n.$$
π : X_{n+m} → X_n : projection defined by
(z_1, ⋯, z_n, t_1, ⋯, t_m) ↦ (z_1, ⋯, z_n).
X_{n,m} : fiber of π.

Φ = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}}
\times \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}}

(multi-valued function on X_{n+m}).
Consider the local system \mathcal{L} associated with Φ.
Notation:
\( W[|\Lambda| - 2m] \) has a basis

\[
F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}
\]

with \( |J| = j_1 + \cdots + j_n = m \) and \( v_{\lambda_j} \in M_{\lambda_j} \) the highest weight vector.
Solutions to KZ equation

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with \(|J| = j_1 + \cdots + j_n = m\) and \(v_{\lambda_j} \in M_{\lambda_j}\) the highest weight vector.

**Theorem (Schechtman-Varchenko, Date-Jimbo-Matsuo-Miwa, ...)**

The hypergeometric integral

\[
\sum_{|J|=m} \left( \int_{\Delta} \Phi R_J(z, t) dt_1 \wedge \cdots \wedge dt_m \right) F^J v
\]

lies in \( N[|\Lambda| - 2m] \) and is a solution of the KZ equation, where \( \Delta \) is a cycle in \( H_m(Y_{n,m}, \mathcal{L}^*) \).
Theorem

There is an isomorphism

\[ N_h[\lambda n - 2m] \cong H_m(Y_{n,m}, \mathcal{L}^*) \]

which is equivariant with respect to the action of the braid group \( B_n \), where \( N_h[\lambda n - 2m] \) is the space of null vectors for the corresponding \( U_h(\mathfrak{g}) \)-module with \( h = 1/\kappa \).
Quantum symmetry for twisted chains

There is the following correspondence:

\[
\text{twisted multi-chains} \iff \text{weight vectors } F^{j_1} v_1 \otimes \cdots \otimes F^{j_n} v_n
\]

\[
\text{twisted boundary operator} \iff \text{the action of } E \in U_h(\mathfrak{g})
\]

\[
H_m(Y_{n,m}, \mathcal{L}^*) \iff N_h[\lambda n - 2m]
\]