INTRODUCTION TO ALGEBRAIC GEOMETRY

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1. Preliminary of Calculus on Manifolds

1.1. **Tangent Vectors.** What are tangent vectors we encounter in Calculus?

(1) Given a parametrised curve $\alpha(t) = \big(x(t), y(t)\big)$ in $\mathbb{R}^2$, $\alpha'(t) = \big(x'(t), y'(t)\big)$ is a tangent vector of the curve.

(2) Given a surface given by a parameterisation
$$\mathbf{x}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),$$
$\mathbf{n} = \dfrac{\partial \mathbf{x}}{\partial u} \times \dfrac{\partial \mathbf{x}}{\partial v}$ is a normal vector of the surface. Any vector perpendicular to $\mathbf{n}$ is a tangent vector of the surface at the corresponding point.

(3) Let $\mathbf{v} = (a, b, c)$ be a unit tangent vector of $\mathbb{R}^3$ at a point $p \in \mathbb{R}^3$, $f(x, y, z)$ be a differentiable function in an open neighbourhood of p, we can have the directional derivative of f in the direction $\mathbf{v}$:
$$D_{\mathbf{v}} f = a\frac{\partial f}{\partial x}(p) + b\frac{\partial f}{\partial y}(p) + c\frac{\partial f}{\partial z}(p). \tag{1.1}$$

In fact, given any tangent vector $\mathbf{v} = (a, b, c)$, not necessarily a unit vector, we still can define an operator on the set of functions which are differentiable in open neighbourhood of p as in (1.1)

Thus we can take the viewpoint that each tangent vector of $\mathbb{R}^3$ at p is an operator on the set of differential functions at p, i.e.
$$\mathbf{v} = (a, b, v) \to a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}|_p,$$
or simply
$$\mathbf{v} = (a, b, c) \to a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \tag{1.2}$$
with the evaluation at p understood. The set $T_p\mathbb{R}^3$ of such tangent vectors is a three-dimensional vector space. If we choose
$$\mathbf{e_1} = (1, 0, 0), \mathbf{e_2} = (0, 1, 0), \mathbf{e_3} = (0, 0, 1)$$

as a basis of the tangent space $T_p\mathbb{R}^3$, under our new viewpoint, $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ correspond to operators $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$ respectively. From now on, we will take a viewpoint that $T_p\mathbb{R}^3$ is a vector space with basis $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$ and each element $\mathbf{v} \in T_p\mathbb{R}^3$ acts on differentiable functions f at p as (1.1).

1.2. **Cotangent Vectors and Total Differential.** Given a vector space V, we have the dual space V^* defined as the vector space of linear transformations from V to $\mathbb{R}$:

$$V^* = \{L \colon V \to \mathbb{R} \mid L \text{ is a linear transformation}\}.$$

Given a basis $\mathbf{e_1}, \ldots, \mathbf{e_n}$ of V, there is a dual basis $\mathbf{e_1}^*, \ldots, \mathbf{e_n}^*$ of V^* defined as

$$\mathbf{e_i}^*(\mathbf{e_j}) = \delta_{i,j}.$$

Now let's apply the concept of linear algebra above to the vector space $T_p\mathbb{R}^3$. For the convenience of notations, we use (x_1, x_2, x_3) to denote the coordinates of $\mathbb{R}^3$. We get the dual space, denoted by $T_p^*\mathbb{R}^3$, called the cotangent space of $\mathbb{R}^3$ at p. Vectors in $T_p^*\mathbb{R}^3$ are called cotangent vectors. If we choose the basis $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}$ and $\dfrac{\partial}{\partial x_3}$ of $T_p\mathbb{R}^3$, its dual basis is denoted by dx_1, dx_2, dx_3, i.e.,

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{i,j}.$$

For a function f differentiable at p, we can define a cotangent vector at p by

$$df|_p\left(a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3}\right) = \left(a\frac{\partial f}{\partial x_1}(p) + b\frac{\partial f}{\partial x_2}(p) + c\frac{\partial f}{\partial x_3}(p)\right). \tag{1.3}$$

In terms of the basis dx_1, dx_2, d_3,

$$df|_p = \frac{\partial f}{\partial x_1}(p)dx_1 + \frac{\partial f}{\partial x_2}(p)dx_2 + \frac{\partial f}{\partial x_3}(p)dx_3. \tag{1.4}$$

This has exactly the same form of the total differential we learned in Calculus. In textbooks of Calculus, dx_1, dx_2 and dx_3 are called symbols representing "THINGS" that are infinitesimal small, and dx_i is the "limit" of Δx_i when Δx_i goes to zero. Similarly for the total differential. In Calculus, we have

$$\Delta f = \frac{\partial f}{\partial x_1}(p)\Delta x_1 + \frac{\partial f}{\partial x_2}(p)\Delta x_2 + \frac{\partial f}{\partial x_3}(p)\Delta x_3 + R,$$

where R is a remainder going to zero as Δx_i goes to zero if f is differentiable at p. When Δx_i goes to zero, we are told that we get the total differential

$$df|_p = \frac{\partial f}{\partial x_1}(p)dx_1 + \frac{\partial f}{\partial x_2}(p)dx_2 + \frac{\partial f}{\partial x_3}(p)dx_3.$$

But when Δx_i goes to zero, everything above is zero, isn't it? Clearly dx_i and the total differential are not properly explained in textbooks of Calculus.

All the discussions above can be done on $\mathbb{R}^n$.

From the discussions in this subsection, we see that just using the new viewpoint of tangent vectors as in (1.2), the concept of dual spaces in linear algebra, and the definition (1.4), all these ambiguously defined dx_i and the total differential are defined precisely.

1.3. **Vector Fields and Differential 1-Forms.** In Calculus, a vector field on $\mathbb{R}^3$ is a vector valued function $\mathbf{v}(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}), h(\mathbf{x}))$ representing a tangent vector at the point $(\mathbf{x})$. Using our new viewpoint (1.1), the vector field can be represented by

$$f(\mathbf{x})\frac{\partial}{\partial x_1} + g(\mathbf{x})\frac{\partial}{\partial x_2} + h(\mathbf{x})\frac{\partial}{\partial x_3}.$$

Similarly, we can also have cotangent vector fields.

$$f(\mathbf{x})dx_1 + g(\mathbf{x})dx_2 + h(\mathbf{x})dx_3$$

represents a cotangent vector at the point $\mathbf{x}$. Thus if f is a differentiable function in an open set $U \in \mathbb{R}^3$, the expression (1.4) gives a cotangent vector field on U

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3. \tag{1.5}$$

Cotangent vector fields are called 1-forms. Thus (1.5) is a 1-form.

1.4. **Multi-linear Algebra and Exterior Algebra of Grassmann.** Given a vector space V, a multi-linear map T from $V^k = \underbrace{V \times \ldots \times V}_{k}$ to $\mathbb{R}$ is a linear transformation in each variable, i.e.,

$$\begin{aligned} &T(\mathbf{v_1}, \ldots, a\mathbf{v_i} + b\mathbf{u_i}, \ldots, \mathbf{v_k}) \\ =&aT(\mathbf{v_1}, \ldots, \mathbf{v_i}, \ldots, \mathbf{v_k}) + bT(\mathbf{v_1}, \ldots, \mathbf{u_i}, \ldots, \mathbf{v_k}). \end{aligned}$$

There are many such multi-linear maps in Calculus and Linear Algebra, such as the dot product (or inner product) and the determinant. The determinant of an $n \times n$ matrix can be regarded as a multi-linear map from n copies of $\mathbb{R}^n$ to $\mathbb{R}$ where the i-th column vector of the matrix is considered as an vector in i-th copy of $(\mathbb{R}^n)^n$.

Multi-linear maps can be described via the language of tensors due to the following universal property of tensor product. Given two vector spaces V and W, there exists a vector space $V \otimes W$, called the tensor product of V and W, together with a multilinear map $\psi\colon V \times W \to V \otimes W$, with the following universal property that for any vector space Z and any multilinear map $\varphi\colon V \times W \to Z$, there exists a unique linear map from $f\colon V \otimes W \to Z$ such that $\varphi = f \circ \psi$. From this definition, we can see that the tensor product is unique up to a unique isomorphism.

We can also define the tensor product via the direct construction: $V \otimes W$ is the vector space generated by symbols (v, w) for $v \in V$ and $w \in W$ quotient out the subspace generated by $(v + v', w) - (v, w) - (v', w)$, $(v, w+w') - (v, w) - (v, w')$, $(av, w) - a(v, w)$ and $(v, aw) - a(v, w)$ for $v, v' \in V$, $w, w' \in W$ and $a \in \mathbb{R}$.

Given vector spaces $V_1, \ldots, V_k$, we can also define similarly the tensor product $V_1 \otimes \ldots \otimes V_k$. For a vector space V, we use $V^{\otimes k}$ to denote the k-fold tensor product of V with itself k times. Given a basis $e_1, \ldots, e_n$ of V, $V^{\otimes k}$ has a basis $e_{i_1} \otimes \ldots e_{i_k}$ for $1 \le i_1, \ldots, i_k \le n$. An element in $V^{*\otimes k}$ gives a k-fold multilinear map on V as follows. Let $\{e_i^*\}$ be the dual basis in V^* of the basis $\{e_i\}$ of V, $e_{i_1}^* \otimes \ldots e_{i_k}^*$ defines a multilinear map on V:

$$e_{i_1}^* \otimes \ldots e_{i_k}^*(v_1, \ldots, v_k) = e_{i_1}^*(v_1) \cdot \ldots \cdot e_{i_k}^*(v_k), \quad \text{for } v_1, \ldots, v_k \in V.$$

One can verify that the vector space of multilinear maps from V^k to $\mathbb{R}$ is isomorphic to the the vector space $V^{*\otimes k}$.

Among multilinear maps on V, there is a special kind of multilinear maps, called alternating linear maps, defined by

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

The determinant is a such map. There is an operation changing a multilinear map T to an alternating linear map as follows:

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k is the symmetric group on k letters. One can compare the expression on the right with the definition of determinants.

Let $\Lambda^k V$ to represent the vector space of all k-fold alternating multilinear maps on V, we can define a wedge product as follows: let $T \in \Lambda^k V$ and $S \in \Lambda^\ell V$,

$$T \wedge S = \frac{(k+\ell)!}{k! \cdot \ell!} \mathrm{Alt}(T \otimes S).$$

We can check that the wedge product is multilinear and anti-commutative:

$$T \wedge S = (-1)^{k\ell} S \wedge T.$$

$\Lambda^k V$ has dimension $\dfrac{n!}{k! \cdot (n-k)!}$ since it has a basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} \,|\, 1 \leq i_1 < \ldots < i_k \leq n\}.$$

Thus $\Lambda^k V = 0$ if $k > n$, $\Lambda^n V \cong \mathbb{R}$ with a basis $e_1 \wedge \ldots \wedge e_n$, and $\Lambda^0 V = \mathbb{R}$ by convention. Therefore the vector space

$$\Lambda V = \Lambda^0 V \oplus \Lambda^1 V + \ldots \oplus \Lambda^n V$$

is an algebra with the wedge as the product, called the exterior algebra, or Grassmann algebra.

1.5. **Differential Forms and Exterior Derivatives.** After a brief excursion to the garden of linear algebra, let's come back to the palace of Calculus.

Let V be the tangent space $T_p\mathbb{R}^n$ of $\mathbb{R}^n$ at any point p with basis $\{\dfrac{\partial}{\partial x_i}\}$ and V^* be the cotangent space with basis $\{dx_i\}$.

$$\omega(p) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k}(p) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

is an element in $\Lambda^k V^*$. When p varies in an open set U of $\mathbb{R}^n$, $f_{i_1 \ldots i_k}$'s are functions of p. It can also regarded as a section of the map

$$\coprod_{p \in U} T_p^* \mathbb{R}^n \to U. \tag{1.6}$$

If $f_{i_1 \ldots i_k}$'s are differentiable, ω is called the differential k-form on U.

There is wedge product on forms. Given a k-form ω and a ℓ-form η on U, we can have $\omega \wedge \eta$ which is a $k+\ell$-form and $\omega \wedge \eta = (-)^{k\ell} \eta \wedge \omega$.

We can also define the exterior derivative d on k-forms to get $(k+1)$-forms:

$$d\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} df_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \tag{1.7}$$

One can check that the exterior derivative satisfies Leibniz rule,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-)^k \omega \wedge d\eta,$$

and

$$d \circ d = 0.$$

The last one can be seen through a simple example. Let f be a smooth function on $\mathbb{R}^2$ with coordinates x, y. We have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

$$\begin{aligned} ddf &= d\frac{\partial f}{\partial x} \wedge dx + d\frac{\partial f}{\partial y} \wedge dy \\ &= \frac{\partial^2 f}{\partial y \partial x}dy \wedge dx + \frac{\partial^2 f}{\partial x \partial y}dx \wedge dy \\ &= (-\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y})dx \wedge dy \\ &= 0. \end{aligned}$$

We see that $dd = 0$ is due to the known fact in Calculus that the multiple partial derivatives of smooth functions are independent of the order of partial differentiations.

Let's come back to Calculus and see how our differential forms and exterior differentiations are related to known quantities and operations there.

Consider a 1-form ω_1, a 2-form ω_2, and a 3-form ω_3 on an open subset U of $\mathbb{R}^3$ with coordinates x_1, x_2, x_3.

$$\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3, \tag{1.8}$$

$$\omega_2 = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 + g_3 dx_1 \wedge dx_2, \tag{1.9}$$

$$\omega_3 = f dx_1 \wedge dx_2 \wedge dx_3. \tag{1.10}$$

$$\begin{aligned} & d\omega_1 = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \qquad (1.11) \\ = \; & \frac{\partial f_1}{\partial x_2}dx_2 \wedge dx_1 + \frac{\partial f_1}{\partial x_3}dx_3 \wedge dx_1 + \frac{\partial f_2}{\partial x_1}dx_1 \wedge dx_2 + \frac{\partial f_2}{\partial x_3}dx_3 \wedge dx_2 \\ & + \frac{\partial f_3}{\partial x_1}dx_1 \wedge dx_3 + \frac{\partial f_3}{\partial x_2}dx_2 \wedge dx_3 \\ = \; & (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})dx_2 \wedge dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})dx_3 \wedge dx_1 + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})dx_1 \wedge dx_2 \end{aligned}$$

If we identify the 1-form (1.8) with the vector field

$$\mathbf{v} = f_1\frac{\partial}{\partial x_1} + f_2\frac{\partial}{\partial x_2} + f_3\frac{\partial}{\partial x_3}$$

and the 2-form (1.9) with the vector field

$$\mathbf{u} = g_1\frac{\partial}{\partial x_1} + g_2\frac{\partial}{\partial x_2} + g_3\frac{\partial}{\partial x_3},$$

then we can see that the 2-form $d\omega_1$ corresponds to the curl of the vector field $\mathbf{v}$, i.e., $\mathbf{curl}(\mathbf{v})$.

$$\begin{aligned} & d\omega_2 = dg_1 \wedge dx_2 \wedge dx_3 + dg_2 \wedge dx_3 \wedge dx_1 + dg_3 \wedge dx_1 \wedge dx_2 \\ = & \frac{\partial g_1}{\partial x_1}dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial g_2}{\partial x_2}dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial g_3}{\partial x_3}dx_1 \wedge dx_2 \wedge dx_3 \\ = & \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}\right)dx_1 \wedge dx_2 \wedge dx_3. \end{aligned} \tag{1.12}$$

If we identify ω_3 with the function f, then the 3-form $d\omega_2$ above corresponds to $\mathbf{div}(\mathbf{u})$.

Given a differentiable function g, gradient of g

$$\mathbf{grad}(g) = \frac{\partial g}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial g}{\partial x_3}\frac{\partial}{\partial x_3}$$

is identified with the 1-form

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + \frac{\partial g}{\partial x_3}dx_3$$

by the previous rule of the identification of forms with vector fields.

Now we see that differential operators in vector calculus such as $\mathbf{grad}, \mathbf{curl}, \mathbf{div}$ are all one single exterior differentiation d.

1.6. **Smooth Manifolds.** We have encountered circles in the plane or the space, and spheres in the space in calculus. The way we study them is via parameterizations. In fact, these are examples of compact manifolds. The very definition of manifolds is hidden implicitly in textbooks of Calculus.

In Calculus, on $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, we can do differentiations and integrals of functions and vector fields since the Euclidean spaces have coordinates. But to find tangent vectors of circles and spheres and do the line integral on circles and the surface integral on surfaces, we need the aid of parameterizatoins to provide coordinates. And very often we need several parameterizations, for example, spheres need more than one parameterizations. The key of the paramerterizations is that, for a sphere for example, they give identifications of some open subsets of the shpere with open sets in $\mathbb{R}^2$. Or loosely speaking, the sphere is locally isomorphic to an open subset of $\mathbb{R}^2$.

Example 1.1. Consider two parameterizations of the unit sphere S given by $x^2+y^2+z^2=1$,

$$\mathbf{z}_+(u,v)=(u,v,\sqrt{1-u^2-v^2}) : W_1=\{(u,v)\,|u^2+v^2<1\}\to S,$$
$$U_1=\mathbf{z}_+(W_1);$$

$$\mathbf{x}_+(s,t)=(\sqrt{1-s^2-t^2},s,t) : W_2=\{(s,t)\,|s^2+t^2<1\}\to S,$$
$$U_2=\mathbf{x}_+(W_2).$$

In the intersection $\mathbf{z}_+^{-1}(U_1\cap U_2)$, we can consider, for $u>0$,

$$\mathbf{x}_+^{-1}\circ\mathbf{z}_+(u,v)=\big(s(u,v),t(u,v)\big)=(v,\sqrt{1-u^2-v^2}), \tag{1.13}$$

called the transition function.

Clearly $s(u,v)$ and $t(u,v)$ are differentiable functions of u,v. The determinant of the Jacobian matrix of the transition function (1.13) is

$$\frac{\partial s}{\partial u}\cdot\frac{\partial t}{\partial v}-\frac{\partial s}{\partial v}\cdot\frac{\partial t}{\partial u}=-\frac{-2u}{2\sqrt{1-u^2-v^2}}=\frac{u}{\sqrt{1-u^2-v^2}}.$$

□

How do we define a differentiable function on S in Caculus? For example, in Calculus we say a function f on $U_1\cap U_2\subset S$ is differentiable if there exists a differentiable function F on an open neighbourhood V of $U_1\cap U_2$ in $\mathbb{R}^3$ such that $f=F|_{U_1\cap U_2}$. This definition needs the ambient space $\mathbb{R}^3$ of the sphere S. In fact, we don't need the help of the ambient space, but using parameterizations as follows. We can use the paramerterization $\mathbf{x}_+$ to define f differentiable if $f\circ\mathbf{x}_+$ is differentiable as a function of s,t. However, there is another parameterization $\mathbf{z}_+$. Could it be that $f\circ\mathbf{z}_+$ is not a differentiable function of u,v? If it were, this approach would depend on parameterizations and thus should be rejected as a definition. Fortunately, due to the change rule,

$$f\circ\mathbf{z}_+=(f\circ\mathbf{x}_+)\circ(\mathbf{x}_+^{-1}\circ\mathbf{z}_+)$$

is differentiable.

Now we can say a manifold X is, just like the sphere, a topological space locally identified with an open subset of $\mathbb{R}^n$ via the parameterizations. If we want more properties of X, we can add requirements of the parameterizations. For example, if we want to define smooth functions on X without the help of an ambient Euclidean space, we just require that transition functions of the parameterizations are all smooth due to the Chain Rule in Calculus.

Definition 1.2. Let X be a topological space with countable bases and Hausdorff. X is said to be a smooth manifold if there exists a covering $\{U_i\}$ of X together with maps $\varphi_i \colon U_i \to \mathbb{R}^n$ such that φ_i is a homeomorphism from U_i to the open subset $\varphi_i(U_i)$ of $\mathbb{R}^n$ and $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \subset \mathbb{R}^n \to \varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$ is a smooth map. If $\{x_1, \ldots, x_n\}$ is a coordinate system for $\mathbb{R}^n$, then it is also called a local coordinate system for U_i (or X). $x_k(\varphi_i)$ is a function on U_i, by abuse of notation, denoted also by x_k.

If X is connected, the dimension of X is defined to the number n above.

Clearly, the inverse φ_i^{-1} of φ_i in the definition is a parameterization we saw in Calculus and discussed above.

Example 1.3. Consider the real projective space $\mathbb{RP}^2$ defined as follows.

$$\mathbb{RP}^2 = \{\text{one dimensional subspaces of } \mathbb{R}^3\}.$$

Since a line passing through the origin in $\mathbb{R}^3$ is determined by a non-zero directional vector and two non-zero vectors give the same line if and only if they are proportional, we can write $\mathbb{RP}^2$ in another way:

$$\mathbb{RP}^2 = \mathbb{R}^3 - \{\mathbf{0}\}/$$

where the equivalence relation is that (a_1, a_2, a_3) is equivalent to (b_1, b_2, b_3) if and only if $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$ for $\lambda \neq 0$. Thus we can write

$$\begin{aligned}\mathbb{RP}^2 = \{[x_1, x_2, x_3] \mid [x_1, x_2, x_3] &= [\lambda x_1, \lambda x_2, \lambda x_3] \\ &\text{for } \lambda \neq 0, (x_1, x_2, x_3) \neq \mathbf{0}\}.\end{aligned}$$

There is clearly a map

$$\pi \colon \mathbb{R}^3 - \mathbf{0} \to \mathbb{RP}^2, \quad \pi((x_1, x_2, x_3)) = [x_1, x_2, x_3].$$

We give $\mathbb{RP}^2$ the quotient topology, i.e., a set $U \in \mathbb{RP}^2$ is defined to be an open set if $\pi^{-1}(U)$ is an open subset of $\mathbb{R}^3 - \mathbf{0}$.

We use coordinate charts on $\mathbb{RP}^2$ as follows. Let $U_1 = \{x_1 \neq 0\} \subset \mathbb{RP}^2$. We define a map

$$\varphi_1 \colon U_1 \to \mathbb{R}^2, \quad [x_1, x_2, x_3] \to (\frac{x_2}{x_1}, \frac{x_3}{x_2}) \in \mathbb{R}^2.$$

We can define similar coordinate charts $\varphi_2 \colon U_2 = \{x_2 \neq 0\} \to \mathbb{R}^2$ and $\varphi_3 \colon U_3 = \{x_3 \neq 0\} \to \mathbb{R}^2$. One can check that φ_i is a homeomorphism. Let's check the transition functions. For example,

$$\varphi_2 \circ \varphi_1^{-1} \colon \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R}^2 - \{\mathbf{0}\},$$

$$(x,y) \to [1,x,y] = [\frac{1}{x}, 1, \frac{y}{x}] \to \left(\frac{1}{x}, \frac{y}{x}\right).$$

Clearly this map is a smooth map. Thus $\mathbb{RP}^2$ is a smooth manifold.

Consider the sphere $S\colon x_1^2 + x_2^2 + x_3^2 = 1$. We have a surjection map $\pi|_S\colon s \to \mathbb{RP}^2$. Since the sphere S is compact and the map $\pi|_S$ is continuous, the real projective space $\mathbb{RP}^2$ is compact. For a point $(x_1, x_2, x_3) \in S$, $(\lambda \cdot x_1, \lambda \cdot x_2, \lambda \cdot x_3) \in S$ if and only if $\lambda = \pm 1$. Thus S is a double cover of $\mathbb{RP}^2$.

How do we define tangent vectors on a smooth manifold?

Assume X is an n-dimensional smooth manifold smoothly embedded in $\mathbb{R}^N$ and φ^{-1}'s is a smooth parameterization of an open subset U of X. Then the tangent space of U at a point p is generated by tangent vectors

$$\frac{\partial \varphi^{-1}}{\partial x_1}, \ldots, \frac{\partial \varphi^{-1}}{\partial x_n}.$$

If $(y_1, \ldots, y_N)$ is a coordinate system on $\mathbb{R}^N$, and we write

$$\varphi^{-1}(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n), \ldots, f_N(x_1, \ldots, x_n)\big),$$

then we have

$$\frac{\partial \varphi^{-1}}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_N}{\partial x_i}\right),$$

which can also be written as, if we regard the tangent vector as an operator on differential functions,

$$\frac{\partial f_1}{\partial x_i}\frac{\partial}{\partial y_1} + \ldots + \frac{\partial f_N}{\partial x_i}\frac{\partial}{\partial y_N}. \tag{1.14}$$

Given a smooth function g on U, we have

$$\left(\frac{\partial f_1}{\partial x_i}\frac{\partial}{\partial y_1} + \ldots + \frac{\partial f_N}{\partial x_i}\frac{\partial}{\partial y_N}\right) \cdot g = \frac{\partial f_1}{\partial x_i}\frac{\partial g}{\partial y_1} + \ldots + \frac{\partial f_N}{\partial x_i}\frac{\partial g}{\partial y_N}$$
$$= \frac{\partial g \circ \varphi^{-1}}{\partial x_i}$$

by the chain rule.

We define differential operators $\dfrac{\partial}{\partial x_i}$ acting on differential functions g on U by

$$\left(\frac{\partial}{\partial x_i}\right)(g) = \frac{\partial g \circ \varphi^{-1}}{\partial x_i}. \tag{1.15}$$

Comparing (1.14) and (1.15), we see that

$$\frac{\partial}{\partial x_i} = \frac{\partial f_1}{\partial x_i}\frac{\partial}{\partial y_1} + \ldots + \frac{\partial f_N}{\partial x_i}\frac{\partial}{\partial y_N}.$$

Thus if we regard tangent vectors as differential operators, the tangent space T_pX is a vector space spanned by the tangent vectors

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \tag{1.16}$$

for all $p \in U$.

The advantage of this viewpoint is that the definition of tangent vectors of a smooth manifold doesn't need the ambient Euclidean space $\mathbb{R}^N$ and hence can be used as the definition of an arbitrary smooth manifold.

We can define the concept of smooth maps between smooth manifolds. Given a map $F\colon M \to N$ between two smooth manifolds M and N. We say F is smooth if for every point $p \in M$, a coordinate chart $\varphi\colon U \to \mathbb{R}^m$ of an open subset U containing p, and a coordinate chart $\psi\colon V \to \mathbb{R}^n$ of an open subset V containing $F(p)$ such that $U \subset F^{-1}(V)$, the map $\psi \circ F \circ \varphi^{-1}$ is a smooth map from $\varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n$.

1.7. **Tangent Bundles and Cotangent Bundles.** In Calculus, for curves in the plane $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$, we learned the concept of tangent lines of curves and tangent planes of surfaces. For example, for a smooth surface S in $\mathbb{R}^3$, for each point $p \in S$, we have the tangent plane T_pS consisting of tangent vectors of S at p. If we vary the point p, we get a subset TS of $\mathbb{R}^3 \times \mathbb{R}^3$ with a map π to S

$$\pi\colon TS = \coprod_{p \in S} T_pS = \{(p, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, p \in S, \mathbf{v} \in T_pS\} \to S.$$

The map π maps a tangent vector $\mathbf{v}$ at p to the point p. Clearly each fiber $\pi^{-1}(p)$ is a vector space.

We can do the same for arbitrary n-dimensional smooth manifold X to get a set

$$TX = \coprod_{p \in X} T_pX,$$

called the tangent bundle together with a natural map $\pi\colon TX \to X$ mapping T_pX to p. Clearly each fiber $\pi^{-1}(p) = T_pX$ is a vector space. Furthermore, we can give a smooth manifold structure to TX as follows. Let $\varphi\colon U \to \mathbb{R}^n$ be a coordinate chart of X. Let $x_1, \ldots, x_n$ be a coordinate system on $\mathbb{R}^n$. At each point p of U, based upon the discussion of the basis (1.16) in §1.6, we can choose a basis of T_pX:

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}.$$

Thus we can define a coordinate chart of $\pi^{-1}(U) = TX|_U$:

$$\Phi\colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n, \quad \left(p, a_1\frac{\partial}{\partial x_1} + \dots a_n\frac{\partial}{\partial x_n}\right) \to \big(\varphi(p), (a_1, \dots, a_n)\big).$$

We can give the topology to TX such that W is an open subset of TX iff $\Phi\big(W \cap \pi^{-1}(U)\big)$ is open for every U.

$\Phi|_p$, the restriction of Φ on the fiber of π over $p \in X$, is clearly an isomorphism of vector spaces.

We can check that TX is a smooth manifold. In fact, let $\psi : V \subset X \to \mathbb{R}^n$ be another coordinate chart with $y_1, \dots, y_n$ as the coordinates of $\mathbb{R}^n$. We have the corresponding coordinate chart for $\pi^{-1}(V)$:

$$\Psi\colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n, \quad \left(p, b_1\frac{\partial}{\partial y_1} + \dots b_n\frac{\partial}{\partial y_n}\right) \to \big(\psi(p), (b_1, \dots, b_n)\big).$$

Let's write

$$\psi \circ \varphi^{-1}(x_1, \dots, x_n) = \big(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)\big).$$

Let g be a differential function on $U \cap V$,

$$\begin{aligned} \left(\frac{\partial}{\partial x_i}\right)g &= \frac{\partial g \circ \varphi^{-1}}{\partial x_i} = \frac{\partial\big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)}{\partial x_i} \\ &= \frac{\partial f_1}{\partial x_i}\frac{\partial g \circ \psi^{-1}}{\partial y_1} + \dots \frac{\partial f_n}{\partial x_i}\frac{\partial g \circ \psi^{-1}}{\partial y_n}. \\ &= \left(\frac{\partial f_1}{\partial x_i}\frac{\partial}{\partial y_1} + \dots \frac{\partial f_n}{\partial x_i}\frac{\partial}{\partial y_n}\right)g. \end{aligned}$$

Thus we get the transformation formula for different bases of the tangent space T_pX for $p \in U \cap V$:

$$\frac{\partial}{\partial x_i} = \frac{\partial f_1}{\partial x_i}\frac{\partial}{\partial y_1} + \dots \frac{\partial f_n}{\partial x_i}\frac{\partial}{\partial y_n}. \tag{1.17}$$

Thus we get

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

We can use $Jac(\psi \circ \varphi^{-1})$ to represent the $n \times n$ matrix above. Thus the transition function $\Psi \circ \Phi^{-1}$ is given by

$$\Psi \circ \Phi^{-1}(p, \mathbf{v}) = \big(\psi \circ \varphi^{-1}(p), Jac(\psi \circ \varphi^{-1})(\mathbf{v})\big),$$

and smooth.

If the surface S is the xy-plane in $\mathbb{R}^3$, then $TS \cong S \times \mathbb{R}^2$, i.e.,the tangent bundle is the product of S with $\mathbb{R}^2$. We call such tangent bundle a trivial bundle. If we consider the sphere $S\colon x^2+y^2+z^2=1$, the tangent bundle TS won't be a trivial bundle. However the tangent bundle of the circle is a trivial bundle.

The tangent bundle of a smooth manifold has some properties shared by many other manifolds, which are called a vector bundle. Let's give a definition of vector bundles.

Definition 1.4. Let X be a smooth manifold and E be another smooth manifold with a smooth map $\pi\colon E \to X$. E is called a vector bundle if for each $p \in X$, the fiber π^{-1} is a vector space of dimension r and there exists an open subset U containing p and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\Phi|_q$, the restriction to each fiber $\pi^{-1}(q)$, is an isomorphism of vector spaces for every $q \in U$.

If we take dual of each fiber of a vector bundle E over X, we get a dual vector bundle E^* over X. The dual tangent bundle is called the cotangent bundle, denoted by T^*X.

A smooth section s of a vector bundle E over X is a smooth map from X to E such that $\pi \circ s(p) = p$, i.e., s maps p to a vector in the fiber $E|_p$ of π over p. Thus we can see that a smooth vector field on X is just a smooth section of the tangent bundle TX, and a smooth 1-form is just a smooth section of the cotangent bundle T^*X.

We can also apply various constructions in linear algebra to vector bundles. For example, we take i-th wedge product $\wedge^i E$ of the vector bundle E. The smooth sections of $\wedge^p T^*X$ are called smooth p-forms. These are the extension of the concept of differential forms on $\mathbb{R}^n$ in 1.5 to manifolds.

One can also extend the exterior differentiations to manifolds. Can you give a try? Do it locally, i.e., using local coordinate charts, then prove that it is independent of the choice of charts.