Optimal Dividend Policy of A Large Insurance Company with Solvency Constraints

Zongxia Liang

Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China
zliang@math.tsinghua.edu.cn

Joint work with Jicheng Yao

Hsu 100 Conference, July 5-7, 2010, Peking University
The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance
The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance
The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance
- Controlling dividends payout
The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance
- Controlling dividends payout
The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance
- Controlling dividends payout
- Controlling bankrupt probability (or solvency) and so on
Cramér-Lundberg model of cash flows

- The classical model with no reinsurance, dividend pay-outs
Cramér-Lundberg model of cash flows

- The classical model with no reinsurance, dividend pay-outs
Cramér-Lundberg model of cash flows

- The classical model with no reinsurance, dividend pay-outs

The cash flow (reserve process) $r_t$ of the insurance company follows

$$r_t = r_0 + pt - \sum_{i=1}^{N_t} U_i,$$
Cramér-Lundberg model of cash flows

- The classical model with no reinsurance, dividend pay-outs

The cash flow (reserve process) \( r_t \) of the insurance company follows

\[
r_t = r_0 + pt - \sum_{i=1}^{N_t} U_i,
\]

where

claims arrive according to a Poisson process \( N_t \) with intensity \( \nu \) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).
Cramér-Lundberg model of reserve process

$U_i$ denotes the size of each claim. Random variables $U_i$ are i.i.d. and independent of the Poisson process $N_t$ with finite first and second moments given by $\mu_1$ and $\mu_2$.

$$p = (1 + \eta)\nu \mu_1 = (1 + \eta)\nu E\{U_i\}$$

is the premium rate and $\eta > 0$ denotes the safety loading.
Diffusion approximation of Cramér-Lundberg model

By the central limit theorem, as $\nu \to \infty$,

$$r_t \overset{d}{\approx} r_0 + BM(\eta \nu \mu_1 t, \nu \mu_2 t).$$
Diffusion approximation of Cramér-Lundberg model

By the central limit theorem, as $\nu \to \infty$,

$$rt \overset{d}{\approx} r_0 + \text{BM}(\eta \nu \mu_1 t, \nu \mu_2 t).$$

So we can assume that the cash flow $\{R_t, t \geq 0\}$ of insurance company is given by the following diffusion process

$$dR_t = \mu dt + \sigma dW_t,$$

where the first term "$\mu t$" is the income from insureds and the second term "$\sigma W_t$" means the company’s risk exposure at any time $t$. 
Making Proportional reinsurance to reduce risk

- The insurance company gives fraction $\lambda(1 - a(t))$ of its income to reinsurance company.
Making Proportional reinsurance to reduce risk

- The insurance company gives fraction $\lambda(1 - a(t))$ of its income to reinsurance company.

- As a return, the reinsurance share with the insurance company’s risk exposure $\sigma W_t$ by paying money $(1 - a(t))\sigma W_t$ to insureds.
Making Proportional reinsurance to reduce risk

- The insurance company gives fraction $\lambda(1 - a(t))$ of its income to reinsurance company.

- As a return, the reinsurance share with the insurance company's risk exposure $\sigma W_t$ by paying money $(1 - a(t))\sigma W_t$ to insureds.

The cash flow $\{R_t, t \geq 0\}$ of the insurance company then becomes

$$dR_t = (\mu - (1 - a(t))\lambda)dt + \sigma a(t)dW_t, \quad R_0 = x.$$  

We generally assume that $\lambda \geq \mu$ based on real market.
Making dividends payout for the company’s shareholders

If \( L_t \) denotes cumulative amount of dividends paid out to the shareholders up to time \( t \), then the cash flow \( \{R_t, t \geq 0\} \) of the company is given by

\[
dR_t = \left( \mu - \left( 1 - a(t) \right) \lambda \right) dt + \sigma a(t) dW_t - dL_t,
\]

\( R_0 = x \),

where \( 1 - a(t) \) is called the reinsurance fraction at time \( t \), the \( R_0 = x \) means that the initial capital is \( x \), the constants \( \mu \) and \( \lambda \) can be regarded as the safety loadings of the insurer and reinsurer, respectively.
Making dividends payout for the company’s shareholders

If $L_t$ denotes cumulative amount of dividends paid out to the shareholders up to time $t$, 

Making dividends payout for the company’s shareholders

If $L_t$ denotes cumulative amount of dividends paid out to the shareholders up to time $t$, then the cash flow $\{R_t, t \geq 0\}$ of the company is given by

$$dR_t = (\mu - (1 - a(t))\lambda)dt + \sigma a(t)dW_t - dL_t, \quad R_0 = x, \quad (1)$$
Making dividends payout for the company’s shareholders

If $L_t$ denotes cumulative amount of dividends paid out to the shareholders up to time $t$, then the cash flow $\{R_t, t \geq 0\}$ of the company is given by

$$dR_t = (\mu - (1 - a(t))\lambda)dt + \sigma a(t)dW_t - dL_t, \quad R_0 = x, \quad (1)$$

where $1 - a(t)$ is called the reinsurance fraction at time $t$, the $R_0 = x$ means that the initial capital is $x$, the constants $\mu$ and $\lambda$ can be regarded as the safety loadings of the insurer and reinsurer, respectively.
Optimal Control Problem for the model (1)

Notations:
Optimal Control Problem for the model (1)

Notations:

A policy $\pi = \{a_\pi(t), L^\pi_t\}$ is a pair of non-negative càdlàg $\mathcal{F}_t$-adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. 
Optimal Control Problem for the model (1)

Notations:

- A policy \( \pi = \{a_\pi(t), L_\pi^t\} \) is a pair of non-negative càdlàg \( \mathcal{F}_t \)-adapted processes defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\).

- A pair of \( \mathcal{F}_t \) adapted processes \( \pi = \{a_\pi(t), L_\pi^t\} \) is called a admissible policy if \( 0 \leq a_\pi(t) \leq 1 \) and \( L_\pi^t \) is a nonnegative, non-decreasing, right-continuous with left limits.
Optimal Control Problem for the model (1)

Notations:

- A policy \( \pi = \{a_\pi(t), L_\pi^t\} \) is a pair of non-negative càdlàg \( \mathcal{F}_t \)-adapted processes defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).
- A pair of \( \mathcal{F}_t \) adapted processes \( \pi = \{a_\pi(t), L_\pi^t\} \) is called an admissible policy if \( 0 \leq a_\pi(t) \leq 1 \) and \( L_\pi^t \) is a nonnegative, non-decreasing, right-continuous with left limits.
- \( \Pi \) denotes the whole set of admissible policies.
Optimal Control Problem for the model (1)

Notations:

- A policy $\pi = \{a_\pi(t), L^\pi_t\}$ is a pair of non-negative càdlàg $\mathcal{F}_t$-adapted processes defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

- A pair of $\mathcal{F}_t$ adapted processes $\pi = \{a_\pi(t), L^\pi_t\}$ is called an admissible policy if $0 \leq a_\pi(t) \leq 1$ and $L^\pi_t$ is a nonnegative, non-decreasing, right-continuous with left limits.

- $\Pi$ denotes the whole set of admissible policies.

- When an admissible policy $\pi$ is applied, the model (1) can be rewritten as follows:

$$dR^\pi_t = (\mu - (1 - a_\pi(t))\lambda)dt + \sigma a_\pi(t)dW_t - dL^\pi_t, \quad R^\pi_0 = x. \quad (2)$$
Optimal Control Problem for the model (1)

General setting:
Optimal Control Problem for the model (1)

General setting:

- The performance function $J(x, \pi)$ defined by

$$J(x, \pi) = \mathbb{E}\left\{ \int_{0}^{\tau_{x}^{\pi}} e^{-ct} dL_{t}^{\pi} \right\}$$

(3)
General setting:

The performance function $J(x, \pi)$ defined by

$$J(x, \pi) = \mathbb{E}\left\{ \int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi \right\}$$

(3)

where $\tau_x^\pi = \inf\{ t \geq 0 : R_t^\pi = 0 \}$ is the time of bankruptcy, $c > 0$ is a discount rate.
Optimal Control Problem for the model (1)

General setting:

- The performance function $J(x, \pi)$ defined by
  \[ J(x, \pi) = \mathbb{E}\left\{ \int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi \right\} \] (3)

  where $\tau_x^\pi = \inf\{ t \geq 0 : R_t^\pi = 0 \}$ is the time of bankruptcy, $c > 0$ is a discount rate.

- The optimal return function $V(x)$ defined by
  \[ V(x) = \sup_{\pi \in \Pi} \{ J(x, \pi) \}. \] (4)
Optimal Control Problem for the model (1)

General setting:

- The performance function $J(x, \pi)$ defined by

$$J(x, \pi) = \mathbb{E}\left\{ \int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi \right\}$$

where $\tau_x^\pi = \inf\{ t \geq 0 : R_t^\pi = 0 \}$ is the time of bankruptcy, $c > 0$ is a discount rate.

- The optimal return function $V(x)$ defined by

$$V(x) = \sup_{\pi \in \Pi} \{ J(x, \pi) \}.$$  \hspace{1cm} (4)

- Optimal control problem for the model (1) is to find the optimal return function $V(x)$ and the optimal policy $\pi^*$ such that $V(x) = J(x, \pi^*)$.
Solution of optimal control problem for the model (1) does not meet safety level

It well known that one can find a dividend level $b_0 > 0$, an optimal policy $\pi^*_b$ and an optimal return function $V(x, \pi^*_b)$ to solve optimal control problem for the model (1), i.e.,

$$V(x) = V(x, b_0) = J(x, \pi^*_b)$$

and $b_0$ satisfies

$$\int_0^\infty \int \left\{ s : R^{\pi^*_b}(s) < b_0 \right\} \ dL_{s}^{\pi^*_b} = 0$$
Solution of optimal control problem for the model (1) does not meet safety level

It well known that one can find a dividend level \( b_0 > 0 \), an optimal policy \( \pi_{b_0}^* \) and an optimal return function \( V(x, \pi_{b_0}^*) \) to solve optimal control problem for the model (1), i.e.,

\[
V(x) = V(x, b_0) = J(x, \pi_{b_0}^*)
\]

and \( b_0 \) satisfies

\[
\int_0^\infty I\{s: R_{\pi_{b_0}^*} (s) < b_0\} \ dL_{\pi_{b_0}^*} = 0
\]

However, the \( b_0 \) may be too low and it will make the company go bankrupt soon
Solution of optimal control problem for the model (1) does not meet safety level

Indeed, we proved that the $b_0$ and $\pi^{*}_{b_0}$ satisfy for any $0 < x \leq b_0$ there exists $\varepsilon_0 > 0$ such that

$$P\{T^{\pi^{*}_{b_0}}_x \leq T\} \geq \varepsilon_0 > 0,$$

where

$$\varepsilon_0 = \min \left\{ \frac{4[1 - \Phi\left(\frac{x}{d\sigma \sqrt{T}}\right)]^2}{\exp\left\{ \frac{x}{\sigma^2} \left( \lambda^2 + \delta^2 \right) T \right\}} , \frac{x}{\sqrt{2\pi}} \int_0^T t^{-\frac{3}{2}} \exp\left\{ -\frac{(x+\mu t)^2}{2\sigma^2 t} \right\} dt \right\},$$

$$T^{\pi}_x = \inf \left\{ t \geq 0 : R^\pi_t = 0 \right\}.$$
Indeed, we proved that the $b_0$ and $\pi_{b_0}^*$ satisfy for any $0 < x \leq b_0$ there exists $\varepsilon_0 > 0$ such that
\[
P\{T_{X_{b_0}}^{\pi_{b_0}} \leq T\} \geq \varepsilon_0 > 0,
\] (5)
where
\[
\varepsilon_0 = \min \left\{ \frac{4[1 - \Phi(\frac{x}{d\sigma\sqrt{T}})]^2}{\exp\{\frac{2}{\sigma^2}(\lambda^2 + \delta^2) T\}} , \frac{x}{\sqrt{2\pi}\sigma} \int_0^T t^{-\frac{3}{2}} \exp\left\{-\frac{(x+\mu t)^2}{2\sigma^2 t}\right\} dt \right\},
\]
\[
T_{X_{b_0}}^{\pi_{b_0}} = \inf \{ t \geq 0 : R_t^{\pi_{b_0}} = 0 \}.
\]

If the company’s preferred risk level is $\varepsilon(\leq \varepsilon_0)$, i.e.,
\[
P[T_{X_{b_0}}^{\pi_{b_0}} \leq T] \leq \varepsilon,
\] (6)
then the company has to reject the policy $\pi_{b_0}^*$ because it does not meet safety requirement (6) by (5), and the insurance company is a business affected with a public interest,
The best way to the company with the model (1)
The best way to the company with the model (1)

and insureds and policy-holders should be protected against insurer insolvencies. So the best policy $\pi^*_b(b \geq b_0)$ of the company should meet the following
and insureds and policy-holders should be protected against insurer insolvencies. So the best policy $\pi^*_b (b \geq b_0)$ of the company should meet the following

- The safety standard (6)
The best way to the company with the model (1)

and insureds and policy-holders should be protected against insurer insolvencies. So the best policy $\pi^*_b(b \geq b_0)$ of the company should meet the following

- The safety standard (6)
- The cost for safety standard (6) being minimal
and insureds and policy-holders should be protected against insurer insolvencies. So the best policy $\pi^*_b(b \geq b_0)$ of the company should meet the following:

- The safety standard (6)
- The cost for safety standard (6) being minimal

We establish setting to solve the problems above as follows.
General setting optimal control problem for the model (1) with solvency constraints

- For a given admissible policy $\pi$ the performance function

$$ J(x, \pi) = \mathbb{E}\{ \int_0^{T_x^\pi} e^{-ct} dL_t^\pi \} $$

- The optimal return function

$$ V(x) = \sup_{b \in \mathcal{B}} \{ V(x, b) \} $$

where $V(x, b) = \sup_{\pi \in \Pi_b} \{ J(x, \pi) \}$, solvency constraint set

$$ \mathcal{B} := \{ b : \mathbb{P}[\tau^\pi_b \leq T] \leq \varepsilon, J(x, \pi_b) = V(x, b) \text{ and } \pi_b \in \Pi_b \} $$

$$ \Pi_b = \{ \pi \in \Pi : \int_0^\infty I_{\{s: R^\pi_s < b\}} dL_s^\pi = 0 \} $$

with property: $\Pi = \Pi_0$ and $b_1 > b_2 \Rightarrow \Pi_{b_1} \subset \Pi_{b_2}$.
Main goal

Finding value function $V(x)$, an optimal dividend policy $\pi_{b^*}$ and the optimal dividend level $b^*$ to solve the sub-optimal control problem (7) and (8), i.e., $J(x, \pi_{b^*}) = V(x)$.

Our main results are the following
Main Results

Theorem

Assume that transaction cost $\lambda - \mu > 0$. Let level of risk $\varepsilon \in (0, 1)$ and time horizon $T$ be given.

(i) If $P[\tau_{b_0}^{\pi^*_b} \leq T] \leq \varepsilon$, then we find $f(x)$ such that the value function $V(x)$ of the company is $f(x)$, and $V(x) = V(x, b_0) = J(x, \pi^*_b) = V(x, 0) = f(x)$. The optimal policy associated with $V(x)$ is $\pi^*_b = \{A^*_b(R^*_{t_{b_0}}, L^*_{t_{b_0}}), L^*_{t_{b_0}}\}$, where $(R^*_{t_{b_0}}, L^*_{t_{b_0}})$ is uniquely determined by the following SDE with reflection boundary:
Main Results

Theorem (continue)

\[
\begin{aligned}
    dR_{t}^{\pi_{0}} &= (\mu - (1 - A_{b_{0}}^{\pi_{0}} (R_{t}^{\pi_{0}}))) \lambda) dt + \sigma A_{b_{0}}^{\pi_{0}} (R_{t}^{\pi_{0}}) dW_{t} - dL_{t}^{\pi_{0}}, \\
    R_{0}^{\pi_{0}} &= x, \\
    0 &\leq R_{t}^{\pi_{0}} \leq b_{0}, \\
    \int_{0}^{\infty} I_{\{t: R_{t}^{\pi_{0}} < b_{0}\}} (t) dL_{t}^{\pi_{0}} &= 0
\end{aligned}
\]

(9)

and \( \tau_{x}^{\pi_{0}} = \inf\{ t : R_{t}^{\pi_{0}} = 0 \} \). The optimal dividend level is \( b_{0} \). The solvency of the company is bigger than \( 1 - \varepsilon \).
Main Results

Theorem (continue)

(ii) If \( P[\tau_{b_0}^{\pi_b} \leq T] > \varepsilon \), then there is a unique \( b^* > b_0 \) satisfying 
\[ P[\tau_{b^*}^{\pi_{b^*}} \leq T] = \varepsilon \] 
and find \( g(x) \) such that \( g(x) \) is the value function of the company, that is,
\[
g(x) = \sup_{b \in \mathcal{B}} \{ V(x, b) \} = V(x, b^*) = J(x, \pi_{b^*}) \tag{10}
\]
and
\[ b^* \in \mathcal{B}, \tag{11} \]

where
\[
\mathcal{B} := \{ b : P[\tau_{b}^{\pi_b} \leq T] \leq \varepsilon, J(x, \pi_b) = V(x, b) \text{ and } \pi_b \in \Pi_b \}.
\]
Main Results

Theorem (continue)

The optimal policy associated with \( g(x) \) is

\[
\pi^*_{b^*} = \left\{ A^*_{b^*} \left( R^*_{\pi^*_{b^*}} \right), L^*_{\pi^*_{b^*}} \right\}, \text{ where } (R^*_{\pi^*_{b^*}}, L^*_{\pi^*_{b^*}}) \text{ is uniquely determined by the following SDE with reflection boundary:}
\]

\[
\begin{cases}
    dR^\pi_{b^*} = (\mu - (1 - A^*_{b^*}(R^*_{\pi^*_{b^*}}))\lambda)dt + \sigma A^*_{b^*}(R^*_{\pi^*_{b^*}})dW_t - dL^\pi_{b^*}, \\
    R^\pi_{b^*} = x, \\
    0 \leq R^\pi_{b^*} \leq b^*, \\
    \int_0^\infty \int_{\{t: R^\pi_{b^*} < b^*\}} (t)dL^\pi_{b^*} = 0
\end{cases}
\]

and \( \tau^\pi_{b^*} = \inf\{t : R^\pi_{b^*} = 0\} \). The optimal dividend level is \( b^* \). The optimal dividend policy \( \pi^*_{b^*} \) and the optimal dividend \( b^* \) ensure that the solvency of the company is \( 1 - \varepsilon \).
Main Results

Theorem (continue)

(iii) \[
\frac{g(x, b^*)}{g(x, b_0)} \leq 1. \tag{13}
\]

(iv) Given risk level $\varepsilon$ risk-based capital standard $x = x(\varepsilon)$ to ensure the capital requirement of can cover the total given risk is determined by $\varphi^b(T, x(\varepsilon)) = 1 - \varepsilon$, where $\varphi^b(T, y)$ satisfies

\[
\begin{aligned}
\varphi_t^b(t, y) &= \frac{1}{2} [A^*_b(y)]^2 \sigma^2 \varphi_{yy}^b(t, y) + (\lambda A^*_b(y) - \delta) \varphi_y^b(t, y), \\
\varphi^b(0, y) &= 1, \text{ for } 0 < y \leq b, \\
\varphi^b(t, 0) &= 0, \varphi_y^b(t, b) = 0, \text{ for } t > 0.
\end{aligned}
\tag{14}
\]
Main Results

Theorem (continue)

where $f(x)$ is defined as follows: If $\lambda \geq 2\mu$, then

$$f(x) = \begin{cases} 
  f_1(x, b_0) = C_0(b_0)(e^{\zeta_1 x} - e^{\zeta_2 x}), & x \leq b_0, \\
  f_2(x, b_0) = C_0(b_0)(e^{\zeta_1 b_0} - e^{\zeta_2 b_0}) + x - b_0, & x \geq b_0.
\end{cases}$$  \tag{15}

If $\mu < \lambda < 2\mu$, then

$$f(x) = \begin{cases} 
  f_3(x, b_0) = \int_0^x X^{-1}(y)dy, & x \leq m, \\
  f_4(x, b_0) = \frac{C_1(b_0)}{\zeta_1} \exp(\zeta_1(x - m)) + \frac{C_2(b_0)}{\zeta_2} \exp(\zeta_2(x - m)), & m < x < b_0, \\
  f_5(x, b_0) = \frac{C_1(b_0)}{\zeta_1} \exp(\zeta_1(b_0 - m)) + \frac{C_2(b_0)}{\zeta_2} \exp(\zeta_2(b_0 - m)) \\
  + x - b_0, & x \geq b_0.
\end{cases}$$
Main Results

Theorem (continue)

\( g(x) \) is defined as follows: If \( \lambda \geq 2\mu \), then

\[
g(x) = \begin{cases} 
  f_1(x, b), & x \leq b, \\
  f_2(x, b), & x \geq b.
\end{cases}
\] (17)

If \( \mu < \lambda < 2\mu \), then

\[
g(x) = \begin{cases} 
  f_3(x, b), & x \leq m(b), \\
  f_4(x, b), & m(b) < x < b, \\
  f_5(x, b), & x \geq b.
\end{cases}
\] (18)
Main Results

Theorem (continue)

\( A^*(x) \) is defined as follows: If \( \lambda \geq 2\mu \), then \( A^*(x) = 1 \) for \( x \geq 0 \). If \( \mu < \lambda < 2\mu \), then

\[
A^*(x) = A(x, b_0) := \begin{cases} 
-\frac{\lambda}{\sigma^2}(X^{-1}(x))X'(X^{-1}(x)), & x \leq m, \\
1, & x > m,
\end{cases} (19)
\]

where \( X^{-1} \) denotes the inverse function of \( X(z) \), and

\[
X(z) = C_3(b_0)z^{-1-c/\alpha} + C_4(b_0) - \frac{\lambda - \mu}{\alpha + c} \ln z, \quad \forall z > 0, \quad m(b_0) = X(z_1)
\]
Main Results

Theorem (continue)

\[ \zeta_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 c}}{\sigma^2}, \quad \zeta_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 c}}{\sigma^2}, \]

\[ b_0 = 2 \ln \left| \frac{\zeta_2 / \zeta_1}{\zeta_2 - \zeta_1} \right|, \quad C_0(b_0) = \frac{1}{\zeta_1 e^{\zeta_1 b_0} - \zeta_2 e^{\zeta_2 b_0}}, \Delta = b_0 - m, \]

\[ z_1 = z_1(b_0) = \frac{\zeta_1 - \zeta_2}{(-\zeta_2 - \lambda/\sigma^2) e^{\zeta_1 \Delta} + (\zeta_1 + \lambda/\sigma^2) e^{\zeta_2 \Delta}}, \]

\[ C_1(b_0) = z_1 \frac{-\zeta_2 - (\lambda/\sigma^2)}{\zeta_1 - \zeta_2}, \quad C_2(b_0) = z_1 \frac{\zeta_1 + (\lambda/\sigma^2)}{\zeta_1 - \zeta_2}, \]

\[ C_3(b_0) = z_1^{1+c/\alpha} \frac{\lambda(c + \alpha(2\mu/\lambda - 1))}{2(\alpha + c)^2}, \quad \alpha = \frac{\lambda^2}{2\sigma^2}, \]

\[ C_4(b_0) = -\frac{(\lambda - \mu)c}{(\alpha + c)^2} + \frac{(\lambda - \mu)\alpha}{(\alpha + c)^2} \ln C_3(b_0) + \frac{(\lambda - \mu)\alpha}{(\alpha + c)^2} \ln \left( \frac{\alpha + c}{(\lambda - \mu)c} \right). \]
For a given level of risk and time horizon, if probability of bankruptcy is less than the level of risk, the optimal control problem of (7) and (8) is the traditional (3) and (4), the company has higher solvency, so it will have good reputation. The solvency constraints here do not work. This is a trivial case.
If probability of bankruptcy is large than the level of risk $\varepsilon$, the traditional optimal policy will not meet the standard of security and solvency, the company needs to find a sub-optimal policy $\pi^{*\text{b}}$ to improve its solvency. The sub-optimal reserve process $R^{\pi^{*\text{b}}}_t$ is a diffusion process reflected at $b^*$, the process $L^{\pi^{*\text{b}}}_t$ is the process which ensures the reflection. The sub-optimal action is to pay out everything in excess of $b^*$ as dividend and pay no dividend when the reserve is below $b^*$, and $A^*(b^*, x)$ is the sub-optimal feedback control function. The solvency probability is $1 - \varepsilon$. 
Economic and financial explanation

- We proved that the value function is decreasing w.r.t $b$ and the bankrupt probability is decreasing w.r.t. $b$, so $\pi_{b^*}$ will reduce the company’s profit, on the other hand, in view of $\mathbb{P}[\tau_{b^*} \leq T] = \varepsilon$, the cost of improving solvency is minimal and is $g(x, b_0) - g(x, b^*)$. Therefore the policy $\pi_{b^*}$ is the best equilibrium action between making profit and improving solvency.
Economic and financial explanation

- We proved that the value function is decreasing w.r.t. $b$ and the bankrupt probability is decreasing w.r.t. $b$, so $\pi_{b^*}$ will reduce the company’s profit, on the other hand, in view of $\mathbb{P}[\tau^{\pi_{b^*}} \leq T] = \varepsilon$, the cost of improving solvency is minimal and is $g(x, b_0) - g(x, b^*)$. Therefore the policy $\pi_{b^*}$ is the best equilibrium action between making profit and improving solvency.

- The risk-based capital $x(\varepsilon, b^*)$ to ensure the capital requirement of can cover the total risk $\varepsilon$ can be determined by numerical solution of $1 - \varphi^{b^*}(x, b^*) = \varepsilon$ based on (14). The risk-based capital $x(\varepsilon, b^*)$ decreases with risk $\varepsilon$, i.e., $x(\varepsilon, b^*)$ increases with solvency, so does risk-based dividend level $b^*(\varepsilon)$. 

Higher risk will get higher return.
Economic and financial explanation

- We proved that the value function is decreasing w.r.t $b$ and the bankrupt probability is decreasing w.r.t. $b$, so $\pi_{b^*}$ will reduce the company’s profit, on the other hand, in view of $P[\tau_{b^*} \leq T] = \varepsilon$, the cost of improving solvency is minimal and is $g(x, b_0) - g(x, b^*)$. Therefore the policy $\pi_{b^*}$ is the best equilibrium action between making profit and improving solvency.

- The risk-based capital $x(\varepsilon, b^*)$ to ensure the capital requirement of can cover the total risk $\varepsilon$ can be determined by numerical solution of $1 - \varphi^{b^*}(x, b^*) = \varepsilon$ based on (14). The risk-based capital $x(\varepsilon, b^*)$ decreases with risk $\varepsilon$, i.e., $x(\varepsilon, b^*)$ increases with solvency, so does risk-based dividend level $b^*(\varepsilon)$.

- The premium rate will increase the company’s profit. Higher risk will get higher return.
8 steps to get solution

Step 1: Prove **the inequality (5)** by Girsanov theorem, comparison theorem on SDE, B-D-G inequality.
8 steps to get solution

- Step 1: Prove the inequality (5) by Girsanov theorem, comparison theorem on SDE, B-D-G inequality.
- Step 2: Prove

**Lemma 1**

Assume that $\delta = \lambda - \mu > 0$ and define $(R_t^{\pi^*_b, b}, L_t^{\pi^*_b})$ by the following SDE:

\[
\begin{cases}
    dR_t^{\pi^*_b, b} = (\mu - (1 - A^*_b(R_t^{\pi^*_b, b}))\lambda)dt + \sigma A^*_b(R_t^{\pi^*_b, b})dW_t - dL_t^{\pi^*_b}, \\
    R_0^{\pi^*_b, b} = b, \\
    0 \leq R_t^{\pi^*_b, b} \leq b, \\
    \int_0^\infty \int_{\{R_t^{\pi^*_b, b} < b\}} (t)dL_t^{\pi^*_b} = 0.
\end{cases}
\]

Then \( \lim_{b \to \infty} \mathbb{P}[\tau_{\pi^*_b}^b \leq T] = 0. \)
8 steps to get solution

- Step 3: Solving HJB equation to determine the value function $g(x, b)$
8 steps to get solution

- **Step 3:** Solving HJB equation to determine the value function $g(x, b)$
- **Step 4:** Prove value function $g(x, b)$ is strictly decreasing w.r.t. $b$
8 steps to get solution

- **Step 3:** Solving HJB equation to determine the value function $g(x, b)$
- **Step 4:** Prove value function $g(x, b)$ is strictly decreasing w.r.t. $b$
- **Step 5:** Prove the probability of bankruptcy $\mathbb{P}[\tau_b^b \leq T]$ is a strictly decreasing function of $b$ by Girsanov theorem, comparison theorem on SDE, B-D-G inequality and strong Markov property.
8 steps to get solution

- **Step 3:** Solving HJB equation to determine the value function $g(x, b)$
- **Step 4:** Prove value function $g(x, b)$ is strictly decreasing w.r.t. $b$
- **Step 5:** Prove the probability of bankruptcy $\mathbb{P}[\tau_b^b \leq T]$ is a strictly decreasing function of $b$ by Girsanov theorem, comparison theorem on SDE, B-D-G inequality and strong Markov property.
- **Step 6:** Prove the probability of bankruptcy $\psi^b(T, b) = \mathbb{P}\{\tau_{\pi^* b}^{\pi b} \leq T\}$ is continuous function of $b$ by energy inequality approach used in PDE theory.
8 steps to get solution

- Step 3: Solving HJB equation to determine the value function $g(x, b)$
- Step 4: Prove value function $g(x, b)$ is strictly decreasing w.r.t. $b$
- Step 5: Prove the probability of bankruptcy $\mathbb{P}[\tau_b^b \leq T]$ is a strictly decreasing function of $b$ by Girsanov theorem, comparison theorem on SDE, B-D-G inequality and strong Markov property.
- Step 6: Prove the probability of bankruptcy $\psi_b^b(T, b) = \mathbb{P}\{\tau_b^{\pi^*_b} \leq T\}$ is continuous function of $b$ by energy inequality approach used in PDE theory.
- Step 7: Economical analysis
8 steps to get solution

- Step 3: Solving HJB equation to determine the value function $g(x, b)$
- Step 4: Prove value function $g(x, b)$ is strictly decreasing w.r.t. $b$
- Step 5: Prove the probability of bankruptcy $\mathbb{P}[^{b} \leq T]$ is a strictly decreasing function of $b$ by Girsanov theorem, comparison theorem on SDE, B-D-G inequality and strong Markov property.
- Step 6: Prove the probability of bankruptcy $\psi^{b}(T, b) = \mathbb{P}\left\{ \tau_{b}^{\pi^{*}} \leq T \right\}$ is continuous function of $b$ by energy inequality approach used in PDE theory.
- Step 7: Economical analysis
- Step 8: Numerical analysis of PDE by matlab and


References


Thank You!