Optimal Risk Probability for First Passage Models

—in Semi-Markov Decision Processes

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Outline

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1. **Motivation**

**Background**: Reliability engineering, and risk analysis

**Problem**: $\sup_{\pi} P_{i}^{\pi}(\tau_{B} > \lambda)$,

- $i$ an initial state
- $\pi$ is a policy
- $B$ is a given target set
- $\tau_{B}$ is a first passage time to $B$
- $\lambda$ is a threshold value.
2. Semi-Markov Decision Processes

The model of SMDP:

\[ \{S, B, (A(i), i \in S), Q(t, j|i, a)\} \]

where

- \( S \): the state space, a denumerable set;
- \( B \): a given target set, a subset of \( S \);
- \( A(i) \): finite set of actions available at \( i \in S \);
- \( Q(t, j|i, a) \): semi-Markov kernel, \( a \in A(i), i, j \in S \);
Notation:

• **Policy** \( \pi \): A sequence \( \pi = \{ \pi_n, n = 0, 1, \ldots \} \) of stochastic kernels \( \pi_n \) on the action space \( A \) given \( H_n \) satisfying

\[
\pi_n(A(i_n)|(0, i_0, \lambda_0, a_0, \ldots, t_{n-1}, i_{n-1}, \lambda_{n-1}, a_{n-1}, t_n, i_n)) = 1
\]

• **Stationary policy**: measurable \( f \), \( f(i, \lambda) \in A(i) \) for all \( (i, \lambda) \)

• \( P^{\pi}_{(i, \lambda)} \): Probability measure on \( (S \times [0, \infty) \times (\cup_{i \in S} A(i)))^\infty \)

• \( S_n, J_n, A_n \): \( n \)-th decision epoch, the state and action at the \( S_n \), respectively.
Assumption A. There exist $\delta > 0$ and $\epsilon > 0$ such that

$$\sum_{j \in S} Q(\delta, j | i, a) \leq 1 - \epsilon, \text{ for all } (i, a) \in K.$$ 

Assumption A $\Rightarrow P_{(i,\lambda)}^\pi(\{S_\infty = \infty\}) = 1$

Semi-Markov decision process $\{(Z(t), A(t), t \geq 0) :$

$$Z(t) = J_n, A(t) = A_n, \text{ for } S_n \leq t < S_{n+1}$$

The first passage time into $B$, is defied by

$$\tau_B := \inf\{ t \geq 0 | Z(t) \in B \}, \text{ (with } \inf\emptyset := \infty),$$
3. Optimality Problems

The risk probability:

\[ F^\pi(i, \lambda) := P^\pi_{(i,\lambda)}(\tau_B \leq \lambda) \]

The optimal value:

\[ F_*(i, \lambda) := \inf_{\pi \in \Pi} F^\pi(i, \lambda), \]

**Definition 1.** A policy \( \pi^* \in \Pi \) is called optimal if

\[ F^{\pi^*}(i, \lambda) = F_*(i, \lambda) \ \forall \ (i, \lambda) \in S \times R. \]

- Existence and computation of optimal policies ???
4. Optimality Equation

For $i \in B^c, a \in A(i)$, and $\lambda \geq 0$, let

$$T^a u(i, \lambda) := Q(\lambda, B \mid i, a) + \sum_{j \in B^c} \int_0^\lambda Q(dt, j \mid i, a) u(j, \lambda - t),$$

with $u \in \mathcal{F}_{[0,1]}$ (the set of measurable functions $0 \leq u \leq 1$),

$$Q(\lambda, B \mid i, a) := \sum_{j \in B} Q(\lambda, j \mid i, a), \quad T^a u(i, \lambda) := 0 \text{ for } \lambda < 0.$$

Then, define operators $T$ and $T^f$:

$$Tu(i, \lambda) := \min_{a \in A(i)} T^a u(i, \lambda); \quad T^f u(i, \lambda) := T^{f(i, \lambda)} u(i, \lambda),$$

for each stationary policy $f$.  

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Theorem 1. Let Under Assumption A, we have

(a) \( F^f = \lim_{n \to \infty} u^f_n \), where \( u^f_n := T^f u_{n-1}, u^f_{-1} := 1 \);

(b) \( F^f \) satisfied the equation, \( u = T^f u \), for all \( f \in F \);

• Theorem 1 gives an approximation of risk probability \( F^f \).

For each \((i, \lambda) \in B^c \times R_+ \) and \( \pi \in \Pi \), let

\[
F^\pi_{-1}(i, \lambda) := 1, \\
F^\pi_n(i, \lambda) := 1 - \sum_{m=0}^{n} P^\pi_{(i, \lambda)}(S_m \leq \lambda < S_{m+1}, J_k \in B^c, 0 \leq k \leq m)
\]
Theorem 2. Let $F_n^*(i, \lambda) := \inf_{\pi} F_n^\pi(i, \lambda)$, then

(a) $F_n^* + 1 = T F_n^*$ for all $n \geq -1$, and $\lim_{n \to \infty} F_n^* = F_*$.

(b) $F_*$ satisfies the optimality equation: $F_* = T F_*$. 

(c) $F_*$ is the maximal fixed point of $T$ in $\mathcal{F}[0,1]$.

Remark 1.

- Theorem 2(a) gives a value iteration algorithm for computing the optimal value function $F_*$. 
- Theorem 2(b) establishes the optimality equation.
5. **Existence of Optimality Policies**

To ensure the existence of optimal policies, we introduce the following condition.

**Assumption B.** For every \((i, \lambda) \in B^c \times R\) and \(f\),

\[ P^f_{(i,\lambda)}(\tau_B < \infty) = 1. \]

To verify Assumption B, we have a fact below:

**Theorem 3.** If there exists a constant \(\alpha > 0\) such that

\[ \sum_{j \in B} Q(\infty, j|i, a) \geq \alpha \] for all \(i \in B^c, a \in A(i)\),

then Assumption B holds.
Theorem 4. Under Assumptions A and B, we have

(a) $F_f$ and $F_*$ are the unique solution in $\mathcal{F}_{[0,1]}$ to equations
\[ u = T_f u \text{ and } u = Tu, \] respectively;

(b) any $f$, such that $F_* = T_f F_*$, is optimal;

(c) there exists a stationary policy $f^*$ satisfying the optimality equation: $F_* = TF_* = T_{f^*} F_*$, and such policy $f^*$ is optimal.

Remark 2.

- Theorem 4(c) shows the existence of an optimal policy.
To give the existence of special optimal policies, let

\[
A^*(i, \lambda) := \{ a \in A(i) \mid F^*(i, \lambda) = T^a F^*(i, \lambda) \}.
\]

\[
A^*(i) := \bigcap_{\lambda \geq 0} A^*(i, \lambda)
\]

**Theorem 5.** If \( \sup_i \sup_{a \in A(i)} Q(t, B^c \mid i, a) < 1 \) for some \( t > 0 \), and Assumptions A and B hold, then,

(a) for any \( g \in G := \{ g \mid g(i) \in A(i) \forall i \in S \} \), \( F^g \) is the unique solution in \( F_{[0,1]} \) to the equation: \( u = T^g u \);

(b) there exists an optimal policy \( f \in G \) if and only if \( A^*(i) \neq \emptyset \) for all \( i \in B^c \).
5. Numerable examples

Example 5.1. Let \( S = \{1, 2, 3\} \), \( B = \{3\} \), where

- state 1: the good state
- state 2: the medium state
- state 3: the failure state

Let \( A(1) = \{a_{11}, a_{12}\} \), \( A(2) = \{a_{21}, a_{22}\} \), \( A(3) = \{a_{31}\} \).

The semi-Markov kernel is of the form:

\[
Q(t, j | i, a) = H(t | i, a)p(j | i, a)
\]
• $H(t \mid i, a)$: the distribution functions of the sojourn time

• $p(j \mid i, a)$: the transition probabilities.

$$H(t \mid 1, a_{11}) := \begin{cases} 1/25, & t \in [0, 25], \\ 1, & t > 25; \end{cases}$$

$$H(t \mid 2, a_{21}) := \begin{cases} 1/20, & t \in [0, 20], \\ 1, & t > 20; \end{cases}$$

$$H(t \mid 3, a_{31}) := 1 - e^{-0.2t}.$$  

$$H(t \mid 1, a_{12}) = 1 - e^{-0.08t};$$

$$H(t \mid 2, a_{22}) = 1 - e^{-0.15t}. $$
\[ p(1 \mid 1, a_{11}) = 0, \ p(2 \mid 1, a_{11}) = \frac{9}{20}, \ p(3 \mid 1, a_{11}) = \frac{11}{20}; \]
\[ p(1 \mid 1, a_{12}) = 0, \ p(2 \mid 1, a_{12}) = \frac{1}{2}, \ p(3 \mid 1, a_{12}) = \frac{1}{2}; \]
\[ p(1 \mid 2, a_{21}) = \frac{1}{5}, \ p(2 \mid 2, a_{21}) = 0, \ p(3 \mid 2, a_{21}) = \frac{4}{5}; \]
\[ p(1 \mid 2, a_{22}) = \frac{1}{4}, \ p(2 \mid 2, a_{22}) = 0, \ p(3 \mid 2, a_{22}) = \frac{3}{4}; \]
\[ p(3 \mid 3, a_{31}) = 1. \]

Using the value iteration algorithm in Theorem 2, we obtain some computational results as in Figure 1 and Figure 2.
the threshold value $\lambda$

$T^a F^*(i, \lambda)$

$T^a_{11} F^*(1, \lambda)$

$T^a_{12} F^*(1, \lambda)$

$T^a_{21} F^*(2, \lambda)$

$T^a_{22} F^*(2, \lambda)$

(18.96, 0.7937)

(21.36, 0.6388)
Figure 1. The functions $T^a F^*(i, \lambda)$

Figure 2. The value function $F^*(i, \lambda)$
More clearly, we have

\[ F^*(1, \lambda) = \begin{cases} 
  T^{a11} F^*(1, \lambda), & 0 \leq \lambda < 21.36, \\
  T^{a11} F^*(1, \lambda) = T^{a12} F^*(1, \lambda), & \lambda = 21.36, \\
  T^{a12} F^*(1, \lambda), & 21.36 < \lambda < 29.3, \\
  T^{a11} F^*(1, \lambda) = T^{a12} F^*(1, \lambda), & \lambda = 29.3, \\
  T^{a11} F^*(1, \lambda)(= 0.7742), & \lambda > 29.3, \\
 \end{cases} \]

\[ F^*(2, \lambda) = \begin{cases} 
  T^{a21} F^*(2, \lambda), & 0 \leq \lambda < 18.54, \\
  T^{a21} F^*(2, \lambda) = T^{a22} F^*(2, \lambda), & \lambda = 18.54, \\
  T^{a22} F^*(2, \lambda), & 18.54 < \lambda < 23.82, \\
  T^{a21} F^*(2, \lambda) = T^{a22} F^*(2, \lambda), & \lambda = 23.82, \\
  T^{a21} F^*(2, \lambda)(= 0.8542), & \lambda > 23.82. \\
\]
Define a policy $f^*$ by

$$f^*(1, \lambda) = \begin{cases} 
    a_{11}, & 0 \leq \lambda \leq 21.36, \\
    a_{12}, & 21.36 < \lambda \leq 29.3, \\
    a_{11}, & \lambda > 29.3,
\end{cases}$$

$$f^*(2, \lambda) = \begin{cases} 
    a_{21}, & 0 \leq \lambda \leq 18.54, \\
    a_{22}, & 18.54 < \lambda \leq 23.82, \\
    a_{21}, & \lambda > 23.82,
\end{cases}$$

Then, we have

- $F^*(i, \lambda) = T^{f^*} F^*(i, \lambda)$ for $i = 1, 2$ and all $\lambda \geq 0$,
- $f^*$ is an optimal stationary policy.
\[ A^*(1, \lambda) = \begin{cases} 
\{a_{11}\}, & 0 \leq \lambda < 21.36, \\
\{a_{11}, a_{12}\}, & \lambda = 21.36, \\
\{a_{12}\}, & 21.36 < \lambda < 29.3, \\
\{a_{11}, a_{12}\}, & \lambda = 29.3, \\
\{a_{11}\}, & \lambda > 29.3, 
\end{cases} \]

\[ A^*(2, \lambda) = \begin{cases} 
\{a_{21}\}, & 0 \leq \lambda < 18.54, \\
\{a_{21}, a_{22}\}, & \lambda = 18.54, \\
\{a_{22}\}, & 18.54 < \lambda < 23.82, \\
\{a_{21}, a_{22}\}, & \lambda = 23.82, \\
\{a_{21}\}, & \lambda > 23.82, 
\end{cases} \]
Hence,
\[ A^*(1) = \bigcap_{\lambda \geq 0} A^*(1, \lambda) = \emptyset, \]
\[ A^*(2) = \bigcap_{\lambda \geq 0} A^*(2, \lambda) = \emptyset, \]
which show there is no optimal policy in $G$.

**Remark 3.** This shows that the assumption in the previous literature is not satisfied for this example !!!
Example 5.2. Let $S = \{1, 2\}$, $B = \{2\}$;

$A(1) = \{a_{11}, a_{12}\}$, $A(2) = \{a_{21}\}$;

$Q(t, j \mid i, a)$ is given by

\[
Q(t, j \mid 1, a_{11}) = \begin{cases} 
\frac{1}{2}, & \text{if } t \geq 1, j = 1, 2, \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
Q(t, j \mid 1, a_{12}) = \begin{cases} 
1, & \text{if } t \geq 2, j = 2, \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
Q(t, j \mid 2, a_{21}) = \begin{cases} 
1 - e^{-t}, & \text{if } t \geq 0, j = 2, \\
0, & \text{otherwise}. 
\end{cases}
\]

Assumptions A and B holds in this example.
We now define a policy $d$ as follows:

$$d(1, \lambda) = \begin{cases} 
a_{12}, & 0 \leq \lambda \leq 2, \\
a_{11}, & \lambda > 2.
\end{cases}$$

Then, by Theorem 1, we have $F^d(1, \lambda) = \lim_{n \to \infty} F^d_n(1, \lambda)$, which yields

$$F^d(1, \lambda) = \begin{cases} 
0, & 0 \leq \lambda < 2, \\
1, & \lambda = 2, \\
1/2, & 2 < \lambda < 3.
\end{cases}$$

Hence, $F^d(1, \lambda)$ is not a distribution function of $\lambda$. 

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Many Thanks !!!