STABILITY OF EQUILIBRIA OF HAMILTONIAN SYSTEMS

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1. OUTLINE OF RESULTS

The idea of this chapter is simple. The second variation of the Hamiltonian about a critical point decomposes as a sum over the real invariant subspaces I_σ of the linearised equations, corresponding to pairs σ, $-\sigma$ of pure imaginary or pure real eigenvalues and quadruplets σ, $-\sigma$, σ^*, $-\sigma^*$ of complex eigenvalues. If σ is a non-zero pure imaginary simple eigenvalue then the energy on I_σ is either positive or negative. The sign cannot change under continuous change of the Hamiltonian. If two such eigenvalues with the same sign of energy collide then loss of linear stability cannot result, because of energy conservation. Stability can be lost only by collision of eigenvalues with opposite sign of energy or by collision of eigenvalues at zero.

We unfold the various cases with double eigenvalues on the imaginary axis to show what will typically happen as one follows a smooth path of equilibria as parameters vary.

(i) Existence of a non-zero imaginary double eigenvalue with mixed signature and non-trivial Jordan normal form is codimension 1, and has unfolding as illustrated in figure 1, so typically gives a transition to instability.

(ii) Existence of a non-zero double imaginary eigenvalue with mixed signature and diagonal Jordan normal form is codimension 3. Its unfolding is a three-parameter version of figure 2, giving 'bubbles of instability' in many one-parameter slices (figure 3).

(iii) Existence of a non-zero double imaginary eigenvalue with definite signature is codimension 3. The Jordan normal form is always diagonal. The unfolding is a three-parameter version of figure 4, giving 'avoided crossings' in most one-parameter slices (figure 5).

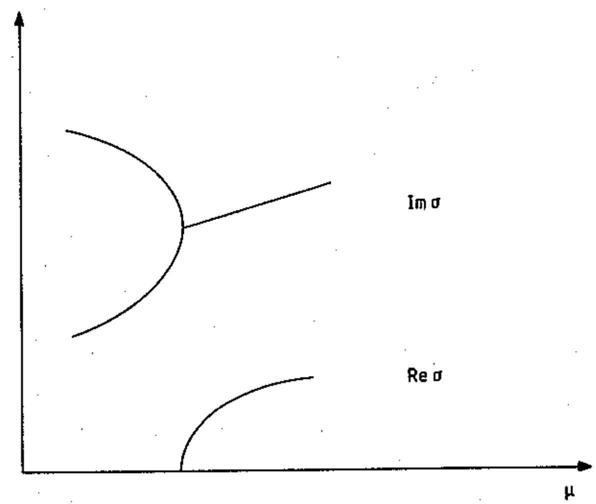

Figure 1. Unfolding a double eigenvalue with mixed signature and non-trivial Jordan normal form.

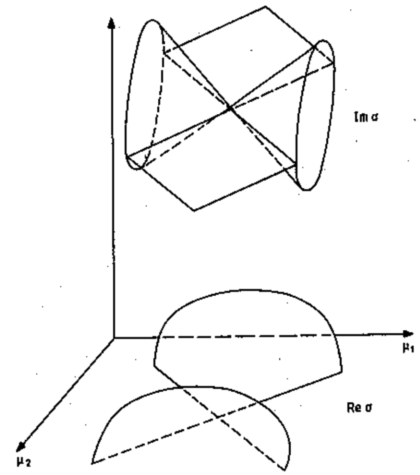

Figure 2.

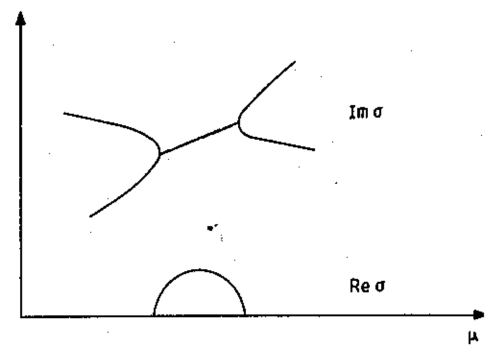

Figure 3.

Figure 2. Unfolding a double eigenvalue with mixed signature and diagonal Jordan normal form.
Figure 3. A bubble of instability.

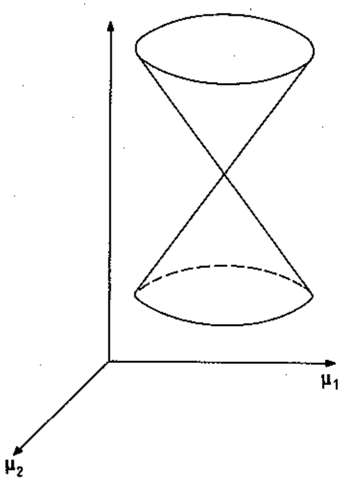

Figure 4.

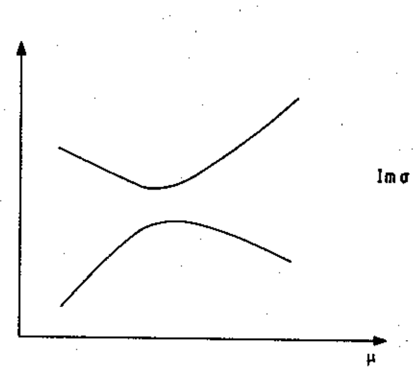

Figure 5.

Figure 4. Unfolding of a double eigenvalue with definite signature.
Figure 5. An avoided crossing.

(iv) Existence of a double eigenvalue at zero with non-trivial Jordan normal form is codimension 1, with unfolding as illustrated in figure 6, so typically gives a transition to instability.

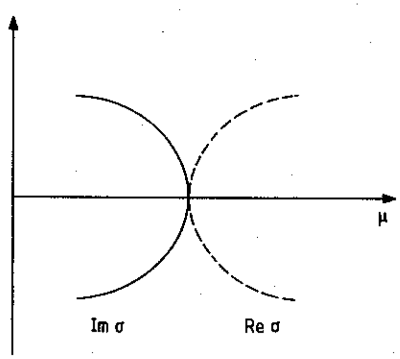

Figure 6. Unfolding of a double eigenvalue at zero with non-trivial Jordan normal form.

(v) Existence of a double eigenvalue at zero with diagonal Jordan normal form is codimension 3. Its unfolding is a three-parameter version of figure 7, giving bubbles of instability in many one-parameter slices.

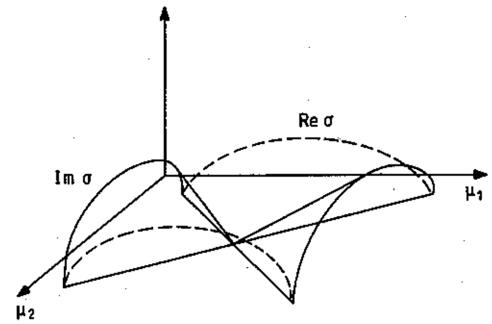

Figure 7. Unfolding of a double eigenvalue at zero with diagonal Jordan normal form.

Note that in cases (iv) and (v) one is not guaranteed persistence of the equilibrium on change of parameters, as it has a zero eigenvalue. However, in case (iv) one typically gets a smooth path of equilibria in the product of the phase and parameter spaces to which the unfolding applies. Case (v) becomes more complicated.

2. REVIEW

We begin by reviewing general notions of Hamiltonian mechanics (see Arnold 1978, for example).

A symplectic form on a manifold M is an antisymmetric, bilinear, closed, non-degenerate function ω from the product of the tangent space TM (infinitesimal displacements) with itself to $\mathbf{R}$. For example

$$\omega((\delta q_1,\delta p_1),\ (\delta q_2,\delta p_2)) = \delta p_1\delta q_2 - \delta p_2\delta q_1 = [\delta q_1\ \ \delta p_1]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\delta q_2\\ \delta p_2\end{bmatrix}$$

for $(\delta q_j,\ \delta p_j) \in \mathbf{R}^n \times \mathbf{R}^n$. In fact, non-degeneracy of ω implies that the dimension of M is even, and, by Darboux's theorem, every symplectic form can be written like this locally in appropriate coordinates.

A symplectic form ω induces an isomorphism from the cotangent space $T'M_z$ (the linear forms from TM_z to $\mathbf{R}$), to the tangent space TM_z at $z \in M$.

We denote it by J , and it is defined as follows:

$$\omega(u, J\cdot L) = L\,u \qquad \text{for} \quad L \in T'M_z \quad u \in TM_z .$$

Think of ω and J as matrices with $J^T\omega = I$, u as a column vector, L as a row vector, $\omega(u,v)$ as $v^T\omega u$, and $J\cdot L$ as JL^T.

A function $H : M \to \mathbb{R}$ (the Hamiltonian) on the manifold, with a symplectic form ω, induces a Hamiltonian flow:

$$z^\cdot = J\cdot DH_z$$

where $z^\cdot$ denotes the derivative with respect to time t and DH_z is the derivative of H at z. The solution z(t) with initial condition z(0) is called the orbit of z(0).

For example,

$$H(q, p) = p^2/2m + V(q)$$

with ω as above, gives the flow

$$q^\cdot = p/m \qquad p^\cdot = -DV_q$$

describing the motion of a particle of mass m in the potential V(q).

A point z_0 where

$$z_0^\cdot = 0$$

is call an equilibrium or stationary point of the flow. Since ω is non-degenerate the equilibria are precisely the critical points of H where

$$DH_{z_0} = 0 .$$

An equilibrium z_0 is said to be stable if for any neighbourhood U of z_0 there is a subneighbourhood V of z_0 such that the orbits of all points of V always remain within U (for both directions of time).

For a Hamiltonian flow, H is conserved

$$H(z)^\cdot = DH_z z^\cdot = \omega(z^\cdot, J\cdot DH_z) = \omega(z^\cdot, z^\cdot) = 0$$

since ω is antisymmetric.

If z_0 is a critical point at which the second derivative $D^2H_{z_0}$ is

definite (i.e. $D^2H_{z_0}(v, v)$ is positive for all non-zero $v \in TM_{z_0}$, or negative for all non-zero $v \in TM_{z_0}$), then the level surfaces of H form locally a foliation by slightly deformed ellipsoids. Thus we obtain a sufficient condition for stability of an equilibrium, due to Dirichlet.

Dirichlet's criterion. If z_0 is an equilibrium at which $D^2H_{z_0}$ is definite, then z_0 is stable.

The second derivative D^2H_z can be thought of either as a symmetric bilinear form on TM_z: $(u,v) \to D^2H(u,v)$, or as a linear map of TM_z to $T'M_z$: $v \to L_v = \{u \to D^2H(u,v)\}$. Linearising about an equilibrium z_0 gives the tangent flow

$$v^\cdot = D(J\cdot DH_{z_0})v = J\cdot D^2H_{z_0} v \qquad v \in TM_{z_0}$$

since $DH_{z_0} = 0$. The eigenvalues of $J\cdot D^2H_{z_0}$ are of vital importance to the stability of z_0. The equilibrium z_0 has an eigenvalue 0 if and only if $D^2H_{z_0}$ is degenerate (i.e. $\exists\, v \in TM_{z_0}, v \neq 0$, such that $\forall\, w \in TM_{z_0}$, $D^2H_{z_0}(v, w) = 0$), because ω is non-degenerate. If z_0 has no eigenvalue 0, then it is isolated from other equilibria (contraction mapping principle).

Now suppose we have a smooth family of Hamiltonian flows, and for parameter value $\mu = 0$ there is an equilibrium z_0. If z_0 has no eigenvalue 0 then, by the implicit function theorem, there is a locally unique equilibrium z_μ for all small enough μ, and it depends smoothly on μ. Thus we get the following corollary to Dirichlet's criterion.

Corollary. An equilibrium z_0 for which $D^2H_{z_0}$ is definite can lose stability as parameters μ vary only by $D^2H_{z_\mu}$ losing definiteness.

The first purpose of this chapter is to generalise Dirichlet's criterion and its corollary to some cases where $D^2H_{z_0}$ is not definite, provided one weakens the notion of stability.

3. A THEOREM IN LINEAR ALGEBRA

Definitions. An equilibrium z_0 is said to be linearly stable if all its tangent orbits are bounded. It is spectrally stable if all its eigenvalues lie on the imaginary axis.

For a Hamiltonian system, if σ is an eigenvalue of z_0 then so are $-\sigma, \sigma^*, -\sigma^*$ (though not necessarily distinct). This is because if L is an eigenform (left eigenvector) with eigenvalue σ, then $J \cdot L$ is an eigenvector with eigenvalue $-\sigma$, L^* has eigenvalue σ^*, and $J \cdot L^*$ has eigenvalue $-\sigma^*$. In fact, they each have the same multiplicity, and by elimination 0 has even multiplicity. Thus, stable implies linearly stable which implies spectrally stable. Furthermore, if z_0 is spectrally stable and all the eigenvalues are simple then z_0 is linearly stable, and remains so for all small enough Hamiltonian perturbations.

Definition. If σ is an eigenvalue let E_σ be the correspondingly eigenspace, i.e. $\{v \in TM_{z_0} : \exists\, n \in \mathbb{Z}_+ \text{ such that } (J \cdot D^2H - \sigma I)^n v = 0\}$.

Lemma 1. For all $v, w \in TM_z$, $D^2H_z(v,w)^\cdot = 0$.

Proof. By differentiating the law of conservation of energy. Equivalently, in the case that z is an equilibrium

$$D^2H(v,w)^\cdot = D^2H(v^\cdot, w) + D^2H(v, w^\cdot)$$

$$= \omega(v^\cdot, J \cdot D^2H\, w) + \omega(w^\cdot, J \cdot D^2H\, v)$$

$$= \omega(v^\cdot, w^\cdot) + \omega(w^\cdot, v^\cdot) = 0$$

still true for Hamiltonian fluid (eg Euler)

by symmetry of D^2H and antisymmetry of ω.

Lemma 2. If $v_1 \in E_{\sigma_1}$, $v_2 \in E_{\sigma_2}$ (σ_1, σ_2 not necessarily distinct), with $\sigma_1 + \sigma_2 \neq 0$, then $D^2H_{z_0}(v_1, v_2) = 0$.

Corollary. For $\sigma \neq 0$, $D^2H(v,v) = 0\ \forall v \in E_\sigma$.

Proof of Lemma 2. If v_1, v_2 are eigenvectors, then $v_j^\cdot = \sigma_j v_j$, so

$$0 = D^2H(v_1, v_2)^\cdot = (\sigma_1 + \sigma_2)\, D^2H(v_1, v_2) .$$

Thus, if $\sigma_1 + \sigma_2 \neq 0$ then $D^2H(v_1, v_2) = 0$.

If v_1 or v_2 is not an eigenvector, then it gives a solution of the form

$$v_j(t) = \sum_{0 \le n \le n_j} v_{jn}\, t^n \exp(\sigma_j t)$$

for some $n_j \in \mathbb{Z}_+$, and vectors v_{jn} with $v_{j0} = v_j$. Then

$$D^2H(v_1(t), v_2(t)) = \exp[(\sigma_1+\sigma_2)t] \sum_{d=0}^{n_1+n_2} t^d \sum_{n+m=d} D^2H(v_{1n}, v_{2m}) .$$

But, by lemma 1, it is constant. Therefore, if $\sigma_1 + \sigma_2 \neq 0$, we get in particular that $D^2H(v_{10}, v_{20}) = 0$, as required.

Definition. If σ is an eigenvalue of z_0, let I_σ be the real invariant space corresponding to eigenvalues $\sigma, -\sigma, \sigma^*, -\sigma^*$.

Lemma 3. If $w = \sum_j v_j$, $v_j \in I_{\sigma_j}$, $(I_{\sigma_j})_j$ distinct, then

$$D^2H(w, w) = \sum_j D^2H(v_j, v_j).$$

Proof. Write $w = \sum_k u_k$, u_k in different eigenspaces E_σ. Then

$$D^2H(w, w) = \sum_{kl} D^2H(u_k, u_l) .$$

By lemma 2, the only terms that contribute are the ones for which $\sigma_k + \sigma_l = 0$. So summing up we get $\sum_j D^2H(v_j, v_j)$, as required.

A particular case of this lemma is that the energy of a superposition of waves is the sum over modes of the energy in each mode.

So we can consider the $D^2H|I_\sigma$ separately.

Lemma 4. If σ is non-zero then $D^2H|I_\sigma$ is non-degenerate.

Proof. If $D^2H|I_\sigma$ is degenerate then $\exists v \in I_\sigma$ such that $\forall w \in I_\sigma$, $D^2H(v,w) = 0$. But by lemma 2, it is also zero for all $w \in I_{\sigma'}$, for any $I_{\sigma'} \neq I_\sigma$. Therefore it is zero for all $w \in TM_{z_0}$. Thus $J \cdot D^2H\, v = 0$, and v has eigenvalue 0.

Consequently, $D^2H|I_\sigma$ can be expressed as a sum of 2s positive and/or negative squares in an appropriate basis, where 2s is the dimension of I_σ (Lagrange's method, e.g. Gantmacher (1959)). The numbers of positive and negative squares are called the signature of $D^2H|I_\sigma$. If σ has a non-zero real part then we know that $D^2H|I_\sigma$ cannot be definite, because energy conservation would imply that all tangent orbits in I_σ are bounded, contradicting the existence of tangent orbits growing exponentially like $\exp[\pm \mathrm{Re}(\sigma)t]$. In fact:

Lemma 5. (i) If σ is pure imaginary and non-zero, then the numbers of positive and negative squares in the diagonal form for $D^2H|I_\sigma$ are both even.

(ii) If σ has non-zero real part then $D^2H|I_\sigma$ has equal numbers of positive and negative squares.

Proof. (i) There is a lot of freedom in the diagonalisation procedure. If σ is non-zero and pure imaginary we shall make a choice that produces squares in pairs of the same sign. As usual, the proof is inductive. We shall describe only the first step.

Choose $\xi_1 \in I_\sigma$ such that $D^2H(\xi_1, \xi_1) = K \neq 0$. There exists such, by non-degeneracy. Then $\exists\ v \in E_\sigma$ such that $\xi_1 = v + v^*$. Let $\xi_2 = -i(v - v^*)$, which is independent of ξ_1. Then, using lemma (i) d

$$\begin{aligned} D^2H(\xi_2, \xi_2) &= -D^2H(v, v) + 2\,D^2H(v, v^*) - D^2H(v^*, v^*) \\ &= 2\,D^2H(v, v^*) \\ &= D^2H(\xi_1, \xi_1)\ . \end{aligned}$$

Similarly, $D^2H(\xi_1, \xi_2) = 0$. Choose $\xi_3, \ldots, \xi_{2s}$ so that $\xi_1, \ldots, \xi_{2s}$ form a basis for I_σ. Then

$$\begin{aligned} D^2H(\textstyle\sum a_j\xi_j, \sum a_j\xi_j) &= K(a_1^2 + a_2^2) + 2\sum_{j>2} a_j[a_1 D^2H(\xi_1,\xi_j) + a_2 D^2H(\xi_2,\xi_j)] \\ &\quad + \sum_{j,k>2} a_j a_k [D^2H(\xi_j, \xi_k)] \\ &= K[a_1 + \sum_{j>2} a_j D^2H(\xi_1,\xi_j)/K]^2 + K[a_2 + \sum_{j>2} a_j D^2H(\xi_2, \xi_j)/K]^2 \\ &\quad + \sum_{j,k>2} a_j a_k [D^2H(\xi_j,\xi_k) - D^2H(\xi_1,\xi_j)D^2H(\xi_1,\xi_k)/K \\ &\qquad - D^2H(\xi_2,\xi_j)D^2H(\xi_2,\xi_k)/K]\ . \end{aligned}$$

The remainder is a quadratic form of dimension 2s-2 with the same properties as $D^2H|I_\sigma$.

(ii) If σ has a non-zero real part then we shall produce squares in pairs of opposite sign. Choose ξ_1 as before. If σ is real then $\exists\ v_\pm \in E_{\pm\sigma}$ such that $\xi_1 = v_+ + v_-$. Let $\xi_2 = v_+ - v_-$. If σ is complex, then $\exists\, v_\pm \in E_{\pm\sigma}$ such that

$$\xi_1 = v_+ + v_- + v_+^* + v_-^*\ .$$

Let

$$\xi_2 = v_+ - v_- + v_+^* - v_-^*\ .$$

In either case

$$D^2H(\xi_2,\xi_2) = -D^2H(\xi_1,\xi_1) \quad \text{and} \quad D^2H(\xi_1,\xi_2) = 0.$$

Proceed as before.

Theorem. If all the eigenvalues of an equilibrium z_0 of a Hamiltonian system with Hamiltonian H are pure imaginary and non-zero, and $D^2H_{z_0}|I_\sigma$ is definite for each eigenvalue σ, then z_0 is linearly stable. The equilibrium can lose spectral stability as parameters vary only by collision of eigenvalues for which $D^2H|I_\sigma$ has opposite signature or by collision of eigenvalues at 0.

Proof. The tangent flow decomposes into the direct sum of the flows on the I_σ. Since H is conserved, definiteness of $D^2H|I_\sigma$ implies that all tangent orbits are bounded, proving the first result. As already remarked, a necessary condition to lose spectral stability is the existence of a multiple eigenvalue σ. But if $D^2H|I_\sigma$ is definite then the eigenvalue cannot split into a pair not purely imaginary because for such a pair, $D^2H|I_\sigma$ has equal numbers of positive and negative squares. The signatures of the $D^2H|I_\sigma$ concatenate on collision of eigenvalues, except possibly at zero, because they are non-degenerate. Hence the second result.

This theorem was essentially known by Weierstrass (1858), and appears in a different guise in Wintner (1935) and the appendix to Moser (1968). An analogous result, 'Krein's theorem' holds for stability of periodic orbits of Hamiltonian systems (see Appendix 29 to Arnold and Avez (1968), references therein and Yakubovitch and Starzhinskii (1975)). It can also be extended to the case of weak dissipation: if one adds negative definite dissipation, then the pure imaginary eigenvalues of positive signature move into the left half-plane (damped) while those of negative signature move into the right half-plane (unstable).

4. APPLICATIONS

The importance of the sign of the energy for small disturbances is already well recognised in some areas of physics. We give some examples.

Nayfeh and Mook (1979) consider the stability of a rotating two-dimensional oscillator. They find that the second-order equations conserve an energy

$$a_1^2 + \nu a_2^2$$

where a_1 and a_2 are the amplitudes of two modes of oscillation, and ν depends on the rotation rate and parameters of the oscillator. Thus if they both have the same sign, then the oscillator is stable, but if they have opposite signs then stability can be lost.

In plasmas (see Hasegawa (1975) and references therein) and fluid flows (Cairns 1979), there can be negative energy waves. These can couple to positive energy waves when the frequencies and wavelengths coincide, leading to instability. Also, if there are dissipative effects, then the negative energy waves can couple to the dissipation, becoming unstable.

The real power of the theorem comes when one is considering possible onset of instability as parameters change, starting from a situation which is easy to analyse, but heading off into an intractable regime. For example, there is a one-parameter family of periodic uniformly travelling water wave solutions for irrotational inviscid flow with a free surface in a gravitational field, which can be parametrised by height. In a frame travelling at the wave speed, they can be described as equilibria of Hamiltonian systems. The linear stability problem splits into subspaces of the form $\exp[i(px+qy)]\, f(x)$, $f(x)$ periodic. The stability is easy to analyse at zero height. All the eigenvalues are pure imaginary. There are some collisions as p,q vary, but they are all innocuous because there is no coupling at zero height. As the height is increased, however, one finds numerically (e.g. Chen and Saffman 1985) that some collisions turn into bubbles of instability, others open up into avoided crossings (figures 3 and 5). By calculating the signatures at zero height and using the above theorem, Saffman and I are able to predict that certain pairs of eigenvalues cannot have a collision leading to instability (MacKay and Saffman 1986). Furthermore, if we make non-degeneracy assumptions, then we can predict an avoided crossing or bubble of instability, using the unfoldings to be given at the end of this chapter. Incidentally, since wave systems are infinite dimensional one might be worried that the above finite-dimensional analysis might not apply, but provided the eigenspaces are finite dimensional and they span the tangent space there is no problem.

As a last general example (Weierstrass 1858), if the Hamiltonian splits as the sum of a positive definite 'kinetic' part depending quadratically on p only and any sort of 'potential' part, depending on q only, with the standard symplectic form, then the signatures of all non-zero eigenvalues on the imaginary axis are positive. This is because for $H(p,q) = K(p) + V(q)$, no eigenvector corresponding to a non-zero eigenvalue can have $\delta p = 0$. Otherwise $\dot{\delta q} = D^2K\,\delta p = 0$, so δq is constant. Similarly for generalised eigenvectors. Hence, in the diagonalisation procedure for pure imaginary eigenvalues we can generate pairs $K_i(\delta p_i^2 + \delta q_i^2)$. Since the kinetic energy is positive definite, the K_i must all be positive. So there is linear stability only at minima of the potential and it can be lost only by an eigenvalue moving to zero. Arnold (1978) shows that the case of a complex quadruplet of eigenvalues cannot occur in this case (§23), and analyses the codimension of all cases of repeated eigenvalues (Appendix 10).

5. ALTERNATIVE DEFINITION OF SIGNATURE

Moser (1958) uses a different definition of signature. We show here that his definition is equivalent to ours. Given a pair σ, $-\sigma$, pure imaginary and non-zero, choose σ to have a positive imaginary part. Then consider the quadratic form

$$i\omega(w, w^*) \qquad w \in E_\sigma .$$

It is real, by antisymmetry of ω. Moser defines the signature of σ and $-\sigma$ to be the signature of this quadratic form

Lemma 6. $i\omega(w, w^*)$, $w \in E_\sigma$ has the same signature as $D^2H|I_\sigma$.

Proof $\forall v \in I_\sigma$, $\exists w \in E_\sigma$ such that $v = w + w^*$, and vice versa. Then since

$$D^2H(w, w) = 0 \qquad \text{for } w \in E_\sigma,$$

we get

$$D^2H(v, v) = 2\, D^2H(w, w^*)$$
$$= (\sigma - \sigma^*)\, \omega(w, w^*).$$

In some cases this is an easier way to calculate the signatures.

6. UNFOLDINGS OF CASES WITH A DOUBLE IMAGINARY EIGENVALUE

There remains the question of whether stability will in fact be lost if the conditions of the theorem are not satisfied. To answer this, given an equilibrium with double non-zero imaginary eigenvalues $\pm\sigma$, and mixed signature, then coordinates can be found on I_σ such that

$$H = q_1p_2 - q_2p_1 + ap_1^2/2$$

with the standard symplectic form (Williamson 1936). We have taken $|\sigma|=1$ by scaling time, and given a slightly different normal form from Williamson. The coefficient a is related to the Jordan normal form: if a=0 it is diagonal, otherwise not. In coordinates (q_1,q_2,p_1,p_2) we get

$$J\cdot D^2H = \begin{bmatrix} 0 & -1 & a & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Now perturb the Hamiltonian by $h(q,p)$, $Dh_{(0,0)} = 0$, depending on a parameter μ, with h=0 when μ=0. To first order in h

$$D = \det J\cdot D^2(H+h) = 1 + 2(h_{q_1p_2} - h_{q_2p_1}) - ah_{q_2q_2}$$

and the sum of the six principal 2×2 minors of $J\cdot D^2(H+h)$ is

$$B = 2 + 2(h_{q_1p_2} - h_{q_2p_1}) + ah_{q_1q_1} .$$

So the characteristic polynomial

$$\sigma^4 + B\sigma^2 + D = 0$$

(the other coefficients are zero since we know that the roots come in pairs $\pm\sigma$) has roots

$$\sigma^2 = -B/2 \pm \sqrt{[B^2/4 - D]}$$

$$= -[1 + h_{q_1p_2} - h_{q_2p_1} + ah_{q_1q_1}/2] \pm \sqrt{[a(h_{q_1q_1} + h_{q_2q_2})]} + O(h^{3/2}) .$$

Thus provided the conditions

$$a \neq 0 \qquad \partial/\partial\mu(h_{q_1q_1} + h_{q_2q_2}) \neq 0$$

are satisfied, we get parabolic collision of the eigenvalues as μ approaches 0 from one side, leading to instability with the real part of σ growing parabolically.

It is interesting to calculate the unfolding for the definite case. Then $D^2H|I_\sigma$ can be put into the form (Williamson 1936)

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) .$$

On adding a perturbation $h(q,p)$, the discriminant $B^2/4 - D$ vanishes to first order, so it is necessary to calculate it to higher order. Then one finds to second order

$$B^2/4 - D = \tfrac{1}{4}(h_{p_1p_1} + h_{q_1q_1} - h_{p_2p_2} - h_{q_2q_2})^2 + (h_{p_1p_2} + h_{q_1q_2})^2 + (h_{q_1p_2} - h_{p_1q_2})^2 \geq 0.$$

Thus using Morse's lemma to take care of higher-order terms, the eigenvalues typically separate linearly with μ, remaining on the imaginary axis. Note, however, that three conditions must be satisfied to get coincident roots, so in one- or even two-parameter families, crossings of eigenvalues of the same signature are typically avoided. Galin (1975)(see Arnold (1978) App 6 for an English summary) has proved that existence of a double eigenvalue with definite signature is indeed codimension 3 as the second-order calculation given above suggests.

There is a subclass of Hamiltonian systems into which many applications fall. They are the reversible Hamiltonian systems, defined by existence of an involution S (S^2 = id) which reverses the flow. For example, with the standard symplectic form, if H is even in the momenta, take $S(p,q) = (-p,q)$. If z_0 is a symmetric equilibrium, i.e. $Sz_0 = z_0$, then the last term in the above expression for $B^2/4 - D$ is always zero. This suggests that possession of a double eigenvalue of definite signature is only codimension 2 in the space of reversible systems. This has been proved by Jimenez and MacKay (1986). But this is still enough to lead typically to avoided crossings in one-parameter families.

There is another case that is worth unfolding, since it occurs quite often. This is the case of mixed signature with diagonal Jordan normal form. We can take

$$H = q_1p_2 - q_2p_1 + h(q,p).$$

To second order one finds

$$B^2/4 - D = \tfrac{1}{4}(h_{p_1p_1}+h_{p_2p_2}+h_{q_1q_1}+h_{q_2q_2})^2 - \tfrac{1}{4}(h_{p_1p_1}+h_{p_2p_2}-h_{q_1q_1}-h_{q_2q_2})^2$$
$$-(h_{p_1q_1}+h_{p_2q_2})^2 .$$

Thus using Morse's lemma again a crossing in the 'uncoupled' case (last two terms equal to zero) typically develops into a bubble of instability when coupling is added. A more natural normal form, perhaps, for this case is

$$H = \tfrac{1}{2}(q_1{}^2+p_1{}^2) - \tfrac{1}{2}(q_2{}^2+p_2{}^2)$$

for which, to second order

$$B^2/4-D = \tfrac{1}{4}(h_{p_1p_1}+h_{q_1q_1}+h_{p_2p_2}+h_{q_2q_2})^2 - (h_{p_1p_2}-h_{q_1q_2})^2 - (h_{p_1q_2}+h_{p_2q_1})^2$$

giving the same result. When $B^2/4 - D = 0$, one can check that the Jordan normal form is non-trivial unless all three squares are zero, so we expect the case of a double eigenvalue with mixed signature and diagonal Jordan normal form to be codimension 3. Again this was proved by Galin (1975). As for the definite case, the codimension drops to 2 when one restricts to reversible systems (Jimenez and MacKay 1986). Also one can check that the signatures of the eigenvalues are exchanged as one passes from one of the stable regions to the other.

Lastly, one might wish to know what happens when a pair of imaginary eigenvalues $\pm\sigma$ collide at zero. A normal form for $D^2H|I_\sigma$ with a double eigenvalue $\sigma = 0$ is (Williamson 1936)

$$H(p,q) = a\ p^2/2$$

giving

$$J \cdot D^2H = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

which has non-trivial Jordan normal form if $a \neq 0$. The eigenvalues of $J \cdot D^2(H+h)$ are given by

$$\sigma^2 = h_{pq}^2 - ah_{qq} + h_{qq}h_{pp} .$$

So the case of non-trivial Jordan normal form is codimension 1 with

$$\sigma^2 = -a\ h_{qq}$$

to first order. The case of diagonal Jordan normal form gives

$$\sigma^2 = h_{pq}^2 + \tfrac{1}{4}(h_{qq}+h_{pp})^2 - \tfrac{1}{4}(h_{qq}-h_{pp})^2 .$$

As mentioned in the outline, one is not guaranteed persistence of an equilibrium as parameters vary when it has an eigenvalue zero. Thus these last two unfoldings are valid only if the equilibria form smooth submanifolds in the product of a parameter space and the phase space. In fact under the following non-degeneracy conditions on a one-parameter family $H(p,q,\mu)$ with quadratic part $ap^2/2$ at $\mu=0$

$$a \neq 0 \quad H_{qqq} \neq 0 \quad H_{q\mu} \neq 0$$

one gets a smooth path of equilibria

$$\mu \approx - H_{qqq}\ q^2/H_{q\mu}$$

$$p \approx [H_{pqq}H_{q\mu}/2H_{qqq} - H_{p\mu}]\mu/a$$

('tangent bifurcation', 'saddle-node bifurcation', 'limit point'). If $a = 0$ then apart from exceptional cases one can choose coordinates so that in the critical case

$$H(p,q) = p^3 \pm pq^2 .$$

These are codimension 3 and lead to a set of equilibria in their unfoldings called 'umbilics'. We leave it to the standard books on catastrophe theory to describe them.

A complete list of normal forms, codimensions and unfoldings for linear Hamiltonian systems has been given by Galin (1975) (see Arnold 1978, Appendix 6), but the importance of the signature does not seem to have been emphasised there. Neither was the effect of reversibility discussed. Analogous results for linear stability of periodic orbits of Hamiltonian systems will be presented in Howard and MacKay (1986).

7. CONCLUSION

The results described in this chapter should be more widely known and used than they are, in particular now that Hamiltonian formulations are known for so many systems.

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CALCULATION OF TRANSPORT COEFFICIENTS IN CHAOTIC SYSTEMS

A S BLAND AND G ROWLANDS

1. INTRODUCTION

It is now recognised that on the one hand complicated behaviour of physical systems can be understood in terms of relatively simple mathematical models whilst on the other hand simple equations can have complicated solutions. A classic example is the logisitc equation

$$x_{n+1} = \lambda x_n(1 - x_n) \,. \qquad (1.1)$$

Here λ is a parameter whilst x_n describes the dynamics of the system. This equation describes, with changing λ, a bifurcation sequence eventually leading to chaos - a complicated solution. Feigenbaum has shown the universality of the behaviour and suggested it as a model for the onset of turbulence in physical systems - a complicated behaviour.

An example more germane to the present discussion is that of a charged particle in a spatially non-uniform magnetic field. In dimensionless form we write the equation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \mathbf{v} \wedge \mathbf{B}(\varepsilon\mathbf{x}) \qquad (1.2)$$

where $\mathbf{B}$, the magnetic field, is a given function of space through its dependence on $\mathbf{x}$. We have introduced the parameter ε as a measure of the non-uniformity. For general fields a single constant of the motion exists, namely the kinetic energy v^2. For a uniform magnetic field ($\varepsilon \equiv 0$) the particle undergoes helical motion along the direction of the magnetic field with the perpendicular energy, $v_\perp^2$, and the parallel energy, $v_\parallel^2$, ($v^2 = v_\perp^2 + v_\parallel^2$) independently constant.

For small ε it may be shown that an adiabatic invariant (or magnetic moment), μ, exists such that $\mu(= v_\perp^2/|\mathbf{B}|)$ is a constant to order ε^2. The